A Min-Max Approach to the Nonuniform N-Dimensional FFT for Rapid Iterative Reconstruction of MR Images

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Introduction

The FFT has been widely used in many areas of signal processing for the efficient computation of the DFT in O(NlogN) computations instead of O(N^2). But in many applications, such as non-rectilinear k-space scanning in MRI [5], the data are not uniformly spaced. Many papers have been written on approximating the nonuniform DFT by interpolating an upsampled version of the FFT, beginning with [3] and including [1,2,4,6,7,8,10,11]. This paper presents an algorithm that provides the best approximation in the min-max sense to the nonuniform DFT (NUFFT), i.e. we minimize the worst case approximation error. We apply the method to a spiral SENSE imaging experiment [9].

Theory

To use the FFT to approximate the nonequispaced DFT, we take a min-max approach. The problem is as follows: Given equally spaced object samples, x_n, for $n=0,1,...,N-1$ with DSFT, $X(\omega)$, we need to compute the DFT at the locations of the nonuniformly spaced sample points in the frequency domain, ω_m for $m=0,1,...,M-1$, $y_m=X(\omega_m)$.

The algorithm is as follows:

1. O(KlogK) Choose some K≥N and compute the K-point FFT, $X_k=X(2\pi k/K)$

2 O(JM) Then interpolate the X_k to approximate the y_m's using the J_m nearest neighbors of the closest point on the regular grid to ω_m. The approximation, y', is given by,

$y'_m=\sum_{(j=1 \text{ to } Jm)} p_{mj} X_{k(m,j)}$,

where $P=\{p_{mj}\}$ is the MxJ_m interpolation matrix and the $k_{(m,j)}$ are the closest points on the regular grid to ω_m.

We used the min-max approach in choosing the interpolation matrix, P. The interpolation matrix is chosen such that we minimize the maximum error of our estimator over all objects, $\underline{x}$. Looking at the m^{th} row of P, $\underline{p}^{(m)}$, we want to minimize the worst case error,

$min_{\underline{p}^{(m)}}\ max_{\underline{x}:||\underline{x}||=1} |y'_m - y_m|$.

Looking at the estimator error, we get

$|y'_m - y_m|=|(\underline{x},\underline{q}^{(m)})|$,

where q is defined by

$q_n^{(m)}=\sum_{(j=1 \text{ to } Jm)}\ p_j^{(m)*}\ e^{i2\pi k_{(m,j)}n/K} - e^{in\omega_m}$

$= W^{(m)}p^{(m)*} - b^{(m)}$

where $W_{nj}^{(m)}=e^{i2\pi k_{(m,j)}n/K}$ and $b_n^{(m)}=e^{in\omega_m}$.

By Cauchy-Shwarz,

$max_{\underline{x}:||\underline{x}||=1} |(\underline{x},\underline{q}^{(m)})|=||\underline{q}^{(m)}||$.

So, the maximum error of our estimator is given by $||\underline{q}^{(m)}||$. Minimizing $\underline{q}^{(m)}$ over $\underline{p}^{(m)}$ gives

$\underline{p}^{(m)*}=(W^{(m)\prime}W^{(m)})^{-1}W^{(m)\prime}\underline{b}^{(m)}$.

In [8], it was shown that if $J_m=J, \forall m$, $W^{(m)\prime}W^{(m)}$ does not depend on the sample locations and hence can be computed once for all M samples, this matrix was termed the regular Fourier Matrix by [8]. Similar to [8], the entries are given by,

$(W^{(m)\prime}W^{(m)})_{j,k} = (1-e^{-i2\pi N(j-k)/K})/(1-e^{-i2\pi(j-k)/K})$

Looking at $W^{(m)}$, it can be divided into a diagonal matrix and a matrix that does not depend on the sample locations,

$W_{n,j}^{(m)}=e^{i2\pi k_{(m)}n/K}\ e^{i2\pi jn/K}$,

where $k_{(m)}$ is the first neighboring grid point used. Letting

$D_{n,j}^{(m)}=e^{i2\pi k_{(m)}n/K}$, and $W_{n,j}=e^{i2\pi jn/K}$,

$W^{(m)}$ can be rewritten as,

$W^{(m)}=D^{(m)}\ W$.

We have derived the following closed form solution for $D^{(m)\prime}\underline{b}^{(m)}$ that only depends on the residual, $r^{(m)}$, between the sample point, ω_m, and the closest point on the regular grid, $2\pi k_m/K$,

$(W^{(m)\prime}\underline{b}^{(m)})_j = (1-e^{i(r(m)-2\pi j/K)N})/(1-e^{i(r(m)-2\pi j/K)})$,

for $j=1,...,J$. Hence, we can find the rows of the weighting matrix P in O(MJ), since $(W^{(m)\prime}W^{(m)})^{-1}$ can be precomputed. An accuracy-computation time tradeoff is available through the choice of values for the sampling factor, K, and the neighborhood size, J.

Results

Unlike the method presented in [8], the algorithm presented here is easily extended to N-dimensional problems encountered in medical imaging. We applied this algorithm to a conjugate gradient method used in reconstructing SENSE images. One spiral of a two interleaved spiral sequence was used in combination with two 5 in. coils placed on either side of a resolution phantom. The reconstruction was performed using the full DFT routine and the NUFFT routine presented here. The full DFT routine took approximately 8min on a Sun Ultra10 to perform 10 iterations to reconstruct a 64x64 image. Figure 1 shows the timing and normalized mean-squared error (NMSE) when the sampling factor, K is 192. Note that the condition number of $(W^{(m)\prime}W^{(m)})$ can become a problem when larger neighborhood sizes are used, but at these sizes, the gain in NMSE may be outweighed by increases in computation time.

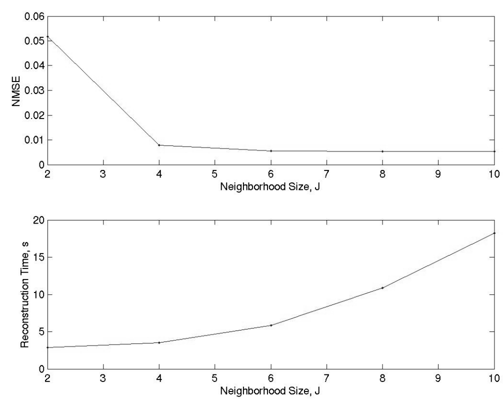

Figure 1: Error and timing results for case of K=192 in both dimensions.

Discussion

We have presented a method that efficiently performs the non-uniform fast Fourier Transform for image reconstruction from data that is not sampled on a Cartesian grid. This method can be used in iterative reconstruction techniques to speed MR image reconstruction, including parallel imaging experiments.

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