On Transformations of Random Vectors

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Aug. 1998, LaTeXed July 24, 2006

Technical Report No. 314
Approved for public release; distribution unlimited.

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July 24, 2006

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Abstract

This technical report treats some technical considerations related to the probability density function of a function of a random vector.

I. Introduction

Let $X \in R^n$ be a continuous random vector with known pdf $\mathsf{p}_X(x)$. In many problems it is necessary to find the pdf $\mathsf{p}_Y(y)$ of a new random vector Y defined as a function $Y = g(X)$ of X, where $g : R^n \to R^n$.

Many textbooks on probability and random variables state the following equality:

$$\mathsf{p}_Y(y) = \mathsf{p}_X\left(g^{-1}(y)\right) |J(y)|, \tag{1}$$

where J is the Jacobian of $g^{-1}(y)$, *i.e.*, the determinant of the gradient of $g^{-1}(y)$.

There is considerable variation in how precisely the textbook authors state the conditions for the above equality. Most books do state the condition that g be one-to-one (and hence invertible). However, the stated conditions on differentiability vary widely.

Many engineering books make *no mention* of the need for g^{-1} to be differentiable, *e.g.* [1–8]. Many books assume that g^{-1} is *globally* differentiable *e.g.* [9–14], but this condition is too restrictive in some applications. Some books [15–17] assume that $g : \mathcal{S} \to g(\mathcal{S})$ is one-to-one and differentiable on some open set $\mathcal{S} \subseteq R^n$, and that the pdf of X vanishes (is zero) outside of $\mathcal{S}$. This is reasonably general, but still inapplicable to problems where, for example, X has a Gaussian pdf and $\mathcal{S}$ is a proper subset of R^n, since the support of the Gaussian pdf is all of R^n.

A more general requirement is to assume that $\mathsf{P}\{X \in \mathcal{S}\} = 1$, for which the condition that $\mathsf{p}_X(x)$ vanishes outside $\mathcal{S}$ is a special case. Hoel, Port, and Stone [18] provide such a theorem without proof. Bickel and Doksum [19] provide a proof of the transformation formula under the condition $\mathsf{P}\{X \in \mathcal{S}\} = 1$, but the proof is not entirely rigorous since the integrals given in [19] can cover points outside $\mathcal{S}$ where the Jacobian need not exist. This technical report provides a rigorous proof of (1), properly handling the technical details of the set $\mathcal{S}$.

This work was motivated by [20], in which a transformation function arises that is differentiable except on a set of hyperplanes of Lebesgue measure zero.

Work supported in part by NIH grants CA-60711 and CA-54362 and the Whitaker Foundation.

II. Theory

The following is simply Theorem 17.2 of [15], included for convenience.

Theorem 1 *Let $h : \mathcal{V} \to h(\mathcal{V})$ be a one-to-one mapping of an open set $\mathcal{V}$ onto an open set $h(\mathcal{V})$. Suppose that (on $\mathcal{V}$) h is continuous and that h has continuous partial derivatives h_{ij} with Jacobian $J(y) \triangleq \mathsf{det}\{h_{ij}(y)\}$. Then for $\mathcal{A} \subseteq \mathcal{V}$, for any nonnegative function f*

$$\int_{\mathcal{A}} f(h(y))|J(y)| \,\mathrm{d}y = \int_{h(\mathcal{A})} f(x) \,\mathrm{d}x \,. \tag{2}$$

The following Theorem is a generalization of (20.20) in [15]. Standard treatments *e.g.*, [14, p. 143] assume that the transformation function is globally differentiable. Our generalization allows for a (measure zero) set where the Jacobian is undefined.

Theorem 2 *Let $g : R^n \to R^n$ be one-to-one and assume that $h = g^{-1}$ is continuous. Assume that on an open set $\mathcal{V} \subseteq R^n$ h is continuously differentiable with Jacobian $J(y)$. Define $J_0 : R^n \to \mathbb{R}$ by*

$$J_0(y) = \begin{cases} J(y), & y \in \mathcal{V} \\ 0, & y \in \mathcal{V}^c, \end{cases} \tag{3}$$

where $\mathcal{V}^c$ is the set complement (in R^n) of $\mathcal{V}$.

Suppose random vector X has pdf $\mathsf{p}_X(x)$ (with respect to Lebesgue measure) with nonzero mass in $h(\mathcal{V}^c)$, i.e., $\mathsf{P}\{X \in h(\mathcal{V}^c)\} = \int_{\mathcal{V}^c} \mathsf{p}_X(x) \,\mathrm{d}x = 0$. *Then the pdf of $Y = g(X)$ is given by*

$$\boxed{\mathsf{p}_Y(y) = \mathsf{p}_X\left(g^{-1}(y)\right)|J_0(y)| = \begin{cases} \mathsf{p}_X\left(g^{-1}(y)\right)|J(y)|\,, & y \in \mathcal{V} \\ 0, & y \in \mathcal{V}^c. \end{cases}} \tag{4}$$

Proof:
For (measurable) $\mathcal{B} \subseteq R^n$

$$0 \le \mathsf{P}\{g(X) \in \mathcal{B} \cap \mathcal{V}^c\} \le \mathsf{P}\{g(X) \in \mathcal{V}^c\} = \mathsf{P}\left\{X \in g^{-1}(\mathcal{V}^c)\right\} = \mathsf{P}\{X \in h(\mathcal{V}^c)\} = 0.$$

Thus $\mathsf{P}\{g(X) \in \mathcal{B} \cap \mathcal{V}^c\} = 0$, so

$$\begin{aligned} \mathsf{P}\{g(X) \in \mathcal{B}\} &= \mathsf{P}\{g(X) \in \mathcal{B} \cap \mathcal{V}\} + \mathsf{P}\{g(X) \in \mathcal{B} \cap \mathcal{V}^c\} \\ &= \mathsf{P}\{X \in h(\mathcal{B} \cap \mathcal{V})\} = \int_{h(\mathcal{B}\cap\mathcal{V})} \mathsf{p}_X(x) \,\mathrm{d}x = \int_{\mathcal{B}\cap\mathcal{V}} \mathsf{p}_X(h(y))\,|J(y)| \,\mathrm{d}y \end{aligned}$$

by Theorem 1, which applies since $\mathcal{B} \cap \mathcal{V} \subseteq \mathcal{V}$. (The set $h(\mathcal{V})$ is open since by assumption $\mathcal{V}$ is open and h is continuous.) Thus by (3):

$$\mathsf{P}\{g(X) \in \mathcal{B}\} = \int_{\mathcal{B}\cap\mathcal{V}} \mathsf{p}_X(h(y))\,|J_0(y)| \,\mathrm{d}y = \int_{\mathcal{B}} \mathsf{p}_X(h(y))\,|J_0(y)| \,\mathrm{d}y - \int_{\mathcal{B}\cap\mathcal{V}^c} \mathsf{p}_X(h(y))\,|J_0(y)| \,\mathrm{d}y,$$

since $\mathcal{B}$ is the union of the disjoint sets $\mathcal{B} \cap \mathcal{V}$ and $\mathcal{B} \cap \mathcal{V}^c$. The second integral above is zero since $|J_0(y)|$ is zero for $y \in \mathcal{V}^c$ by (3). Thus

$$\mathsf{P}\{g(X) \in \mathcal{B}\} = \int_{\mathcal{B}} \mathsf{p}_X(h(y))\,|J_0(y)| \,\mathrm{d}y,$$

for $\mathcal{B} \subseteq R^n$, proving that (4) is a pdf of $g(X)$. □

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