System Modeling for Gamma–Ray Imaging Systems

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Dec. 2011, Revised March 12, 2012

Technical Report No. 411
Approved for public release; distribution unlimited.

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Original version: Dec. 2011. Revised: March 12, 2012

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I. Introduction

Gamma–ray imaging systems measure a subset of all emitted photons. The fraction of recorded photons, or system sensitivity, depends on the detector geometry and photon energy. Models for the distribution of gamma–ray measurements should incorporate the knowledge that the photon was detected because the observed measurements are drawn from the distribution of measurements given that the measurements are recorded. This report gives a useful result that simplifies the calculation of the conditional distribution of the measurements given that they are recorded. We also give an example that illustrates an application of the result to a system modeling problem.

A modeled likelihood for a 3D position–sensitive gamma–ray detector was proposed in [1], but the expressions are computationally difficult. A similar system was also modeled in [2] without explicitly conditioning on the event that a photon is detected. This work provides a general result that simplifies the model in [1]and shows that model proposed in [2] correctly conditions on the event that a photon is detected.

This paper is organized as follows: Section II shows how the conditional likelihood given that a photon is detected is related to the likelihood not conditioned on the event that a photon is detected, Sections III and IV give example applications of the results in Section II to a modeling problem, and Section V gives our conclusions.

II. Theory

In this section, we derive a formula for the density of photon measurements given that they are recorded. We consider an attribute vector $\boldsymbol{r}$ corresponding to one photon interaction sequence, where $\boldsymbol{r} \in \mathcal{R}$, which is the set of all recordable attribute vectors. A photon interaction sequence is defined as the ordered list of all interactions resulting from a single photon. The attribute vector $\boldsymbol{r}$ may include recorded interaction positions, recorded energy depositions, or other information about the photon interaction. Let k be the number of recorded photon interactions for the photon interaction sequence described by the attribute vector $\boldsymbol{r}$. In some systems, the number of photon interactions is a random variable, and the number of interactions k is a function of the recorded attribute vector $\boldsymbol{r}$. For example, in a Compton imaging system, the attribute vector $\boldsymbol{r}$ contains the positions and energies of each interaction, so k is a function of the length of $\boldsymbol{r}$ in this case.

A. The event that a photon is detected

When a photon interaction is recorded, the photon must have been incident on the detector and it must have interacted somewhere in the detector. Let D_0 be the event that a photon is incident on the detector and let D_i, $i = 1, 2, \ldots, k$ be the event that the photon has at least i recorded interactions. Let the event that a k–interaction event is recorded be

$$D^{(k)} \triangleq \left(\bigcap_{i=0}^{k} D_i \right) \cap D_{k+1}^{\mathrm{c}} = D_k \cap D_{k+1}^{\mathrm{c}}, \tag{1}$$

where D_{k+1}^{c} is the event that there are no further recorded interactions after the kth interaction.

Let the event that a photon interaction occurs be

$$D \triangleq \bigcup_{j=1}^{\infty} D^{(j)} = D_1, \tag{2}$$

which is the union of all k-interaction events.

B. The likelihood or a photon interaction sequence

The likelihood of a photon interaction sequence is an important component of maximum–likelihood estimation for gamma–ray imaging systems, e.g., [2, 3]. Let $\boldsymbol{\theta}$ be a vector of unknown and nonrandom parameters. In this work, we focus on computing a particular component of the list–mode likelihood in [3], which is the likelihood[1] of a single photon interaction sequence $\mathsf{p}\left(\boldsymbol{r}|D;\boldsymbol{\theta}\right)$. We use the notation $\mathsf{p}\left(\cdot\right)$ to denote *modeled* probability densities and $\mathsf{Pr}\left(A\right)$ to denote the probability of an event A.

[1]The likelihood is defined as the Radon-Nikodym density [4] of the probability measure corresponding to the modeled distribution of recorded attributes with respect to the appropriate base measure. This generality allows the attribute vectors $\boldsymbol{r}$ to be discrete or continuous random vectors.

Theorem 1 shows that the conditional density given that a photon is detected is the quotient of the unconditional density and the sensitivity. This results simplifies the calculation of the system model by separating the conditional probability density into the quotient of the joint density of the recorded attributes and the system sensitivity.

Theorem 1. *For all attribute vectors* $\boldsymbol{r} \in \mathcal{R}$*,*

$$\mathsf{p}\left(\boldsymbol{r}|D;\boldsymbol{\theta}\right) = \frac{\mathsf{p}\left(\boldsymbol{r},D;\boldsymbol{\theta}\right)}{\mathsf{Pr}\left(D;\boldsymbol{\theta}\right)}, \tag{3}$$

up to a set of measure zero, where $\mathsf{Pr}\left(D;\boldsymbol{\theta}\right)$ *is the system sensitivity.*

Proof. Because $\mathsf{p}\left(\boldsymbol{r}|D;\boldsymbol{\theta}\right)$ is the Radon-Nikodym density of the conditional distribution of recorded attributes given that they are detected, for any measurable set $S \subset \mathcal{R}$

$$\mathsf{Pr}\left(\boldsymbol{r} \in S|D;\boldsymbol{\theta}\right) = \int_S \mathsf{p}\left(\boldsymbol{r}|D;\boldsymbol{\theta}\right) d\mu(\boldsymbol{r}), \tag{4}$$

by the definition of the Radon–Nikodym density, where μ is an appropriate base measure. By the definition of conditional probability,

$$\mathsf{Pr}\left(\boldsymbol{r} \in S|D;\boldsymbol{\theta}\right) = \frac{\mathsf{Pr}\left(\{\boldsymbol{r} \in S\} \cap D;\boldsymbol{\theta}\right)}{\mathsf{Pr}\left(D;\boldsymbol{\theta}\right)}.$$

Let $\mathsf{p}\left(\boldsymbol{r},D;\boldsymbol{\theta}\right)$ be the Radon–Nikodym density of $\mathsf{Pr}\left(\{\boldsymbol{r} \in V\} \cap D;\boldsymbol{\theta}\right)$ for a set $V \subset \mathcal{R}$, which exists by the Radon-Nikodym theorem [4]. By the definition of the Radon–Nikodym density,

$$\mathsf{Pr}\left(\boldsymbol{r} \in V|D;\boldsymbol{\theta}\right) = \int_V \frac{\mathsf{p}\left(\boldsymbol{r},D;\boldsymbol{\theta}\right)}{\mathsf{Pr}\left(D;\boldsymbol{\theta}\right)} d\mu(\boldsymbol{r}) \text{ for all } V \subset \mathcal{R}. \tag{5}$$

By the Radon–Nikodym theorem [4], which states that the Radon–Nikodym density is unique up to a set of measure zero, and by (4) and (5),

$$\mathsf{p}\left(\boldsymbol{r}|D;\boldsymbol{\theta}\right) = \frac{\mathsf{p}\left(\boldsymbol{r},D;\boldsymbol{\theta}\right)}{\mathsf{Pr}\left(D;\boldsymbol{\theta}\right)} \text{ except on a set of } \mu\text{–measure zero.}$$

□

The term in the numerator of (3) is complex because D is the event that a photon interaction sequence with *any* number of interactions is recorded. We make the following assumption in an attempt to simplify this term:

Assumption 1. *There exists a function* $f : \mathcal{R} \to \mathbb{N}$ *such that* $k = f(\boldsymbol{r})$ *is the number of photon interactions in the photon interaction sequence described by* $\boldsymbol{r}$*.*

Assumption 1 is satisfied by most systems that record multiple interactions of a single photon. For example, an attribute vector $\boldsymbol{r}$ corresponding to a photon interaction sequence in a 3D position–sensitive semiconductor detector, e.g., [5], contains a position and deposited energy for each interaction. If each position occupies three elements in $\boldsymbol{r}$ and each energy occupies one element, then $f(\boldsymbol{r}) = \mathrm{length}(\boldsymbol{r})/4$, where $\mathrm{length}(\boldsymbol{r})$ is the number of elements in the vector $\boldsymbol{r}$.

Lemma 1 allows us to simplify the numerator of (3).

Lemma 1. *If Assumption 1 is satisfied, then* $\mathsf{p}\left(\boldsymbol{r},D;\boldsymbol{\theta}\right) = \mathsf{p}\left(\boldsymbol{r},D^{(k)};\boldsymbol{\theta}\right)$*, where* $k = f(\boldsymbol{r})$*.*

Proof. Since k is a deterministic function of $\boldsymbol{r}$,

$$\mathsf{p}\left(\boldsymbol{r},D^{(i)};\boldsymbol{\theta}\right) = 0 \text{ for all } i \neq k.$$

□

Combining the results of Lemma 1 and Theorem 1 gives the formula for the likelihood of a single k–interaction event given that it was recorded:

$$\mathsf{p}\left(\boldsymbol{r}|D;\boldsymbol{\theta}\right) = \frac{\mathsf{p}\left(\boldsymbol{r}, D^{(k)};\boldsymbol{\theta}\right)}{s(\boldsymbol{\theta})}, \tag{6}$$

where

$$s(\boldsymbol{\theta}) \stackrel{\triangle}{=} \mathsf{Pr}\left(D;\boldsymbol{\theta}\right)$$

is the system sensitivity.

III. Example: Near field disk detector

We consider a hypothetical 2D disk detector of radius R that records the (x, y) position of gamma–ray interactions [6]. We assume that the source energy and detector material are such that photoelectric absorption occurs with probability one for any interacting photon. Let the detector record photons emanating from an isotropic point–source at a distance r from the center of the detector. We assume that r is the only unknown parameter, so the parameter vector $\boldsymbol{\theta} = r$. Let μ be the linear attenuation coefficient for photoelectric absorption in the detector. A diagram of this detector is shown in Figure 1.

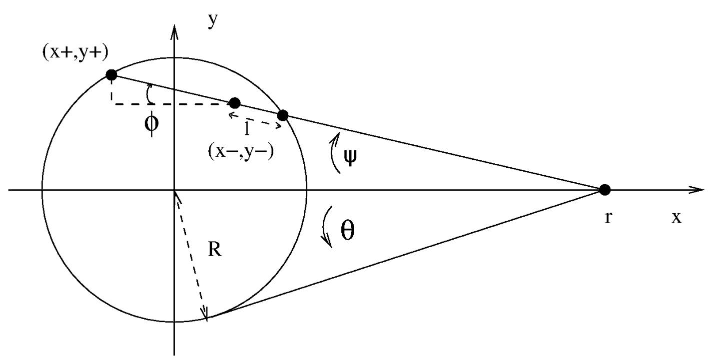

Figure 1: Diagram of near–field disk detector

The near–field likelihood of a recorded photon is difficult to compute in cartesian coordinates, so we parameterize the interaction positions by the observed emission angle $\phi \in [0, 2\pi)$ and interaction depth $l \in \mathbb{R}^+$. The attribute vector for a single photon interaction is $\boldsymbol{r} = (l, \phi)$. The observed emission angle $\phi = 0$ corresponds to the direction of the vector from the source to the center of the detector. Let ψ be the unobserved true emission angle, where the observed emission angle ϕ is a noisy measurement of ψ. Let θ_r be the largest emission angle such that an emitted photon strikes the detector.

We wish to compute the likelihood conditioned on the event D that a photon is detected $\mathsf{p}\left(\boldsymbol{r}|D;\boldsymbol{\theta}\right)$. Since photoelectric absorption is the only possible mode of interaction, we focus on the case where the number of interactions $k = 1$. By Theorem 1 and (6),

$$\begin{aligned}
\mathsf{p}\left(\boldsymbol{r}|D;r\right) &= \frac{\mathsf{p}\left(\boldsymbol{r}, D^{(1)};r\right)}{s(r)} \\
&= \frac{1}{s(r)}\mathsf{p}\left(l,\phi,D_0,D_1,D_2^{\mathrm{c}};r\right) \qquad (7)\\
&= \frac{1}{s(r)}\int_0^{2\pi}\mathsf{p}\left(l,\phi,D_0,D_1,D_2^{\mathrm{c}}|\psi;r\right)\mathsf{p}\left(\psi;r\right)d\psi \\
&= \frac{1}{s(r)}\int_0^{2\pi}\mathsf{p}\left(D_2^{\mathrm{c}}|l,\psi,D_0,D_1;r\right)\mathsf{p}\left(l|\psi,D_0,D_1;r\right)\mathsf{p}\left(D_1|\psi,D_0;r\right)\mathsf{p}\left(\phi|\psi,D_0;r\right)\mathsf{p}\left(D_0|\psi;r\right)\mathsf{p}\left(\psi;r\right)d\psi. \qquad (8)
\end{aligned}$$

The step from (7) to (8) uses the chain rule of conditional probability [7, p. 58] to separate (7) into computable terms. There are numerous other expansions of (7) using the chain rule, but not all of these expansions are easily computable. One must use trial and error to find an application of the chain rule that results in more easily computable terms.

We now derive expressions for each term in (8). Because we assumed that photoelectric absorption is the only possible mode of interaction, the probability of a second interaction is zero, thus

$$\mathsf{p}\left(D_2^{\mathsf{c}}|l,\psi,D_0,D_1;r\right)=1. \tag{9}$$

By the Beer–Lambert law [8, p. 89],

$$\mathsf{p}\left(l|\psi,D_0,D_1;r\right)=\frac{\mu e^{-\mu\left(l-r\cos\psi+\sqrt{R^2-r^2\sin^2\psi}\right)}}{1-e^{-2\mu\sqrt{R^2-r^2\sin^2\psi}}}\mathbb{I}_{\left\{r\cos\psi-\sqrt{R^2-r^2\sin^2\psi}\leq l\leq r\cos\psi+\sqrt{R^2-r^2\sin^2\psi}\right\}}, \tag{10}$$

where $\mathbb{I}_A$ is the indicator function on the set A. Also by the Beer-Lambert law,

$$\mathsf{p}\left(D_1|\psi,D_0;r\right)=1-e^{-2\mu\sqrt{R^2-r^2\sin^2\psi}}. \tag{11}$$

By the detector setup and the definition of θ_r, a photon is incident on the detector if $|\psi|\leq\theta_r$. Thus,

$$\mathsf{p}\left(D_0|\psi;r\right)=\mathbb{I}_{\{|\psi|\leq\theta_r\}}. \tag{12}$$

Since the source is isotropic, the emission angle is uniform over 2π,

$$\mathsf{p}\left(\psi;r\right)=\frac{1}{2\pi}\;\;0\leq\psi<2\pi. \tag{13}$$

For simplicity, we assume that the observed emission angle is equal to the true emission angle with probability one, i.e.,

$$\mathsf{p}\left(\phi|\psi,D_0;r\right)=\delta(\phi-\psi). \tag{14}$$

Combining (8)–(14), we obtain

$$\mathsf{p}\left(\boldsymbol{r}|D;r\right)=\frac{\mu e^{-\mu\left(l-r\cos\phi+\sqrt{R^2-r^2\sin^2\phi}\right)}}{2\pi s(r)}\mathbb{I}_{\{|\phi|\leq\theta_r\}}\mathbb{I}_{\left\{r\cos\phi-\sqrt{R^2-r^2\sin^2\phi}\leq l\leq r\cos\phi+\sqrt{R^2-r^2\sin^2\phi}\right\}}. \tag{15}$$

IV. Example: Far–field 3D position–sensitive detector

In this example, we will derive the likelihood of a two–interaction photon interaction sequence in a 3D position–sensitive detector. This example is applicable to 3D CdZnTe detectors, such as those presented in [5]. The purpose of this example is to use Theorem 1 to simplify the complex derivation of the likelihood for position–sensitive detectors in Section 2.4.3 of [1].

As in [1], let $\boldsymbol{x}_1$ be the true location of the first interaction, e_1^t be the energy deposited in the first interaction, ψ be the angle that enumerates the scatter direction around the Compton cone, and let ρ be the distance between the two interactions. Let $\boldsymbol{r}$ be the vector of observations for one interaction sequence, where

$$\boldsymbol{r}=[\rho,\psi,\boldsymbol{x}_1,e_1^t].$$

Let $\mathcal{R}$ be the set of recordable events. We assume that the nonrandom unknown parameters are the source energy e_0 and source position $\boldsymbol{\phi}$. The purpose of this example is not to estimate the unknown parameters, but rather construct a likelihood that is a function of those parameters.

Two–interaction photoelectric absorption events are detected if the photon is incident (D_0), undergoes one Compton scatter (D_1), undergoes a photoelectric absorption (D_2), and does not interact again (D_3^{c}). Using Theorem 1 and the chain rule of probability,

$$\begin{aligned}\mathsf{p}\left(\boldsymbol{r}|D;e_0,\boldsymbol{\phi}\right) \;=\; & \frac{1}{\mathsf{p}\left(D;e_0,\boldsymbol{\phi}\right)}\mathsf{p}\left(D_c^{\mathsf{c}}|\rho,\psi,\boldsymbol{x}_1,e_1^t,D_2,D_1,D_0;e_0,\boldsymbol{\phi}\right)\mathsf{p}\left(D_2|\rho,\psi,\boldsymbol{x}_1,e_1^t,D_1,D_0;e_0,\boldsymbol{\phi}\right)\\ & \mathsf{p}\left(\rho|\psi,\boldsymbol{x}_1,e_1^t,D_1,D_0;e_0,\boldsymbol{\phi}\right)\mathsf{p}\left(\psi|\boldsymbol{x}_1,e_1^t,D_1,D_0;e_0,\boldsymbol{\phi}\right)\mathsf{p}\left(D_1|\boldsymbol{x}_1,e_1^t,D_0;e_0,\boldsymbol{\phi}\right)\\ & \mathsf{p}\left(\boldsymbol{x}_1|e_1^t,D_0;e_0,\boldsymbol{\phi}\right)\mathsf{p}\left(e_1^t|D_0;e_0,\boldsymbol{\phi}\right)\mathsf{p}\left(D_0;e_0,\boldsymbol{\phi}\right),\end{aligned} \tag{16}$$

where $\mathsf{p}\left(D_2|\rho,\psi,\boldsymbol{x}_1,e_1^t,D_1,D_0;e_0,\boldsymbol{\phi}\right) = 1$ and $\mathsf{p}\left(D_1|\boldsymbol{x}_1,e_1^t,D_0;e_0,\boldsymbol{\phi}\right) = 1$ for all attribute vectors $\boldsymbol{r}$ inside the detector, and $\mathsf{p}\left(D_c^{\mathsf{c}}|\rho,\psi,\boldsymbol{x}_1,e_1^t,D_2,D_1,D_0;e_0,\boldsymbol{\phi}\right) = 1$ because a subsequent interaction occurs with probability zero after a photoelectric absorption.

By the Beer–Lambert attenuation law,

$$\mathsf{p}\left(\rho|\psi,\boldsymbol{x}_1,e_1^t,D_1,D_0;e_0,\boldsymbol{\phi}\right) = \mu_p(e_0-e_1^t)e^{-\mu_p(e_0-e_1^t)\rho} \quad 0 \le \rho \le \rho_{\max}(\psi,e_1^t,\boldsymbol{x}_1,e_0,\boldsymbol{\phi}), \tag{17}$$

where $\rho_{\max}(\psi,e_1^t,\boldsymbol{x}_1,e_0,\boldsymbol{\phi})$ is the distance from the first interaction to the edge of the detector along direction of the Compton scattered photon, and $\mu_p(e_0-e_1^t)$ is the linear attenuation for photoelectric absorption.

The distribution of scatter angles around the Compton cone is uniform:

$$\mathsf{p}\left(\psi|\boldsymbol{x}_1,e_1^t,D_1,D_0;e_0,\boldsymbol{\phi}\right) = \frac{1}{2\pi} \text{ for } 0 \le \psi < 2\pi. \tag{18}$$

The distribution of the first interaction location is

$$\mathsf{p}\left(\boldsymbol{x}_1|e_1^t,D_0;e_0,\boldsymbol{\phi}\right) = \mathsf{p}\left(\boldsymbol{x}_1|D_0;e_0,\boldsymbol{\phi}\right) = \frac{\mu_c(e_0)e^{-\mu_c(e_0)d(\boldsymbol{x}_1;\boldsymbol{\phi})}}{\Omega(\boldsymbol{\phi})} \text{ for } \boldsymbol{x}_1 \text{ inside the detector,} \tag{19}$$

where $d(\boldsymbol{x}_1;\boldsymbol{\phi})$ is the distance from $\boldsymbol{x}_1$ to the edge of the detector along a line with direction $\boldsymbol{\phi}$ toward the source, $\Omega(\boldsymbol{\phi})$ is the surface area of the detector that is exposed to direct source photons, and $\mu_c(e_0)$ is the linear attenuation coefficient due to Compton scattering.

The density of the energy deposited in the Compton scatter interaction, $\mathsf{p}\left(e_1^t|D_0;e_0,\boldsymbol{\phi}\right)$, is based on the Klein-Nishina formula [9] and is given by (2.15) in [1].

The probability that a photon is incident on the detector is

$$\mathsf{p}\left(D_0;e_0,\boldsymbol{\phi}\right) = \frac{\Omega(\boldsymbol{\phi})}{\Omega_T}, \tag{20}$$

where Ω_T is the surface area exposed to the source, which is independent of source position.

Combining (16)–(20),

$$\mathsf{p}\left(\boldsymbol{r}|D;e_0,\boldsymbol{\phi}\right) = \frac{1}{2\pi\Omega_t}\mu_p(e_0-e_1^t)e^{-\mu_p(e_0-e_1^t)\rho}\mu_c(e_0)e^{-\mu_c(e_0)d(\boldsymbol{x}_1;\boldsymbol{\phi})}\mathsf{p}\left(e_1^t|D_0;e_0,\boldsymbol{\phi}\right) \text{ for } \boldsymbol{r} \in \mathcal{R}. \tag{21}$$

This expression is simpler and easier to compute than the likeihood in Section 2.4.3 of [1], and it agrees with the analysis in [2].

V. Conclusion

We showed that the conditional density of recorded gamma–ray photon attributes given that they are detected is the quotient of the attribute density not conditioned on the event that a photon is detected and the system sensitivity. We used this result to simplify the formulas for Compton interaction probabilities proposed in [1]. The theory and examples presented in this report may also aid in understanding probabilistic models for Compton imaging in the literature, e.g., [2].

VI. Acknowledgment

The authors thank Weiyi Wang and Jason Jaworski for their comments.

This work was supported by the U.S. Department of Homeland Security-DNDO by Grant Award Number 2008-DN-077-ARI007-03, by a MIT Lincoln Laboratory Fellowship, and by an Innovative Signal Analysis Fellowship.

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