TRIANGLE TILING: THE CASE $3\alpha + 2\beta = \pi$

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ABSTRACT. An N-tiling of triangle ABC by triangle T (the "tile") is a way of writing ABC as a union of N copies of T overlapping only at their boundaries. Let the tile T have angles (α, β, γ), and sides (a, b, c). This paper takes up the case when $3\alpha + 2\beta = \pi$. Then there are (as was already known) exactly five possible shapes of ABC: either ABC is isosceles with base angles α, β, or $\alpha+\beta$, or the angles of ABC are $(2\alpha, \beta, \alpha+\beta)$, or the angles of ABC are $(2\alpha, \alpha, 2\beta)$. In each of these cases, we have discovered, and here exhibit, a family of previously unknown tilings. These are tilings that, as far as we know, have never been seen before. We also discovered, in each of the cases, a Diophantine equation involving N and the (necessarily rational) number $s = a/c$ that has solutions if there is a tiling using tile T of some ABC not similar to T. By means of these Diophantine equations, some conclusions about the possible values of N are drawn; in particular there are no tilings possible for values of N of certain forms. We prove, for example, that there is no N-tiling with N prime when $3\alpha + 2\beta = \pi$. These equations also imply that for each N, there is a finite set of possibilities for the tile (a, b, c) and the triangle ABC. (Usually, but not always, there is just one possible tile.) These equations provide necessary, and in three of the five cases sufficient, conditions for the existence of N-tilings.

1. INTRODUCTION

This paper is part of a series of papers on triangle tiling, continuing a research program begun by Laczkovich [4]. An N-tiling of triangle ABC by triangle T (the "tile") is a way of writing ABC as a union of N copies of T overlapping only at their boundaries. The general aim of this research program is to understand the nature of triangle tilings, which can be amazingly complex. We can exhibit quite a few families of triangle tilings, some of which are very familiar, but at least six new families of tilings have been discovered in the course of our work, five of which are described in this paper. It would be too much to ask that every triangle tiling belongs to one of these families, as sometimes it is possible to rearrange some of the tiles within a given tiling, and also there are systematic ways to combine tilings. Our aim instead is to completely classify the triples (ABC, T, N) such that there exists a tiling of ABC by T using N tiles. In particular, this classification should enable us to answer more specific questions, such as, for which N does there exist a triangle ABC and a tile T and an N-tiling of ABC by T?

Cases of specific N can be quite interesting. The question that first sparked our interest was whether any triangle can be 7-tiled. We gave a long Euclidean-style proof that no 7-tiling exists, but it was clear that a similar proof for 11-tilings might be a thousand pages long. Now we know that there is also no 11-tiling. A self-contained proof of those two theorems is presented in [1]. But there *does* exist a 28-tiling, which we think is a new discovery. We generalize the 28-tiling, showing

it to be the simplest member of a new family of tilings. These are the "triquadratic tilings"; they exist when the "tiling equation"

$$M^2 + N = 2K^2$$

has a solution in integers (K, M), such that M divides K and K divides N, or equivalently K divides M^2. When we speak of a "solution of the tiling equation", we mean to include the divisibility conditions just mentioned. Each such solution determines the tile of the corresponding triquadratic tiling: the tile must be similar to the triangle with sides $a = M$, $c = K$, and $b = K - M^2/K$. All three sides of that tile are integers, and the tile then satisfies the condition $3\alpha + 2\beta = \pi$.

In previous work of others ([4, 6]) and the present author ([1]) the questions of triangle tiling have been successfully divided into a small finite number of cases according to the shapes of the tile (a, b, c), whose angles are (α, β, γ), and the triangle ABC. In this paper we take up the case $3\alpha + 2\beta = \pi$ (although a few lemmas are proved also for the case $\gamma = 2\pi/3$, when the same proof applies). In case $3\alpha + 2\beta = \pi$, according to [4], there are just five possible shapes of ABC: either ABC is isosceles with base angles α, β, or $\alpha + \beta$, or the angles of ABC are $(2\alpha, \beta, \alpha + \beta)$, or the angles of ABC are $(2\alpha, \alpha, 2\beta)$. In this paper, we give new tilings in each of the last two cases, and complement those constructions with theorems showing that tilings of ABC using a tile with $3\alpha + 2\beta = \pi$ exist only for those pairs (N, ABC) in which the new tilings exist. We now state our results more precisely.

Our results for the case when ABC has angles $(2\alpha, \beta, \alpha + \beta)$ are as follow:

(i) If the tiling equation $M^2 + N = 2K^2$ has a solution (K, M) in positive integers such that K divides M^2 and $M^2 < N$, then there is a triquadratic N-tiling of ABC. The exact description and pictures of these new tilings are given in the paper. The smallest N allowing a triquadratic tiling is $N = 28$.

(ii) The tiling equation has, for each N, at most finitely many solutions (M, s). Each solution with K dividing N determines a possible tile shape, similar to the triangle with sides $a = M$, $c = K$, and $b = K - M^2/K$, of which all three sides are integers. Each of these finitely many tiles may be used for a triquadratic tiling.

(iii) Except for the "quadratic tilings" (that always exist when N is a square), any N-tiling of ABC must use a tile similar to one of the tiles in this finite set. In particular, if the tiling equation has no solution, and N is not a square, then there do not exist any N-tilings of ABC.

(iv) If the tiling equation has solutions, then N is a square times a product of distinct primes of the form $8n \pm 1$, or 2.

These results provide a complete classification of the triples (ABC, N, T) such that ABC can be N-tiled by the tile T, in case ABC has angles $(2\alpha, \beta, \alpha + \beta)$. In case the tiling equation has no solution, then unless N is a square, there are no N-tilings of ABC; and if N is a square, there are no N-tilings except the quadratic tilings by a tile similar to ABC, which always exist when N is a square. Since we can easily check for a particular N whether the triangle equation is solvable, we find immediately that there are no N-tilings for $N = 7, 14, 31, 41, 63$, etc. Remember that this statement applies only to tilings of ABC with angles $(2\alpha, \beta, \alpha + \beta)$.

Now consider the case when ABC has angles $(2\alpha, \alpha, 2\beta)$. Then there is another equation, the "second tiling equation", that governs the existence of tilings. That

equation is

$$N = M^2 \frac{(2 - s^2)(3 - s^2)}{(1 - s)^2(2 + s)^2}$$

where $s = 2\sin(\alpha/2)$ determines the angle α and the ratio of sides $a/c = s$, and the tile (a, b, c) is given by $b = c - a^2/c$. Here M has to be a positive integer and s a rational between 0 and 1. There is a tiling of ABC for each solution of the second tiling equation, and if N is not a perfect square, there are tilings of ABC only if the second tiling equation has a solution. Some of these tilings belong to the new family we call the four-component tilings; pictures are provided.

The second tiling equation has infinitely many solutions, since we can start with the tile (a, b, c) such that a divides c and c divides a^2, and produce a tiling. Conversely, we can determine by an algorithm whether the equation is solvable for a particular N, although we do not have a concise number-theoretic characterization of those N for which it is solvable. We used that algorithm to compute a table of solutions for N up to several thousand.

We now sketch how the particular shapes $(2\alpha, \beta, \alpha + \beta)$ and $(2\alpha, \alpha, 2\beta)$ arise naturally in this subject. Let the sides of the tile opposite angles α, β, and γ respectively be a, b, and c. Between the three numbers α, β, and γ, we have two linear relations. The first one is $\alpha + \beta + \gamma = \pi$, because they are the three angles of a triangle. The second relation arises from the "vertex splitting" at the vertices of the tiled triangle ABC. Consider all the tiles that share any of the three vertices, some (or none) have α angles at the vertex, some (or none) have β angles, and some (or none) have γ angles. Thus for some nonnegative integers P, Q, and R, we have

$$P\alpha + Q\beta + R\gamma = \pi.$$

If $P = Q = R = 1$ then ABC is similar to T; that case was treated in [3, 6, 4]. In this paper we assume T is not similar to ABC. Hence the two linear relations are independent relations; substituting $\gamma = \pi - \alpha - \beta$ we have a non-trivial linear relation between α, β, and π. If we could get a third independent linear relation between the angles, we could solve for α, β and γ. At each vertex of the tiling, some angles add up to either π (for a boundary vertex, or a "non-strict" interior vertex, i.e. one lying on a side of another tile), or to 2π, for a strict interior vertex. If α is not a rational multiple of β, it will never be the case that the same angle can be composed of some α angles and also of some β angles. Hence the possibilities for angles meeting at interior vertices can be controlled to some extent. This analysis was carried out by Laczkovich [4], who showed in this way that when the tile T is not a right triangle, nor similar to ABC, there are only a few possibilities: either ABC is equilateral and $\beta = \pi/3$, or $\gamma = 2\alpha$ (which is another way of saying $3\alpha + \beta = \pi$) and ABC is isosceles with base angles α, or $\gamma = 2\pi/3$, or $3\alpha + 2\beta = \pi$.

It follows from work of Laczkovich [4] (see Lemma 1 below for details) that if $3\alpha + 2\beta = \pi$ then α and β are not rational multiples of π and either ABC is isosceles (in which case its base angles are α, β, or $\alpha + \beta$), or the angles of ABC are $(\alpha, 2\alpha, 2\beta)$, or the angles of ABC are $(2\alpha, \beta, \alpha + \beta)$. All five cases are taken up in this paper. From $3\alpha + 2\beta = \pi$ we have $\gamma = 2\alpha + \beta$ and $\alpha < \pi/3$ and $\beta < \pi/2$.

We next explain the meaning of the integer M in the first and second tiling equations. Our work in this paper begins with the introduction of colored tilings, in which the tiles are colored black and white alternately, and a "coloring equation" is derived that involves the difference M between the numbers of black and white

tiles. (Fig. 2 illustrates the kind of coloring in question.) The coloring equation is then combined with the "area equation", that equates the area of triangle ABC, as computed from the sine of one angle and the lengths of two sides, to N times the area of the tile. Eliminating one variable from these two equations, we obtain the first and second tiling equations discussed above.

The next important step is to prove that the tile has to be rational (that is, the ratios of its sides are rational), so in suitable units the sides can be taken to be integers. For this, we use the coloring equation together with an analysis of the "edge relations" in a tiling. An "edge relation" arises when a line (segment) interior to the tiling has different edges on one side than on the other. For example, we might have 15 b edges on one side, and 18 a edges on the other side, giving rise to the edge relation $15b = 18a$. The coloring equation and the analysis of edge relations work for all five possible shapes of ABC (subject to $3\alpha + 2\beta = \pi$, including the three isosceles shapes. After we prove that the tile has to be rational, then the tiling equations can be used to analyze the possible values of N and (a, b, c) that can lead to tilings. In short, when the tiling equation has no solutions, there are no tilings.

When the tiling equation does have solutions, are there tilings? We answer this question in the affirmative by explicitly constructing the tilings in question. None of these tilings were previously known. Many illustrations of these tilings are given in the figures below. In the $(2\alpha, \beta, \alpha + \beta)$ case we call these the "triquadratic tilings." The smallest of these is a 28-tiling. In the $(2\alpha, \alpha, 2\beta)$ case, we call these the "four-component tilings".

For a given N, if either of the tiling equations has a solution, one can explicitly compute a finite set of possible tiles (one for each solution of the tiling equation) and for each tile, just one triangle ABC, such that if any N-tiling exists, it uses one of those tiles and tiles the corresponding ABC.

The first tiling equation does sometimes have more than one solution (M, K) for a given N, but the smallest such examples is $N = 87800$. We do not know if there can be more than one solution of the second tiling equation for a given N. By computation we know there is no such example with $N \leq 2000$.

Finally, we treat the three cases when ABC is isosceles. In each of these cases we follow the same method as for the cases already described. Namely, we write down the "area equation" that says the area of ABC is equal to N times the area of the tile. Then we write down the "coloring equation" that relates the coloring number M to the lengths X and Y of the sides of ABC. Then we eliminate X and Y from those two equations, yielding a Diophantine equation in N, M, and the rational number $s = a/c$. This is the "tiling equation" for that shape of ABC. We make a table of its solutions for N up to some limit (such as 200 or 500 or 1000), and then exhibit such tilings as we can, and rule out by computational search for "boundary tilings" such N as we can, leaving question marks in the table for other entries. But for values of N that do not occur in any of the five tables, we can definitely assert there is no N-tiling of any triangle by a tile satisfying $3\alpha + 2\beta = \pi$.

By an analysis of each of the five tiling equations, we were able to prove that, if $3\alpha + 2\beta = \pi$ and ABC is N-tiled by a tile with angles (α, β, γ), then N cannot be prime.

2. Some basic lemmas

A basic fact is that in a tiling with $3\alpha + 2\beta = \pi$ (and $\alpha \neq \pi/4$), the angles α and β are not rational multiples of π. This is a consequence of a more general theorem of Laczkovich [4]. We here give the details as to how this fact follows from the theorem stated by Laczkovich, which is Theorem 5.1 of [4].

Lemma 1. *Let $3\alpha + 2\beta = \pi$. Suppose there is an N-tiling of triangle ABC by tile T with angles (α, β, γ). Suppose also that ABC is not similar to T. Then α and β are not rational multiples of π, and the every linear relation between π, α, and β is a multiple of $3\alpha + 2\beta = \pi$.*

Proof. The last statement follows from the first, since a linear relation different from $3\alpha + 2\beta = \pi$ would permit solving for α and β in terms of π.

Theorem 5.1 of [4] does not mention the relation $3\alpha + 2\beta = \pi$. Instead, it assumes that triangle ABC can be dissected into similar (not necessarily congruent) triangles with angles (α, β, γ) that are rational multiples of π. The conclusion is that (α, β, γ) belongs to a certain finite list of possible values. So we have to check if any of the triples in that list satisfy $3\alpha + 2\beta = \pi$. We do not reproduce the entire list here, but the reader may easily check that the only triple that does satisfy $3\alpha + 2\beta = \pi$ is $(\pi/4, \pi/8, 5\pi/8)$. But Laczkovich's Theorem 5.3 shows that that triple is impossible for dissections into *congruent* triangles. That completes the proof.

Definition 1. *Let a triangle have angles (α, β, γ). We define*
$$s = 2\sin(\alpha/2).$$

This definition is useful because the ratios a/c and b/c can be expressed simply in terms of s, as shown in the following lemma.

Lemma 2. *Suppose $3\alpha + 2\beta = \pi$. Let $s = 2\sin\alpha/2$. Then we have*
$$\sin\gamma = \cos\frac{\alpha}{2}$$
$$\frac{a}{c} = s$$
$$\frac{b}{c} = 1 - s^2$$

Proof. Since $\gamma = \pi - (\alpha + \beta)$, we have
$$
\begin{aligned}
\sin\gamma &= \sin(\pi - (\alpha + \beta)) \\
&= \sin(\alpha + \beta) \\
&= \cos(\pi/2 - (\alpha + \beta)) \\
&= \cos\frac{\alpha}{2} \qquad \text{since } \pi/2 - \beta = 3\alpha/2
\end{aligned}
$$
Then $c = \sin\gamma = \cos\alpha/2$, and $a = \sin\alpha = 2\sin(\alpha/2)\cos(\alpha/2)$. Hence
$$\frac{a}{c} = 2\sin\alpha/2.$$
Since $3\alpha + 2\beta = \pi$, we have
$$
\begin{aligned}
\sin\beta &= \sin(\pi/2 - 3\alpha/2) \\
&= \cos(3\alpha/2) \\
&= 4\cos^3\frac{\alpha}{2} - 3\cos\frac{\alpha}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
b/c &= 4\cos^2(\alpha/2) - 3 \\
&= 4(1 - \sin^2 \alpha/2) - 3 \\
&= 1 - 4\sin^2 \alpha/2
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\frac{a}{c} &= s \\
\frac{b}{c} &= 1 - s^2
\end{aligned}
$$

establishing the second equation of the lemma. That completes the proof of the lemma.

Lemma 3. *Suppose triangle ABC is N-tiled by a tile in which $3\alpha + 2\beta = \pi$. Then $\gamma > \pi/2$.*

Proof.

$$
\begin{aligned}
\pi &= 3\alpha + 2\beta \\
&= \alpha + 2(\alpha + \beta) \\
&= \alpha + 2(\pi - \gamma) \\
2\gamma &= \alpha + \pi \\
2\gamma &> \pi \qquad \text{since } 0 < \alpha \\
\gamma &> \tfrac{\pi}{2}
\end{aligned}
$$

That completes the proof.

Lemma 4. *Let triangle ABC be N-tiled by a tile with angles (α, β, γ). Suppose that either $3\alpha + 2\beta = \pi$ and ABC is not isosceles with base angles α, or $\gamma = 2\pi/3$. Then no tile has its γ angle at a vertex of ABC.*

Proof. By Lemma 1, α and β are not rational multiples of π. Hence the angles of ABC are linear integral combinations of α, β, and γ. First assume $3\alpha + 2\beta = \pi$. Then the angles of ABC are each equal to α, 2α, $\alpha + \beta$, β, or 2β. Of these angles, all but 2β are less than γ, as we now show. Then $\gamma = \beta + 2\alpha$, and

$$
\begin{aligned}
\alpha &< \beta + 2\alpha = \gamma \\
\beta &< \beta + 2\alpha < \gamma \\
\alpha + \beta &< \beta + 2\alpha = \gamma \\
2\alpha &\ |< \beta + 2\alpha = \gamma.
\end{aligned}
$$

We claim that none of the angles of ABC has a γ vertex of a tile at any of the vertices of ABC. Since ABC is not similar to the tile, there cannot be a γ angle alone at any vertex, since that would leave $\alpha + \beta$ for the other two vertices, making ABC similar to the tile, since α is not a rational multiple of β.

Since all the possible angles but 2β are less than γ, it only remains to deal with the case where angle C is equal to 2β and $\gamma < 2\beta$, and there is a tile with its γ angle at C. We do not have $2\beta = \gamma$, by Lemma 1. Then there must be another tile at C as well. If the angle of that tile at C is α, then the total angle at C is at least $\gamma + \alpha = 2\alpha + \beta + \alpha = 3\alpha + \beta$, leaving only β for the other two angles of ABC. But that is impossible, since α is not a rational multiple of β. If the second angle at C

is β, then the total angle at C is at least $\gamma + \beta = 2\alpha + 2\beta$, leaving just α for the other two angles, which is again impossible. Hence the second angle at C cannot be β. That completes the proof under the assumption $3\alpha + 2\beta = \pi$.

We now take up the case $\gamma = 2\pi/3$. Then the possible angles of ABC are α, β, $\alpha + \beta$, $\alpha + 2\beta$, $2\alpha + \beta$, 3α, and 3β. All but 3α and 3β are less than $2\alpha + 2\beta = \gamma$, so a γ tile can occur, if at all, only at a vertex angle of 3α or 3β. Suppose vertex C has angle 3α and there is a γ angle of a tile at C. Then $\gamma < 3\alpha$ and angles A and B together are $\pi - 3\alpha < \pi - \gamma$, which is impossible since the three angles of ABC add up to π. Similarly if vertex C has angle 3β and $\gamma < 3\beta$. That completes the proof of the lemma.

Lemma 5. *Suppose triangle ABC is N-tiled by a tile with angles (α, β, γ) and $\gamma > \pi/2$. Suppose all the tiles along one side of ABC do not have their c sides along that side of ABC. Then there is a tile with a γ angle at one of the endpoints of that side of ABC.*

Proof. Let pq be the side of ABC with no c sides of tiles along it. Then the γ angle of each of those tiles occurs at a vertex on pq, since the angle opposite the side of the tile on pq must be α or β. Let n be the number of tiles along pq; then there are $n - 1$ vertices of these tiles on the interior of pq. Since $\gamma > \pi/2$, no vertex on the boundary has more than one γ angle. By the pigeonhole principle, there is at least one tile whose γ angle is not at one of those $n - 1$ interior vertices; that angle must be at P or Q. That completes the proof of the lemma.

Lemma 6. *Suppose triangle ABC is N-tiled by a tile T with angles (α, β, γ). Suppose*

    (i) $\gamma > \pi/2$, and

   *(ii) α is not a rational multiple of π, and*

    (iii) Every angle of triangle ABC is less than γ, and fh (iv) One of the following two conditions holds: Either b is not a multiple of a, or the tiling does not have two equal angles of tiles at A or at C, i.e. two α or two β angles.

    Then there are at least two c edges of tiles on side AC.

Remarks. One can prove by the same method that the c edges must occur in adjacent blocks of at least two edges, but we found no use for that result.

Proof. The proof is given in [1]. But it is short, so for the reader's convenience we repeat it here. By hypothesis (ii), every boundary vertex P (except A, B, and C) that has a γ angle (i.e., some tile with a vertex at P has its γ angle at P) touches exactly three tiles, which contribute angles of α, β, and γ. By Lemma 5, each side of ABC has at least one c edge. The present lemma, however, claims more: there must be at least two c edges. Suppose, to the contrary, that there is just one c tile, Tile 1, with an edge on one side EF of triangle ABC. Then all the other tiles with an edge on EF have a γ angle on EF. We visualize EF as horizontal with triangle ABC above, and use the word "north" and "northwest" accordingly. See Fig. 1. Since there cannot be a γ angle at the vertices of ABC, it follows that both the tiles on AC adjacent to Tile 1 (if there are two, or otherwise, only the one) have their γ angles adjacent to Tile 1. Let pq be the c edge of Tile 1 lying on AC. Let R be the northern vertex of Tile 1. Suppose (without loss of generality) that Tile 1

FIGURE 1. Proof of Lemma 6: another tile won't fit next to Tile 2
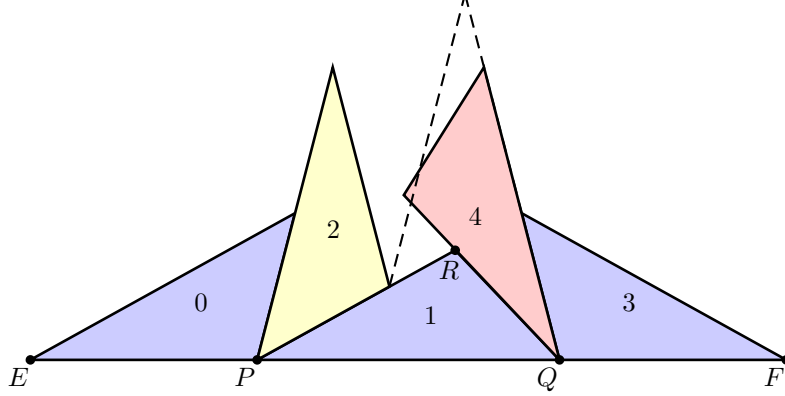
has its β angle at Q. Then the side PR of Tile 1, opposite Q, has length b. Let Tile 2 be the tile adjacent to PR.

Since the hypotheses of the theorem remain true if (the names of) α and β are interchanged, we may assume without loss of generality that $\alpha < \beta$. Then by the law of sines, $a < b$. Since $\gamma > \pi/2$ we also have $a < c$ (by the law of cosines).

Assume, for proof by contradiction, that neither P nor Q is a vertex of ABC. Then there exist Tile 0 and Tile 3 on AC sharing vertices P and Q with Tile 1. Tile 2, between Tile 0 and Tile 1, must have its β angle at P, since Tile 1 has its α angle there and Tile 0 has its γ angle at P. There is an open α angle between Tile 1 and Tile 3; let Tile 4 be the tile that fills that notch. Then Tile 4 has its b or c edge along qr. Since Tile 1 has its a edge along qr and $a < b$ and $a < c$, the edge of Tile 4 on qr extends past R. Then the segment PR is of length b and its northwest side is composed of a number of tile edges, starting with Tile 2 at P. These must all be a edges, since a is the only edge less than b. Since the tiles northwest of PR all have their a edges on PR, they all have a γ angle on PR. But Tile 2 does not have its γ angle at P, since Tile 0 has its γ angle at P. And the last tile cannot have its γ angle at R, since Tile 4 extends along qr past R, and Tile 1 has its γ angle at R. So if there are n tiles northwest of PR, there are only $n-1$ possible places for their γ angles, contradicting the pigeon-hole principle. This contradiction proves that one of P or Q is a vertex of ABC.

Now we argue by cases.

Case 1: Q is a vertex of ABC, i.e., $Q = C$. If the angle of ABC at Q is strictly between β and 2β, then Tile 4 must have its α angle at Q, and we argue exactly as before. If the angle of ABC at Q is exactly β, then we argue as above, except that RQ is now extended past R by one side of ABC rather than an edge of Tile 4. The argument about the γ angles of the tiles northwest of PR is unchanged, if P is not a vertex of ABC. If P is a vertex of ABC, then we still can argue that Tile 2 must have its a side on PR, because it cannot fit next to Tile 1 with its b or c side on PR. (Hypothesis (iii) is not needed here.)

Therefore we may assume that the angle of ABC at Q is at least 2β, and that Tile 4 has its β angle at Q and its a edge against Tile 1. Hence there is a double angle at Q. Then by hypothesis (iv), b is not a multiple of a. Tile 3 cannot have its γ angle at Q, by hypothesis (iii). Therefore Tile 3 has its γ angle at R, and since

$\gamma > \pi/2$ by hypothesis (i), PR does not extend past R as part of the tiling. The tiles northwest of PR must all have their a edges on PR, since a is the only edge less than b. Then b is a multiple of a, contradiction. That completes Case 1.

Case 2: P is a vertex of ABC, and Q is not. Then Tile 4 is placed as shown in the figure. Therefore the angle of ABC at vertex P must be greater than α, since if it were equal to α, Tile 4 would not lie inside ABC. Then Tile 2 exists, and Tile 2 must have its a side on PR, because it cannot fit next to Tile 1 with its b or c side on PR. From there the argument proceeds as before. That completes Case 2.

That completes the proof of the lemma.

Lemma 7. *Let T be a triangle with sides are a, b, and c, and let α and β be the angles opposite a and b respectively. Then $3\alpha + 2\beta = \pi$ if and only if $b = c - a^2/c$.*

Proof. Suppose $3\alpha + 2\beta = \pi$. By Lemma 2, $b/c = 1 - s^2 = 1 - (a/c)^2$. Multiplying by c we have the desired conclusion, $b = c - a^2/c$.

It remains to prove the right-to-left implication. Assume $b = c - a^2/c$. Since the condition $b = c - a^2/c$ is invariant if the triangle is re-scaled, we may as well assume $c = 1$. Then we have $b = 1 - a^2$, and after the rescaling we will have $a < 1$. We could not find a high-school-trigonometry proof of this lemma; we had to use a little calculus. By the law of cosines we have the following two equations:

$$
\begin{aligned}
a^2 &= 1 + (1 - a^2)^2 - 2(1 - a^2)\cos\alpha \\
(1 - a^2)^2 &= 1 + a^2 - 2a\cos\beta
\end{aligned}
$$

Solving for α and β we have

$$
\begin{aligned}
\alpha &= \arccos\Big(\frac{1 + (1 - a^2)^2 - a^2}{2(1 - a^2)}\Big) = \arccos(1 - a^2/2) \\
\beta &= \arccos\Big(\frac{1 + a^2 - (1 - a^2)^2}{2a}\Big) = \arccos\Big(\frac{3a - a^3}{2}\Big)
\end{aligned}
$$

Now we form the expression $3\alpha + 2\beta$ and differentiate it with respect to a. When we differentiate arccos we get an algebraic function. After the differentiation we simplify and show that we get zero. That will prove that $3\alpha + 2\beta$ is a constant. Then we evaluate the constant using one particular triangle and find that it is zero.

Here are the details:

$$
\begin{aligned}
3\alpha + 2\beta &= 3\arccos(1 - a^2/2) + 2\arccos\left(\frac{3a - a^3}{2}\right) \\
\frac{d}{da}(3\alpha + 3\beta) &= 3\frac{a}{\sqrt{1 - (1 - a^2/2)^2}} - 2\frac{(3 - 3a^2)/2}{\sqrt{1 - (3a - a^3)^2/4}} \\
&= \frac{6}{\sqrt{4 - a^2}} - \frac{3 - 3a^2}{\sqrt{1 - (3a - a^3)^2/4}} \\
&= \frac{6}{\sqrt{4 - a^2}} - \frac{3 - 3a^2}{\sqrt{(1 - (3a - a^3)/2)(1 + (3a - a^3)/2)}} \\
&= \frac{6}{\sqrt{4 - a^2}} - \frac{3 - 3a^2}{\frac{1}{2}\sqrt{(a^3 - 3a + 2)(a^3 + 3a + 3)}} \\
&= \frac{6}{\sqrt{4 - a^2}} - \frac{6(1 - a^2)}{\sqrt{(a + 2)(1 - a)^2(a - 2)(a + 1)^2}} \\
&= \frac{6}{\sqrt{4 - a^2}} - \frac{6(1 - a^2)}{(1 - a)(1 + a)\sqrt{(a + 2)(a - 2)}} \\
&= \frac{6}{\sqrt{4 - a^2}} - \frac{6}{\sqrt{4 - a^2}} \\
&= 0
\end{aligned}
$$

as promised. Hence $3\alpha + 2\beta$ is a constant. Now, to show that the constant in question is zero, it suffices to evaluate $3\alpha + 2\beta = 0$ for a particular value of a. Or, approaching the matter another way, let $\alpha = 30°$ and $\beta = 45°$, so $3\alpha + 2\beta = \pi$, and let us check that the triangle with those angles and long side 1 has sides a and $1 - a^2$ for some number a. Let a and b be the sides of that triangle opposite the $30°$ and $45°$ angle respectively. We must show $b = 1 - a^2$. We have

$$
\begin{aligned}
\sin 105° &= \sin 75° \\
&= \sin(30° + 45°) \\
&= \sin 30° \cos 45° + \cos 30° \sin 45° \\
&= \frac{1}{2}\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}} \\
&= \frac{1 + \sqrt{3}}{2\sqrt{2}}
\end{aligned}
$$

By the law of sines we have

$$
\frac{b}{\sin 45°} = \frac{a}{\sin 30°} = \frac{1}{\sin 105°}
$$

Putting in the values of the trig functions we have

$$\frac{b}{1/\sqrt{2}} = \frac{a}{1/2} = \frac{2\sqrt{2}}{1 + \sqrt{3}}$$

$$b\sqrt{2} = 2a = \frac{2\sqrt{2}}{1 + \sqrt{3}}$$

$$a = \frac{2}{1 + \sqrt{3}}$$

$$b = \frac{2}{1 + \sqrt{3}}$$

$$1 - a^2 = \frac{(1 + \sqrt{3})^2 - 2}{(1 + \sqrt{3})^2}$$

$$= \frac{2}{1 + \sqrt{3}}$$

$$= b$$

as claimed. That completes the proof of the lemma.

Lemma 8. *Suppose $3\alpha + 2\beta = \pi$, and (a, b, c) are the sides of a triangle with angles (α, β, γ). Suppose (a, b, c) are integers with no common factor. Let $g = \gcd(a, c)$. Then $c = g^2$, and g is squarefree.*

Remark. The fact that g is necessarily squarefree went unnoticed for several years, but eventually turned out to be the key to finding a necessary and sufficient condition in Theorem 14. That part of the lemma will not be needed until then.

Proof. Let $\hat{a} = a/g$ and $\hat{c} = c/g$. Then g is relatively prime to both $\hat{a}$ and $\hat{c}$. We have

$$b = c - \frac{a^2}{c} \qquad \text{by Lemma 7}$$

$$= g\hat{c} - \frac{\hat{a}^2 g}{\hat{c}}$$

Since b and $g\hat{c}$ are integers, and $\hat{a}$ and c are relatively prime, $\hat{c}$ must divide g. Let $\ell = g/\hat{c}$; then ℓ is an integer, and we have

$$b = g\hat{c} - \frac{\hat{a}^2 g}{\hat{c}}$$

$$= g\hat{c} - \ell\hat{a}^2$$

$$= \frac{g}{\hat{c}}\hat{c}^2 - \ell\hat{a}^2$$

$$= \ell\hat{c}^2 - \ell\hat{a}^2$$

$$= \ell(\hat{c}^2 - \hat{a}^2)$$

Therefore ℓ divides b. But also ℓ divides c, since

$$\ell\hat{c}^2 = \frac{g}{\hat{c}}\hat{c}^2 = g\hat{c} = c.$$

But b and c are relatively prime, since c divides a^2 and (a, b, c) have no common factor. Then since ℓ divides both b and c, we must have $\ell = 1$. Since by definition $\ell = g/\hat{c}$, then $g = \hat{c}$. Therefore $c = g\hat{c} = g^2$. That is the first assertion of the theorem.

It remains to prove that g is squarefree. Suppose, to the contrary, that g is not squarefree. Let e be the largest integer such that $e^2 | g$. The integers ℓ, $\bar{a}$ and $\bar{c}$ are defined by

$$\begin{aligned} g &= e^2 \ell \\ a &= e^2 \bar{a} \\ c &= e^2 \bar{c} \end{aligned}$$

Since $c = g^2$, we have $c = e^4 \ell^2$. Hence

$$\bar{c} = e^2 \ell^2 \tag{1}$$

We have

$$\begin{aligned} b &= c - a^2/c \\ &= e^2 \bar{c} - \frac{e^4 \bar{a}^2}{e^2 \bar{c}} \\ &= e^2 (\bar{c} - \bar{a}^2/\bar{c}) \end{aligned}$$

I say that $\bar{a}^2/\bar{c}$ is an integer. If not, there is a prime p dividing $\bar{c}$ (say to the power j) that either does not divide a, or divides a^2 only to a lower power than j. Since p^j divides $\bar{c}$, it divides c too, and hence also $p^j | a^2$, (since a^2/c is an integer). Since p^j does not divide $\bar{a}^2$, but does divide $a^2 = e^4 \bar{a}^2$, p must divide e^4, and hence e. Hence p^4 divides $a^2 = e^4 \bar{a}$. Hence p^2 divides a. By (1), $e^2 | \bar{c}$. Hence $p^2 | \bar{c}$. Thus p^2 divides both $\bar{a}$ and $\bar{c}$. But that contradicts the definition of g, according to which $\bar{a}$ and $\bar{c}$ have no common square factor. Therefore, as claimed, $\bar{a}^2/\bar{c}$ is an integer. But then e^2 divides b. Since e^2 also divides a and c, that contradicts the hypothesis that (a, b, c) have no common factor. Hence g is squarefree. That completes the proof of the lemma.

3. Types of vertices

Suppose $3\alpha + 2\beta = \pi$ and α is not a rational multiple of $\pi/2$. The "type" of a vertex V in a tiling is a triple (n, k, j), meaning that there are n tiles with an α angle at V, k tiles with a β angle at V, and j tiles with a γ angle at V. Since α is not a rational multiple of π, it follows from $3\alpha + 2\beta = \pi$ that α is not a rational multiple of β or γ, and $\gamma = \beta + 2\alpha$ is not a rational multiple of β. Each vertex is therefore of one of the types $(1, 1, 1)$, $(2, 2, 2)$, $(0, 1, 3)$, $(3, 2, 0)$, $(6, 4, 0)$, and $(4, 3, 1)$. Vertices of types $(1, 1, 1)$ and $(2, 2, 2)$ are called "standard vertices". A vertex of type $(0, 1, 3)$ is called a "center". Vertices of types $(3, 2, 0)$, $(6, 4, 0)$, and $4, 3, 1)$ are called "sporadic vertices". (They do occur in some tilings.)

Lemma 9. *Suppose $3\alpha + 2\beta = \pi$. Suppose triangle ABC is N-tiled by a tile with angles α and β. Then the number of centers is $N_C = 1 + N_1 + 2N_2$, where N_1 is the number of vertices of type $(3, 2, 0)$ and N_2 is the number of vertices of type $(6, 4, 0)$.*

Proof. By [4], α is not a rational multiple of π. Hence there are exactly five tiles with vertices at the vertices of ABC. In an N-tiling there are N triangles. Each has one α, one β, and one γ angle, so the total number of each kind of angle is N. At the vertices of ABC we have three α angles and two β angles, making an excess of three α angles over γ angles at the vertices of ABC. Similarly, at the sporadic vertices we get an excess of $3N_1 + 6N_2$. At the centers, the number of γ angles

exceeds the number of α angles by 3 per center, for a total of $3N_C$. Therefore the total excess of α and β angles over γ angles is

$$0 \quad = \quad 3 + 3N_1 + 6N_2 - 3N_C$$

Solving for N_C we have

$$N_C \quad = \quad 1 + N_1 + 2N_2$$

as claimed. That completes the proof of the lemma.

4. Coloring tilings black and white

We think of coloring each tile black or white, in such a way that tiles touching along a line segment have different colors. Technically we can represent the two colors as ± 1 and speak of the "sign" of a triangle. We show that the equation $3\alpha + 2\beta = \pi$ implies that tilings can be 2-colored in this way. See Fig. 2.

FIGURE 2. A tiling colored so that touching tiles have different colors.
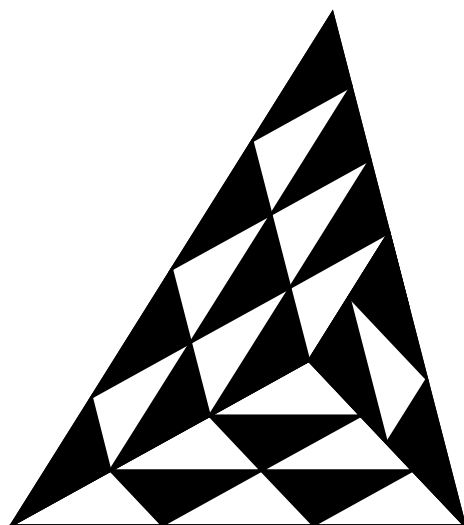
Definition 2. *Let triangle ABC be N-tiled by some tile. The tiling is said to satisfy the* **coloring condition** *if every interior vertex of the tiling has an even number of tiles meeting at that vertex, and every boundary vertex has an odd number.*

Lemma 10. *Suppose $3\alpha + 2\beta = \pi$, or $3\beta + 2\alpha = \pi$, and suppose triangle ABC is tiled by a tile with angles (α, β, γ). Then the coloring condition is satisfied.*

Proof. By Lemma 1, α is not a rational multiple of β. Then there can be at most one linear relation between α, β, and π. Suppose $3\alpha + 2\beta = \pi$. Then there are only the following ways to write 2π as a sum of α, β, and γ angles:

$$
\begin{aligned}
2\pi \quad &= \quad 6\alpha + 4\beta \\
&= \quad 4\alpha + 3\beta + \gamma \\
&= \quad 2\alpha + 2\beta + 2\gamma \\
&= \quad \beta + 3\gamma
\end{aligned}
$$

Since there are an even number of angles on the right side in each of these lines, the interior-vertex part of the tiling condition is satisfied. Now consider a boundary vertex. Here are the ways to write π:

$$\begin{aligned} \pi &= \alpha + \beta + \gamma \\ &= 3\alpha + 2\beta \end{aligned}$$

Both of these lines have an odd number of angles on the right. That completes the proof in the case $3\alpha + 2\beta = \pi$. The case $3\beta + 2\alpha = \pi$ is a notational variant.

Lemma 11. *Suppose triangle ABC is N-tiled by a tile with angles (α, β, γ) not rational multiples of π, and*

(i) $3\alpha + 2\beta = \pi$, and

(ii) the tiling satisfies the coloring condition.

Then it is possible to assign a color "black" or "white" to each of the N tiles in such a way that along each interior edge of the tiling, the triangles on opposite sides of the edge receive opposite colors, and the color of one tile at vertex B is specified to be black.

Proof. Since α, β, and γ are not rational multiples of π, there are only five tiles total at the vertices of ABC, three α angles and two β angles, at least one of which must stand alone at a vertex.

Let T_k be any one of the N tiles, and let P be a point in T_k. Let σ be a path from vertex B to P that does not pass through any vertex of the tiling, and passes transversally through each edge it crosses. (Transversally means it is not tangent to the edge.) Then the sign we wish to assign to T_k is the number of edges crossed by σ. We claim this sign does not depend on which path σ is chosen, but only on the tile T_k. To show that it suffices to show that the number is invariant under homotopies of σ fixing the two endpoints. That in turn follows from the fact that an even number of (segments of) edges meet at each interior vertex. We will now prove that. Since α is not a rational multiple of π, also α is not a rational multiple of β (since if it were, then it would follows from $3\alpha + 2\beta = \pi$ that α is a rational multiple of π). Therefore there are only the following types of interior vertices: (1) vertices where two each of α, β, and γ angles meet; (2) vertices where there are three γ angles and one β angle; (3) vertices with six α angles and four β angles; vertices with four α angles, three β angles, and one γ angle; (4) non-strict vertices occurring on an edge of some tile (rather than at its vertex) and with one each of α, β, and γ; (5) non-strict vertices occurring on an edge with three α and two β angles. Any other combination of angles at the vertex will give another linear relation between α and β besides $3\alpha + 2\beta = \pi$, which will imply that α is a rational multiple of π. In each of the five cases enumerated, there are an even number of segments of edges meeting at the vertex. That completes the proof of the lemma.

Theorem 1 (Coloring theorem)**.** *Let $3\alpha + 2\beta = \pi$, and assume triangle ABC is N-tiled by a tile with angles (α, β, γ), and ABC is not similar to the tile. Let the tiles be assigned signs (or colors) in accordance with Lemma 11. Let X and Z be the two sides of ABC adjacent to the vertex with just one tile. Then we have the "coloring equation"*

$$M(a + b + c) = X \pm Y + Z$$

where M is the number of black (or positive) tiles minus the number of white (or negative) tiles.

More specifically, the sign denoted by the $\pm$ sign is determined by the shape of ABC. If ABC is isosceles with base angles α or β then we have

$$M(a + b + c) = X + Y + Z.$$

If ABC has angles $2\alpha, \alpha, 2\beta$ or $2\alpha, \beta, \alpha + \beta$, or is isosceles with base angles $\alpha + \beta$, then we have

$$M(a + b + c) = X - Y + Z.$$

Proof. The statement mentions "the vertex with just one tile." The fact that there is such a vertex has to be proved. By Lemma 1, which summarizes work of Laczkovich [4], α and β are not rational multiples of π, and hence α is not a rational multiple of β, because of the relation $3\alpha + 2\beta = \pi$. Since ABC is not similar to the tile, no vertex of ABC has a γ angle, as then the other two angles would have to be α and β. Therefore the tiles at the vertices of ABC have three α angles and two β angles. That is five angles–not enough to provide two angles to each of three vertices. Hence one vertex has only one tile. We think of the triangle with that vertex at the north, and colored black. By Lemma 10 and Lemma 11, all the tiles can be colored such that adjacent tiles have opposite colors.

Consider a "maximal segment" pq in the tiling (if any exist), i.e. a part of a straight line consisting of interior edges of the tiling, which cannot be extended to a longer such segment (and hence has its endpoints at strict vertices P and Q, where the angle sum of the angles at the vertex is 2π). Then all the tiles on one side of pq are positive (black), and all the tiles on the other side are negative (white). The signed sum of the lengths of the edges all these triangles share with pq is zero. Hence, the total length of the positive interior edges equals the total length of the negative interior edges.

On the boundary of ABC, all the tiles sharing edges with sides $Z = AB$ and $X = BC$ are black, since the standalone tile at B is black. The color of the tiles on side $Y = AC$ depends on whether there are two tiles at A and C, or only one. If there is only one then AC will be black; if there are two then AC will be white. Let M be the number of positive tiles minus the number of negative tiles. Then the difference between the total length of the positive edges and the total length of the negative edges is M times the perimeter $a + b + c$ of a tile. Since the positive and negative interior lengths are equal, we have

$$M(a + b + c) \quad = \quad X \pm Y + Z$$

The plus sign will apply if AC is white, which happens when there is just one tile at A and C. That happens just when ABC is isosceles with base angles α or base angles β. Otherwise there are two tiles at each of A and C, and AC is black, so the minus sign is needed. That completes the proof of the lemma.

Lemma 12. *Let $3\alpha + 2\beta = \pi$ and assume triangle ABC is N-tiled by a tile with angles (α, β, γ). Let the tiles be assigned signs (or colors black and white) in accordance with Lemma 11, and let M be the number of positive (black) tiles minus the number of negative (white) tiles. Then M is not zero.*

Proof. Depending on the shape of ABC, by Theorem 1 we have $M(a + b + c) = X \pm Y + Z$, where (X, Y, Z) are the lengths of the sides of ABC. Assume, for proof by contradiction, that $M = 0$. Then either $X + Y + Z = 0$ (which is absurd, since all the sides are positive), or $X - Y + Z = 0$, which contradicts the proposition of

Euclid that says one side of a triangle is less than the other two together. That completes the proof of the lemma.

5. The connected components of a tiling

Suppose given an N-tiling of triangle ABC, by the tile with angles α, β, and γ, where $3\alpha + 2\beta = \pi$ and α is not a rational multiple of π. We define a graph $\mathbb{H}$ whose nodes are the tiles of the tiling. Since the word "edges" is already in use for the sides of a tile, we shall refer to the edges of this graph as "connections" instead, and speak of one tile being connected to another.

Definition 3. *Given a tiling of ABC, the graph $\mathbb{H}$ has for its nodes the tiles used in the tiling, and its edges (connections) are defined as follows: Tiles T and S are* **connected** *if T and S share two vertices (and hence the edge between those vertices), and S and T do not have the same angles at the vertices of their shared side. A* **connected component***, or just a* **component***, of a tiling is a maximal connected set in the graph $\mathbb{H}$.*

Examples. Any quadratic tiling has only one connected component. Any bi-quadratic tiling has two connected components, one for each of the two quadratic tilings that it contains. Fig. 3 illustrates the connected components of two different 28-tilings, both different from the one in Fig. 6.

FIGURE 3. Components of two 28-tilings
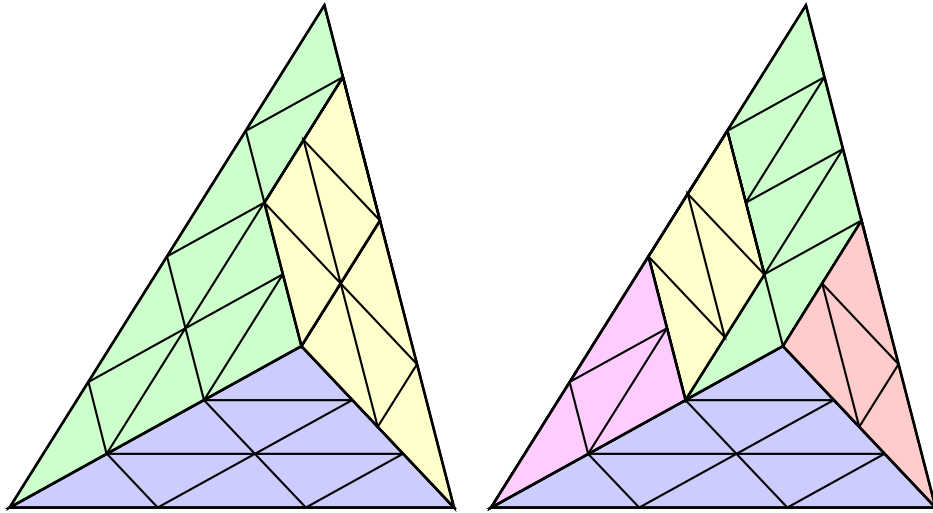
Remarks. We could have also allowed two tiles to be connected if they lie on opposite sides of an interior segment L and have edges of the same length lying on L, i.e. both have their a edges on L, or both have their b edges on L, or both have their c edges on L. This less stringent definition results in a graph with more connected components. For example, consider the partial tiling illustrated in Fig. 4. There are four components of the graph $\mathbb{H}$; but with the less restrictive definition of connection, there would be only three. The segment FJ is a component boundary

in $\mathbb{H}$, but with the less restrictive definition of connection, the tiles with c edges on on FJ would be connected. It will turn out to be important that they *not* be connected.

FIGURE 4. There are four connected components in this partial tiling, not three.
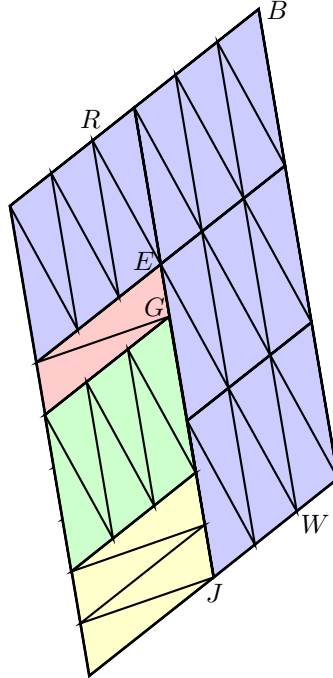
Another variation on the definition of $\mathbb{H}$ is the graph $\mathbb{G}$, defined by removing the requirement that connected tiles have different angles at their common vertices. In this graph, lines that have all the same length edges on both sides, with matching vertices but the tiles have equal angles at each vertex, are not component boundaries as they are in $\mathbb{H}$. For example, in the 28-tiling shown in in Fig. 6, such a line occurs as the angle bisector of angle A, and the two components of $\mathbb{H}$ above and below that line will join into one $\mathbb{G}$-component. We will make no use of these variations on the definition of $\mathbb{H}$, and the word "component" will be used according to the definition.

Lemma 13. *Assume $a \neq b$, i.e. the tile is not isosceles. In any connected set $\mathcal{C}$ in a tiling, any two edges of the same length are parallel or on the same line. Two tile edges of different lengths lying on the same line do not belong to the same connected component,*

Remark. Of course a and b are always unequal if $3\alpha + 2\beta = \pi$ and α is not a rational multiple of π, as we generally assume in this paper; but the lemma does not require those assumptions.

Proof. The second sentence of the lemma follows from the first. We prove the first by induction on the number n of tiles in $\mathcal{C}$. In the base case, when $n = 1$, if we

have two edges of the same length they are the same edge, so they lie on the same line. That completes the base case. Now for the induction step. Suppose $\mathcal{C}$ has $n + 1$ tiles. Let T be one of those tiles, and let S be a tile in $\mathcal{C}$ that is directly connected to T; then by the definition of the graph $\mathbb{H}$, the two tiles S and T form a parallelogram, whose opposite sides are parallel. Since $\alpha \neq \beta$, the sides of S and T of the same lengths are parallel (or on the same line, in the case of the shared side). We can apply the induction hypothesis to the set obtained by deleting T from $\mathcal{C}$; then all the other tiles in $\mathcal{C}$ have their corresponding edges parallel to or collinear with the edges of S, and hence to those of T. That completes the proof.

Definition 4. *A* **lattice tiling** *(of some region, not necessarily a triangle) is a tiling whose vertices lie on the lattice points of some lattice.*

For example, a quadratic tiling, or any subtiling of a quadratic tiling, is a lattice tiling.

Lemma 14. *Suppose there is an N-tiling of ABC by the tile with angles α, β, and γ, and $\alpha \neq \beta$. Let $\mathcal{E}$ be a connected component of the tiling, or more generally, the union of any connected set of tiles. Then the tiling (restricted to $\mathcal{E}$) is a lattice tiling of $\mathcal{E}$.*

Remark. The 28-tiling shows that the lemma fails if the definition of $\mathbb{H}$ is changed to permit tiles with the same angles at their shared vertices to be connected.

Proof. We proceed by induction on the number of tiles in the connected set $\mathcal{E}$. When this number is 1, the result is trivially true. Suppose $\mathcal{E}$ is a connected set of $n + 1 \geq 2$ tiles. Let T be a tile on the boundary of $\mathcal{E}$. Remove T from $\mathcal{E}$, and call what is left $\mathcal{E}'$. By the induction hypothesis, the conclusion of the lemma holds for $\mathcal{E}'$. Let S be a tile in $\mathcal{E}$ that shares an edge and its vertices with T. Then by the induction hypothesis, the vertices of S lie in a lattice that contains all the vertices of $\mathcal{E}'$. Since S and T form a parallogram, that lattice also contains the third vertex of T. That completes the proof of the lemma.

The "type" of a component is specified by giving the directions of its a, b, and c sides. If a particular triangle ABC is understood in the context, and a tile with angles (α, β, γ) is also understood, with α and β not rational multiples of π, and ABC having angle α or β at vertex B, then the following terminology makes sense. When we say a line segment has "direction AB" we mean that it is parallel to AB, and similarly "direction BC". In other words, a "direction" is given by an equivalence class of parallel lines.

Definition 5. *If a tile is placed at vertex B with two sides on AB and BC respectively, then its third side will be in* **Direction** A *if its γ angle is on BC, or in* **Direction** C *if its γ angle is on AB.*

Definition 6. *We give names to the following types of components:*

Type I: c edges have direction BC, a edges have direction AB, and b edges have Direction C.

Type II: c edges have direction AB, a edges have direction BC, and b edges have Direction A.

These types of components are all illustrated in the 28-tiling and the other triquadratic tilings. Further types also arise. For example, along the base AC of

the quadratic tiling are Type III tiles. There are in principle infinitely many types, but they do not actually arise in the known tilings, so there is not much use in classifying them.

Technically, components are of Type I or Type II, not tiles. But we say that a tile (considered as part of a tiling) is "of Type I" if it belongs to a component of Type I, and similarly for Type II.

Please have another look at Fig. 4. Note that there are two different Type I components in that figure, that are not connected even though some of their tiles share a common boundary with tile edges of the same length on that boundary. They still do not connect, since the tiles on that boundary do not share their vertices. This motivates the following definition:

Definition 7. *Two components $\mathcal{C}$ and $\mathcal{D}$ of the same type are said to be **out of sync** if they share a common boundary segment with edges of the same length on that boundary segment that do not share vertices.*

6. The tile is rational

By saying "the tile is rational", we mean that the ratios of its sides are rational. By Lemma 2, the tile is rational if and only if s is rational.

We call a vertex in a tiling "of type 2π" if the sum of the angles at that vertex is 2π, and "of type π" if the sum of the angles is π. The latter occurs when the vertex is on the boundary of ABC, but it can also occur in the interior of ABC. In a tiling it may happen there exist line segments pq bounding several tiles on one or both sides, such that the vertices of tiles on one side are not necessarily vertices of tiles on the other side. For example, one side of pq might have three edges of length b and the other side might have two edges of length c. Such segments are called "essential segments", and the associated linear relations between the edge lengths are central to the subject of triangle tiling. We now give several basic definitions that will be used extensively.

Definition 8. *A line (segment) pq **supports** a tile T if a side, or a positive-length part of a side, of T lies on the segment pq.*

Definition 9. *An **interior segment** is a line segment formed of boundaries of tiles in a tiling, such that its endpoints are vertices of tiles, and it does not lie on the boundary of the tiled triangle ABC.*

*An interior segment pq is **terminated at** q on each side of pq there is a tile supported by pq with a vertex at q. An interior segment terminated at both endpoints is called **doubly terminated**.*

*A **maximal segment** is an interior segment that does not lie on any longer interior segment, i.e. cannot be extended (in either direction).*

*An interior segment pq that has the same number of edges of length a on each side of pq, and the same number of edges of length b on each side, and the same number of edges of length c on each side, is called **inessential**. An interior segment that is not inessential is **essential**.*

Remarks. Equating the sum of the edges on one side of a doubly terminated interior segment to the sum of the edges on the other side gives rise to an integral relation (linear relation with integral coefficients) between a, b, and c. That relation will

be trivial for an inessential interior segment, and nontrivial for an essential interior segment.

Examples. The 28-tiling illustrated in Fig. 6 contains two essential maximal segments. The one at the lower right in the figure has three a edges on one side and two b edges on the other. Thus for this tiling we have $3a = 2b$. which you can see near the lower right of the figure. Each biquadratic tiling contains a maximal segment on the altitude of ABC connecting the right angle to the hypotenuse.

Definition 10. *Given a tiling of triangle ABC by triangle T,*

(i) An interior segment pq is said to **correspond to**, *or to* **witness**, *a relation $jb = ua + vc$ (with $j > 0$ and $u, v \geq 0$) if there are tiles on both sides of pq with vertices at p, and tiles on both sides with vertices at q, and on one side of pq there are j more b edges than on the other, and on the other side, there are u more a edges and v more c edges than on the first side. Similarly for a relation $ja = ub + vc$ or $jc = ua + vb$.*

(ii) An **essential edge relation** *is a relation of the form $jb = ua + vc$, or $ja = ub + vc$, with nonnegative integers j, u, and v and $j > 0$, that corresponds to an essential interior segment pq in the tiling.*

Remark. The definition requires more than that a number-theoretical relation between a, b, and c exists: it must actually be realized (witnessed) in the tiling.

6.1. **Existence of essential edges.** For purposes of describing configurations, we think of ABC as oriented with B upwards, or to the "north", and AC horizontal, with A at the "west" and C at the "east". Also sometimes we use "above" and "below" as synonyms for "north of" and "south of", and "right" and "left" as synonyms for "east" and "west." The "horizontal" is the direction of BC, and the "positive horizontal" is the direction of the ray BC. The meanings of "Direction A" and "Direction C" are given in Definition 5.

Definition 11. *Given a tiling of triangle ABC, with angle β at B filled by just one tile, we we define Ω to be the set of all (closed) Type I and Type II tiles that contains B and such that each tile in Ω can be connected to B by a chain of Type I and Type II tiles, each sharing (part of) an edge with the next.*

Then Ω is a connected set, since any two tiles in Ω can be connected to the tile at B, and hence to each other. (We have not claimed that Ω is simply connected.) The boundary of Ω is a closed polygon $\mathcal{C}$ (or theoretically it might be a union of closed polygons, if Ω is not simply connected), including vertex B. The definition of Ω implies that the Ω stops not only at the end of tiles of Type I and Type II, but when we reach a component boundary between two out-of-sync components.

Fig. 5 shows the region Ω in the tiling from Fig. 2. Black and dark gray tiles are Type I, red and pink tiles are Type II. Their union is Ω. The white and light blue tiles at the bottom are not in Ω. The boundary of Ω in this example consists of two segments, AF in Direction A and FC in Direction C. The segment FC in Direction C witnesses the relation $2b = 3a$.

The reader may also wish to look at Fig. 4 for an example of how Type I and Type II tiles can fit together. That entire figure would be contained within Ω, as it is composed entirely of Type I and Type II tiles. The complexities of that figure

FIGURE 5. Black and dark gray tiles are Type I, red and pink tiles are Type II. Their union is Ω.
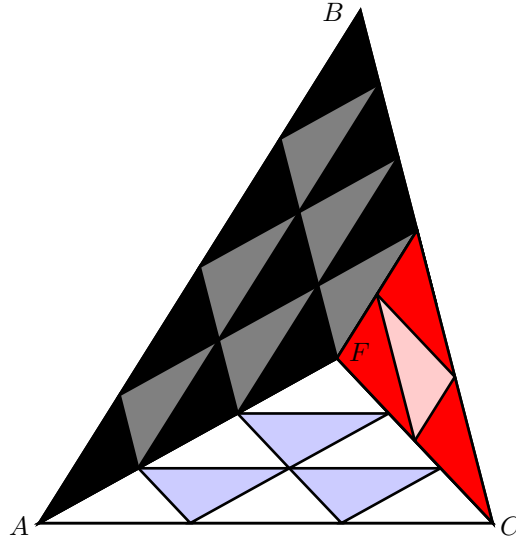
depend on the existence of essential segments with c edges on one side and a, or a and c, edges on the other, for short "a/c edges".

The edges of Type I and Type II tiles lie in just four directions: Directions AB and BC have a and c edges (only), and Directions A and C have b edges (only).

Definition 12. *An* **arrow** *is a segment pq of the tiling, terminated at both ends and supporting only tiles with their b edges on pq, such that the α angles of two tiles sharing a b edge on pq are at the same vertex.*

For example, AF in Fig. 5 is an arrow.

Definition 13 (boundary segment). *The directed segment pq is a* **boundary segment** *if it is part of a tiling, and the tiles supported or partially supported by pq on its left side belong to Ω, and those supported or partially supported on its right side do not belong to Ω, and pq cannot be extended in either direction maintaining this property.*

In other words, the boundary of Ω makes a turn at p and a turn at q. Note that there must be tiles on both sides of pq; a piece of AB or BC does not count as a boundary segment.

Theorem 2. *Let triangle ABC be N-tiled by a tile with sides (a, b, c) and angles (α, β, γ). Suppose*

(i) ABC is not similar to the tile, and none of its angles is greater than γ.

(ii) either $3\alpha + 2\beta = \pi$ or $3\beta + 2\alpha = \pi$, and

(iii) There is just one tile at vertex B, and the angle there is β.

Then there exists an essential segment in one of the four directions, A, C, AB, or BC. The corresponding edge relation is of the form $jb = ua + vc$, with $j > 0$.

Remark. In most of the paper we assume $3\alpha + 2\beta = \pi$. However, this theorem applies more generally, with either β or α at vertex B, so to avoid duplicating the argument for those two cases, we treat them both at once, by allowing the angle at the top to be called β. In other words, if $3\alpha + 2\beta = \pi$, but α is the angle at B, then we rename α and β, so that β is the angle at B. The price of this renaming is that now we have $3\beta + 2\alpha = \pi$ instead of $3\alpha + 2\beta = \pi$. That won't matter anyway for this proof.

Proof. For simplicity we think of the triangle with vertex B drawn at the top, or "north". The tile at the top vertex B has a β angle at B, so it has its a and c sides on AB and BC, or vice-versa. According to Definition 6 that tile is either Type I or Type II. We may rename A and C if necessary so that angle A is less than or equal to angle C. They could be equal if both are $\alpha + \beta$ and $2\alpha + 3\beta = \pi$. Otherwise the angle at A is strictly less than the angle at C.

Assume, for proof by contradiction, that there is no essential segment in any of the four directions mentioned. Let Ω be as defined in Definition 11. Suppose pq is a segment on the boundary of Ω, with Ω on the north, in Direction A or Direction C. Then the tiles on the north have their b edges on pq. If pq terminates at both ends, then all the tiles supported by pq on the south side also have their b edges on pq. Since pq is on the boundary of Ω, those tiles have the opposite orientation from those in Type I or Type II, that is, their α angles point in the same direction as those to the north of pq. Then pq is an arrow, as defined in Def. 12. To summarize, if pq is a segment of the boundary of Ω in Direction A or Direction C, then either pq is an arrow, or it is not terminated at both ends.

If pq is a segment on the boundary of Ω parallel to AB or BC, then there are no a or c edges of tiles on the non-Ω side of pq, since tiles with their a or c edges in those directions are of Type I or Type II, and hence those tiles would lie in Ω, and pq would not lie on the boundary. Hence only b edges occur on the non-Ω side of pq. Hence the segment pq cannot be terminated at both ends, or it would witness a relation $jb = ua + vc$.

We begin by proving that there is a segment of the boundary of Ω in Direction A or Direction C, with one endpoint on AB. Case 1, there is a tile supported by AB that is not in Ω. It is not the top tile (at B), since that tile does belong to Ω. Let Tile 3 be the northernmost tile supported by AB that is not in Ω, and let Tile 1 be the tile supported by AB just north of Tile 3; let E be the vertex they share on AB. Assume Tile 1 is of Type I. Then it has its α angle at E and its b edge on EF. Then either EF lies on the boundary of Ω, in which case we have established our claim, or Tile 2, the next tile south of Tile 1, is also of Type I, in which case it has its γ angle at E. But then Tile 3 must have its β angle at E, which implies that it is of Type I or Type II, contradiction. Therefore we are finished in case Tile 1 is of Type I. Now assume Tile 1 is of Type II. Then Tile 1 has its γ angle at E and its b edge on EF. Again, if Tile 2 (to the south of Tile 1) is in Ω, it is of Type II with its b edge on EF and its α angle at E, so Tile 3 has its β angle at E and must be in Ω, contradiction.

So, we may assume that all the tiles supported by AB are in Ω, i.e., they are of Type I or Type II. Then they all have their a or c edge on AB and their β angle to the north, and their γ or α angle to the south. The angle of ABC at vertex A cannot be γ or α, since that would make ABC similar to the tile, because the angle at B is assumed to be β. Therefore there is another tile with a vertex at A,

not supported by AB. Let the tile at A supported by AB be Tile 1. Then Tile 1 has its southern edge in direction A, since if it were direction C, the angle at A would be at least γ, contradiction. Let Tile 2 be the tile south of Tile 1. I say that Tile 2 is not of Type I or Type 2. Suppose it is; then since its northern border is in Direction A, that must be its b edge, so it is Type I, and since it is to the south of its Direction A edge, it forms a parallelogram with Tile 1. Hence it has its γ angle at A. But then the angle of ABC is at least $\alpha + \gamma$, contradiction. Hence, as I said, Tile 2 is not of Type I or Type II. Therefore the southern border of Tile 1 lies on the boundary of Ω, under the assumption that all tiles supported by AB are of Type I or Type'II. (This is the situation you see in Fig. 5.) That completes the proof that there is some segment EF of the boundary of Ω with E on AB (possibly with $E = B$).

We define a directed graph Γ by defining its nodes and links. We use the word "link" because "edge" is already in use for the edges of tiles. The nodes of Γ are the vertices of tiles in Ω. Two nodes p and q are connected by a link in Ω if and only if

(i) The (directed) line segment pq is a boundary segment, as defined in Def. 13, and

(ii) Either pq is an arrow (as defined in Def. 12) with its tail at q, or pq is not terminated at q.

If p and q are connected by a link in Γ, then we say for short "pq is in Γ."

Now we claim that if pq is in Γ, then there is a segment qr in Γ. That is, if the in-degree of q is positive, so is the out-degree. Moreover, the segment qr is specifically the "next" boundary segment after pq. That is, it is the boundary segment such that all the tiles with vertices at q on the left side of angle pqr belong to Ω. To prove this claim, we argue by cases on the direction of pq (as a directed link).

Case 1: pq is an arrow with tail at q. Then pq is in Direction A northeast, or Direction A southwest, since b edges of tiles in Ω occur only in Directions A and C, and those in Direction C have their α angles forwards when Ω is on the left. Then the tile on the right side of pq at q has its γ angle at q. Call this Tile 1. pq lies on the boundary of Ω, by definition of Γ, but also because the tiles on the right of pq are not of Type I or Type II. Let qr be the next segment of the boundary of Ω. We claim qr is in Γ. If qr is in Direction C, then all of the tiles on its left side are in Ω, and hence have their b edges on qr. Since qr is terminated at q, it is not terminated at R, since if terminated at R it would be an essential segment in Direction C, since Tile 1 is supported by qr but does not have its b edge on qr. Therefore qr is in Γ. If, on the other hand, qr is not in Direction C, then it must be in direction AB or BC northwards, since Tile 1 blocks the southerly directions. Therefore it has all b edges on the right, since a tile with its a or c edge in direction AB or BC would belong to Ω. Since qr is terminated at q, either it is an essential segment or is unterminated at R; but we have assumed there are no essential segments in the four relevant directions; hence qr is unterminated. Hence qr belongs to Γ. That completes Case 1.

Case 2: pq is not terminated at q. Then the next boundary segment qr is terminated at q. If it lies in Direction A or C then it all the tiles it supports on the left have their b edges on qr. It cannot then be terminated at r, since that would make it an essential segment. Therefore we may assume that qr lies in direction

AB or BC. In that case all the tiles supported on the right by qr have their b edges on qr, since if they had an a or c edge on qr, they would belong to Ω. Therefore qr cannot be terminated at r, since that would make it an essential segment. That completes Case 2.

That completes the proof that for every in-link pq to node q of Γ, the next segment qr is an out-link.

Now we claim that the in-degree of any node of Γ is less than or equal to the out-degree. Let q be a node of Γ and let pq be an incoming link. As we have seen, the next segment qr of the boundary of Ω is an outgoing link. As far as we have proved, there might be another piece of the boundary of Ω, say PqR, with the same q. But then, if Pq is a link in Γ, so is qR. The point R is not equal to r, since then $qr = qR$ would have Ω on both sides. Hence the in-degree of q is less than or equal to the out-degree.

Since there are finitely many nodes, the total in-degree is equal to the total out-degree. That is only possible if the in-degree of each node is equal to the out-degree. Therefore, the in-degree of every node is equal to its out-degree.

Now let E be the southernmost point on AB such that AE lies on the boundary of Γ. Then as shown above, there is an outgoing link EF in Direction A or Direction C. Hence, there is an incoming link KE. Since E lies on AB, KE is terminated at E. The angle at A is less than γ, by hypothesis. We have $E \neq A$, since if $E = A$ there is only on direction available, Direction A, for both the incoming and outgoing link, as Direction C makes angle Γ with AB. The direction of KE cannot be BC northwest, as that would mean that the tile below E has two edges in directions AB and BC, so it would belong to Ω, contradicting the definition of E. Therefore KE must be in Direction C northwest, while EF is in Direction A. Then KE has Ω on the left, which is the south. Therefore all the tiles supported by KE on the south have their b edges on KE and their α angles to the northwest. In particular the tile at E on the south of KE has its α angle at E. But angle AEK is equal to $\alpha + \beta$, since by definition of Direction C, angle AEK is γ. Hence the tile supported by AB just south of E has to fill an angle β, with one edge parallel to BC and the other on AB. Therefore that tile belongs to Ω, contradicting the definition of E. That completes the proof of the theorem.

Credit. The introduction of the graph Γ was inspired by the use of a graph in [5], although this is a different graph, and the proof is different. The use of a graph is not absolutely essential, but it makes the proof more concise and elegant. (Without mentioning the graph, we would be proving that a traversal of the boundary of Ω must encounter an essential segment.)

6.2. The tile is rational.

Theorem 3 (Rational tile). *Let ABC be N-tiled by a Let triangle ABC be N-tiled by a tile with sides (a, b, c) and angles (α, β, γ). Suppose $3\alpha + 2\beta = \pi$, and ABC is not similar to the tile. Then the tile is rational; specifically $s = a/c$ is rational and b/c is rational.*

Proof. By Lemma 1, α is not a rational multiple of β and there is no linear relation between α, β, and π except $3\alpha + 2\beta = \pi$ and its multiples. Since ABC is not similar to the tile, no γ angle of a tile occurs at a vertex of ABC, as that would leave only α and β for the other two vertices. Therefore the angles of tiles at the vertices of ABC are three α and two β angles, otherwise we have another linear

relation between α, β, and π. Therefore one of the vertices of ABC has only one tile.

Suppose, for proof by contradiction, that a/c is not rational. By Theorem 2, there is a relation $jb = ua + vc$ with $j > 0$ and $u, v \geq 0$, witnessed on some segment of the tiling, in one of the four directions mentioned in the theorem. If there is any linear relation between a, b, and c, with a nonzero coefficient of b, then it must be a multiple of $jb = ua + vc$, otherwise we could solve for b and get a relation between a and c only, which would contradict our assumption that a/c is not rational. Hence, the relation $jb = ua + vc$ is the *only* linear relation between (a, b, c), up to a constant multiple.

That edge relation might correspond to several different essential segments within the tiling. Each essential edge has a "b-side" (on which j more b edges occur than on the other side). Let the tiling be colored black and white, with the top tile black. Then each essential edge has one black side and one white side (since every vertex P on the interior of an essential segment has an odd number of tiles with a vertex at P on one side of the segment). Let L be the difference between the number of essential segments whose b side is black and the number of essential segments whose b side is white. Then when we compute the number of black edges minus white edges of triangles, from the interior segments of the tiling we get $L(jb - ua - vc)$. On the boundary, we note that AB and BC are black, since there is just one tile at B, while AC may be black or white, depending how many tiles are at vertices A and C. Recall (X, Y, Z) are the lengths of the sides opposite (A, B, C). Therefore

$$X \pm Y + Z \quad = \quad L(jb - ua - vc)$$

Let M be the coloring number of the tiling. Then by Theorem 1, we have $M > 0$ and

$$X \pm Y + Z \quad = \quad M(a + b + c)$$

Therefore

$$M(a + b + c) \quad = \quad L(jb - ua - vc)$$
$$(Lj - M)b \quad = \quad (M + Lu)a + (M + Lv)c$$

That relation must be a multiple of $jb = ua + vc$. Therefore

$$\frac{Lj - M}{j} \quad = \quad \frac{M + Lu}{u} \qquad \text{if } u \neq 0$$
$$\frac{Lj - M}{j} \quad = \quad \frac{M + Lv}{v} \qquad \text{if } v \neq 0$$

If $u \neq 0$ we have

$$L - M/j \quad = \quad M/u + L$$

which contradicts the inequalities $j > 0$, $M > 0$, and $u \geq 0$. Hence $u = 0$. But then $v > 0$, and

$$L - M/j \quad = \quad M/v + L$$

which contradicts the inequalities $M > 0$, $j > 0$, and $v > 0$. That contradiction shows that a/c is rational.

It follows from $jb = ua + vc$ that $b/c = ua/(jc) + v/j$ is also rational. That completes the proof of the theorem.

7. THE CASE WHEN ABC HAS ANGLES $(2\alpha, \beta, \alpha + \beta)$

In this section, we derive the "tiling equation" for ABC of the shape mentioned, and prove that it is a necessary and sufficient condition for the existence of an N-tiling of some triangle ABC of the specified shape.

Our convention throughout this section is that the angle at A is 2α, the angle at B is β, and the angle at C is $\alpha + \beta$. The lengths of the sides opposite vertices (A, B, C) are X, Y, and Z respectively. In pictures, we draw the vertex B at the top.

The $\mathbf{d}$ matrix (which occurs in [6] and the first edition of [7]) is defined by

$$\mathbf{d}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

where X is the length of BC (opposite the 2α angle), Y is the length of AC (opposite the angle at B) and Z is the length of AB (opposite the angle at C). We always draw our pictures with AC horizontal and A at the lower left, or southwest, and B at the top (or north) of the picture. The $\mathbf{d}$ matrix tells how the sides of ABC are composed of edges of tiles (although it contains no information about the order of those edges). We use the following letters for elements of the $\mathbf{d}$ matrix:

$$\mathbf{d} = \begin{pmatrix} p & d & e \\ g & m & f \\ h & \ell & r \end{pmatrix}$$

In case the tile is similar to ABC, then there is an eigenvalue equation involving the $\mathbf{d}$ matrix:

$$\mathbf{d}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

since $(X, Y, Z) = \lambda(a, b, c)$ in that case. In the present case we can make the equation look something like that:

Lemma 15. *Let the angles of ABC be $(2\alpha, \beta, \alpha + \beta)$. Let X, Y, and Z be the lengths of BC, AC, and AB respectively. Let $s = 2\sin(\alpha/2)$. Then for a suitable "scaling factor" μ,*

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \mathbf{d}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mu \begin{pmatrix} (2 - s^2)a \\ b \\ c \end{pmatrix}$$

Remark. Because of the factor $2 - s^2$, this is not quite an eigenvalue problem. When triangle ABC is similar to the tile, the factor $2 - s^2$ does not appear, but still we can make this problem look *almost* like an eigenvalue problem.

Proof. According to the law of sines, for a certain positive λ we have

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \lambda \begin{pmatrix} \sin A \\ \sin B \\ \sin C \end{pmatrix} = \lambda \begin{pmatrix} \sin 2\alpha \\ \sin \beta \\ \sin(\alpha + \beta) \end{pmatrix} = \lambda \begin{pmatrix} 2\cos\alpha\sin\alpha \\ \sin\beta \\ \sin\gamma \end{pmatrix}$$

Also by the law of sines, for some positive κ we have $(a, b, c) = \kappa(\sin \alpha, \sin \beta, \sin \gamma)$. Define $\mu = \lambda/\kappa$. Then we have

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \mu \begin{pmatrix} 2(\cos \alpha)a \\ b \\ c \end{pmatrix} = \mu \begin{pmatrix} 2(1 - 2\sin^2(\frac{\alpha}{2}))a \\ b \\ c \end{pmatrix} = \mu \begin{pmatrix} (2 - s^2)a \\ b \\ c \end{pmatrix}$$

That completes the proof of the lemma.

7.1. **The tiling equation for the shape** $(2\alpha, \beta, \alpha + \beta)$. The "tiling equation" is a Diophantine equation whose solvability controls whether there exists an N-tiling of ABC or not.

Lemma 16. *Let $3\alpha + 2\beta = \pi$ and assume triangle ABC has angles $(2\alpha, \beta, \alpha + \beta)$ and is N-tiled by a tile with angles α and β. Let the tiles be assigned signs (or colors) in accordance with Lemma 11. Let $s = 2\sin(\alpha/2)$ and let M be the number of black (positive) tiles minus the number of white (negative) tiles. Then $M > 0$, and we have*

$$s^2 = \frac{2M^2}{M^2 + N}$$

Proof. By Theorem 1, we have $M(a + b + c) = X - Y + Z$. By Lemma 15, we have

$$(2) \qquad\qquad \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \mu \begin{pmatrix} a(2 - s^2) \\ b \\ c \end{pmatrix}$$

Hence $M(a + b + c) = \mu(a(2 - s^2) - b + c)$. Dividing by c we have

$$\begin{aligned} M(s + (1 - s^2) + 1) &= \mu(s(2 - s^2) - (1 - s^2) + 1 \\ M(s + 2 - s^2) &= \mu(2s - s^3 + s^2) \\ &= \mu s(2 - s^2 + s) \end{aligned}$$

Dividing both sides by $2 - s^2 + s$ (which cannot be zero since $0 < s < 1$) we have

$$(3) \qquad\qquad\qquad M = \mu s$$

Since μ and s are both positive, this implies $M > 0$. We claim $N = \mu^2(2 - s^2)$. To prove this we use the area equation

$$Nbc \sin \alpha = YZ \sin 2\alpha,$$

whose left side is twice the area of N tiles, and whose right side is twice the area of triangle ABC. According to Lemma 38,

$$\sin 2\alpha = (2 - s^2) \sin \alpha.$$

Putting that into the area equation just above, we have

$$Nbc \sin \alpha = YZ(2 - s^2) \sin \alpha.$$

Cancelling $\sin \alpha$ we have

$$Nbc = YZ(2 - s^2).$$

Replacing Y and Z by their values from (2), we have

$$Nbc = \mu^2 bc(2 - s^2)$$

Cancelling bc we have the claimed equation $N = \mu^2(2 - s^2)$.

Then squaring (3) we have

$$
\begin{aligned}
M^2 &= \mu^2 s^2 \\
&= s^2 \frac{N}{2 - s^2} \\
&= N \frac{s^2}{2 - s^2} \\
\frac{N}{M^2} &= \frac{2 - s^2}{s^2} \\
&= \frac{2}{s^2} - 1
\end{aligned}
$$

Solving for s^2 we have

(4)
$$
s^2 = \frac{2M^2}{M^2 + N}
$$

That completes the proof of the lemma.

Theorem 4. *Let $3\alpha + 2\beta = \pi$ and assume triangle ABC has angles $(2\alpha, \beta, \alpha + \beta)$ and is N-tiled by a tile with angles α and β. Then*

(i) N satisfies the equation

$$
M^2 + N = 2K^2
$$

for some positive integers M and K, with $M^2 < N$, where

$$
s = 2\sin(\alpha/2) = M/K,
$$

and

(ii) N is a square times a product of distinct primes of the form $8n \pm 1$.

Proof. Recall that s is rational if and only if the tile is rational, since $a/c = s$ and $b/c = 1 - s^2$. If there is an N-tiling as in the theorem, then by Lemma 16, there exists an integer M such that $M^2 + N = 2M^2/s^2$. Setting $K = M/s$ we have $M^2 + N = 2K^2$. Then K is the square root of a half-integer (not yet an integer).

Now to prove $M^2 < N$. Recall from (4) that

$$
s^2 = \frac{2M^2}{M^2 + N}
$$

But $s = 2\sin(\alpha/2)$. Both α and β are less than γ, so both α and β are less than $\pi/3$. Hence $\alpha/2 < \pi/6$, so $\sin(\alpha/2) < 1/2$, and $s < 1$. Hence $2M^2/(M^2 + N) < 1$. Hence $2M^2 < M^2 + N$; hence $M^2 < N$ as claimed in the statement of the lemma. Since $M^2 + N = 2K^2$ we have

$$
s^2 = \frac{2M^2}{M^2 + N} = \frac{M^2}{K^2}.
$$

Taking the square root of both sides we have $s = M/K$. That completes the proof of the first part of the lemma, but we still have to prove that K is an integer.

Let $K = M/s$. Then $M^2 + N = 2K^2$; then we have $s = M/K$, and $a/c = s$, $b/c = 1 - s^2$, so a, b, and c are proportional to $(M, (N/K) - K, K)$. We have

$$
\begin{aligned}
2 - s^2 &= (2K^2 - M^2)/K^2 = N/K^2 \\
\end{aligned}
$$

(5)
$$
= s^2 \frac{N}{M^2} \qquad \text{since } s = M/K.
$$

By Theorem 3, s is rational. Then $K = M/s$ is rational. Then $2(M^2 + N) = 4K^2$ is a rational square, and hence it is an integer square. Hence $2K$ is an integer. If K is not an integer, then $2K^2$ is also not an integer. But $2K^2 = M^2 + N^2$ is in fact an integer. Hence K is an integer.

Since $M^2 + N = 2K^2$ is solvable for K with $K > 0$, 2 is a square mod p for each p dividing N but not M. But 2 is square mod p just in case p is congruent to ± 1 mod 8. Hence, primes dividing N but not M are congruent to ± 1 mod 8. If p is a prime that divides both N and M, but divides N to a power p^j that does not divide M^2, then similarly 2 is a square mod p. That completes the proof of the theorem.

Remark. Thus any odd primes p dividing N that are not congruent to ± 1 mod 8, must occur to even powers and divide M^2 to at least the power they divide N.

Question: If I divide M and K both by 2 (assuming they are both even), then $s = M/K$ does not change, and since $s = 2\sin(\alpha/2)$ determines the shape of the tile, the new tile is similar to the old one, just half the linear dimensions, so one-quarter the area. But N is divided by 4 according to the tiling equation, whereas, it should take four times as many tiles to tile ABC with these smaller tiles. What is going on here? *Answer*: When you divide M and K by 2, the size of the tiled triangle ABC also changes. Since the length Y of side AC is λb, and $\lambda = K$, if we divide K and b both by 2 then side Y is divided by 4, so the area of ABC is 16 times smaller. The tiles are four times smaller, so indeed it takes only one-fourth as many of them to tile the new ABC, not four times as many.

Lemma 17. *Assume triangle ABC has angles $(2\alpha, \beta, \alpha + \beta)$, and there is an N-tiling of ABC by the tile with angles (α, β, γ), and $N + M^2 = 2K^2$ where $K = M/s$. Then for some $\lambda > 0$ we have $(a, b, c) = \lambda(M, N/K - K, K)$. Then*

$$(X, Y, Z) = \lambda(MN/K, N - K^2, K^2)$$

and the $\mathbf{d}$-matrix equation takes the form

$$
\begin{aligned}
MN/K &= pM + d(N/K - K) + eK \\
N - K^2 &= gM + m(N/K - K) + fK \\
K^2 &= hM + \ell(N/K - K) + rK
\end{aligned}
$$

Proof. Since $K = M/s$, we have $s = M/K$. Since $a/c = s$ and $b/c = 1 - s^2$, a, b, and c are proportional to $(M, N/K - K, K)$. Thus, as claimed, we can choose

$$(6) \qquad (a, b, c) = \lambda(M, N/K - K, K).$$

According to (3), we have $\mu = M/s$. Since $s = M/K$ that implies

$$(7) \qquad \mu = K$$

Since $s = M/K$ we have

$$(8) \qquad 2 - s^2 = \frac{2K^2 - M^2}{K^2} = \frac{N}{K^2}$$

By Lemma 15 we have

$$
\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \mathbf{d} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mu \begin{pmatrix} (2-s^2)a \\ b \\ c \end{pmatrix}
$$

$$
= K \begin{pmatrix} (2-s^2)a \\ b \\ c \end{pmatrix} \qquad \text{since } \mu = K
$$

$$
= K \begin{pmatrix} (N/K^2)a \\ b \\ c \end{pmatrix} \qquad \text{by (8)}
$$

$$
= \lambda K \begin{pmatrix} (N/K^2)M \\ N/K - K \\ K \end{pmatrix} \qquad \text{by (6)}
$$

$$
= \lambda \begin{pmatrix} MN/K \\ N - K^2 \\ K^2 \end{pmatrix}.
$$

This gives the left-hand side of the equation in the theorem. The right-hand side is obtained by substituting the values of (a, b, c) from (6) into the $\mathbf{d}$-matrix. That completes the proof of the theorem.

Lemma 18. *Suppose given an N-tiling of a triangle ABC with angles $(2\alpha, \beta, \alpha+\beta)$, and integers M and K satisfying the tiling equation with $s = M/K$. Then $m < K$, where m is the number of b edges on the side AC of ABC opposite the β angle.*

Proof. From the $\mathbf{d}$-matrix equation we have

$$Y = ga + mb + fc,$$

and $Y = \mu b$ by Lemma 15. Therefore

$$\mu b = ga + mb + fc.$$

The first and third terms on the right are nonnegative; and third one is not just nonnegative, it is positive, since $f > 0$ by Lemma 6. Therefore $m < \mu$. Since $\mu = K$ by (7), we have $m < K$. That completes the proof of the lemma.

Lemma 19. *Assume triangle ABC has angles $(2\alpha, \beta, \alpha + \beta)$, and there is an N-tiling, and $N + M^2 = 2K^2$. Then the length of BC is given by $X = M(b + c)$.*

Proof. We give two proofs. The length of side BC (opposite vertex A) is $X = \mu a(2 - s^2)$ according to the $\mathbf{d}$ matrix equation, and according to Lemma 17), $X = \lambda MN/K$. We have $a = \lambda M$, $c = \lambda K$, $b = c(1 - s^2) = \lambda K(1 - (M/K)^2) = \lambda(K^2 - M^2)/K = \lambda(N - K^2)/K = \lambda N/K - K$. Hence $b + c = \lambda(N/K - K) + \lambda K = \lambda N/K$. Hence

$$BC = X = \lambda MN/K = M(\lambda N/K) = M(b + c).$$

That completes the first proof.

The same result can also be obtained directly from the relation $X - Y + Z = M(a + b + c)$, since $Y = \mu b = Kb$ and $Z = \mu c = Kc$ (by Lemma 15). So $X = BC = M(a + b + c) - Kc + Kb$. Now put $a = \lambda M$, $c = \lambda K$, and $b = \lambda(K - M^2/K)$, we

have

$$
\begin{aligned}
X &= M(b+c) + Ma - Kc + Kb \\
&= M(b+c) + \lambda M^2 - \lambda K^2 + \lambda(K^2 - M^2) \\
&= M(b+c)
\end{aligned}
$$

That completes the second proof of the lemma.

7.2. **For each N, there are only a few possibilities for the tile and for ABC.**
In this section, we assume that triangle ABC has angles $(2\alpha, \beta, \alpha + \beta)$. Then each tiling corresponds to a solution of the tiling equation $M^2 + N = 2K^2$ with $M < N$, since M is the number of positive tiles minus the number of negative tiles, so it is less than the total number of tiles. Hence, for a given N, there are finitely many solutions. In practice, there are comparatively few such M and K, often just one for a given N. Since $s = 2\sin(\alpha)/2 = M/K$, each solution of the tiling equation determines one possible value of α (which may or may not actually correspond to a tiling). Then the numbers μ is also determined as $\mu = M/s$, by (3).

We note that the equation does sometimes have more than one solution. The least such N is $N = 119$, where we have $(M, K) = (3, 8)$ or $(9, 10)$. These yield tiles $(357, 440, 512)$ and $(1071, 190, 1000)$. In that case K does not divide N, and as we will see below, no tiling exists in that case. The least N such that the equation has more than one solution in which K divides N is $N = 87808$. In that case there are two solutions with K dividing N, namely $M = 112, K = 224$ and $M = 208, K = 256$. These correspond to the tiles $(2, 3, 4)$ and $(208, 87, 256)$ respectively. The corresponding triangles ABC are quite large; it is not practical to make a picture that fits on one page. There are many more N with two solutions, but none with three solutions less than ten million.

7.3. **No N-tilings for $N < 28$.** Logically, this section is unnecessary, since the result is a special case of Theorem 6 below. Nevertheless it is of some interest that it can be proved without the concepts that we introduce to prove that theorem. In fact we give two direct proofs for $N = 7$ here.

Lemma 20. *Let $3\alpha + 2\beta = \pi$ and suppose triangle ABC has angles 2α and β, and α is not a rational multiple of π. Then there is no N-tiling of ABC for $N < 28$.*

Proof. For $N < 28$, the equation $M^2 + N = 2K^2$ has solutions with $M < K$ (corresponding to $s < 1$) only for $N = 7, 14, 17$, and 23. Since N must be a square times a product of primes congruent to $\pm 1 \bmod 8$, we can eliminate 14 and 17; only 7 and 23 need to be considered. Note that we have now narrowed the search so much that there is only one possible tile (for each N) and only one possible triangle ABC to consider.

First assume $N = 7$. Then the only solution of $M^2 + N = 2K^2$ is $M = 1$ and $K = 2$, so the equations above show that a triangle similar to the tile will have sides $MK = 2$, $N - K^2 = 3$, and $K^2 = 4$, and with this tile ($(2, 3, 4)$ for short), the triangle ABC will have sides $MN, K(N - K^2), K^3$, or $(7, 6, 8)$.

There are very few possibilities for the $\mathbf{d}$ matrix. The first row of the $\mathbf{d}$ matrix must express 7 as an integral combination (with nonnegative coefficients) of 2, 3, and 4, which can only be done as $7 = 3 + 4$, so the first row can only be $(0, 1, 1)$. The second row must express AC, which is 6, as an integral combination of 2, 3, and 4, so it could be $(3, 0, 0)$ or $(0, 2, 0)$ or $(1, 0, 1)$. But since at vertex B there are

two α angles, the triangles at vertex B have their a sides in the interior, so there must be at least one b or c side along BC. Hence $(3, 0, 0)$ can be ruled out. Suppose the second row is $(1, 0, 1)$. Then the tile T_1 at vertex B has its c side along BC. Since $c = 4$ and $BC = 6$, there is room for only one more tile T_2 along BC, and it must have its a side on BC, so it cannot have its α angle at C; it must have its β angle there, since vertex C is composed of an α and a β angle. Then T_2 must have its c side on BC, and its γ angle at the vertex P on AC that it shares with T_1. The third tile T_3 sharing vertex P therefore has its α angle at P, since T_1 and T_2 contribute β and γ respectively. Then T_3 has either its b or its c side shared (partly) with the a side of T_1; but then this side of T_3 extends beyond T_1. Now there is not enough room for the tile T_4 that must share vertex A with tile T_1. It must have its α angle at vertex A, so its a side cannot be shared with T_1. Hence the side it shares with T_1 must be its b or c side, which is impossible because that would meet the interior of T_3. This contradiction shows that the second row of the $\mathbf{d}$ matrix cannot be $(1, 0, 1)$. The only remaining possibility is $(0, 2, 0)$, i.e., side AC is composed of two b sides. Then tile T_1 has its α angle at vertex A, its b side along AC, and its γ angle at the center P of side AC. Tile T_2 has its b side equal to PC, and hence it does not have its β angle at C. Hence it has its α angle at C. Since its β angle is opposite PC, it must have its γ angle at P. But now there are two γ angles at P, which is impossible, since $\gamma > \pi/2$. That completes the proof that there is no 7-tiling.

Now assume $N = 23$. The only solution of $M^2 + 23 = 2K^2$ with $M^2 < 23$ is $M = 3$ and $K = 4$, so the shape of the tile is $(MK, N - K^2, K^2)$, which is 12-7-16, and triangle ABC has the shape $(MN, K(N - K^2), K^3)$, which is 69-28-64. Note that here we have $a > b$, which is allowed since in this section we are not assuming $\alpha < \beta$. The second row of the $\mathbf{d}$ matrix has to express 28 as an integral combimation of 12, 7, and 16. So the possibilities are $(1, 0, 1)$ and $(0, 4, 0)$. We can rule out $(0, 4, 0)$ since the tile T_1 at vertex A has to have its b or c side on AC. Hence the second row of the $\mathbf{d}$ matrix is $(1, 0, 1)$, and as in the case $N = 7$ there are just two tiles along AC. Tile T_1 thus has its c side on AC, and tile T_2 has its a side on AC; those two tiles share vertex P, where T_1 has a β angle (since its γ is opposite AP and its α is at A), and tile T_2 has its a side on PC, and hence does not have its α angle at C; so it must have its β angle at C and its α angle opposite PC, so it must have its γ angle at P. Now T_1 has a β angle at P and T_2 has a γ angle there, so the third tile T_3 sharing vertex P has an α angle there, and hence does not have its a side shared with T_1. Consider the other tile T_4 at vertex A. Like T_1, it has its α angle at A, so it has its b or c side shared with T_1. If it shares the b side, then both T_1 and T_2 have their γ angles at a shared vertex Q; if T_4 has its c side along T_1, then it extends beyond T_1. In either case then, it is not possible for tile T_3 to extend past T_1 along their shared boundary. Since tile T_1 has its a side there, tile T_3 either has its a or b side along T_1. But we already proved it does not have its a side there. Hence it has its b side there. But that leaves a section of the boundary of T_1 that does not touch T_3, of length $a - b$. That is $12 - 7 = 5$. Since 5 is less than the shortest side b of the tile, there is no way to place a tile along this segment. That completes the proof that there is no 23-tiling, and that in turn completes the proof of the lemma.

7.4. **The case** $N = 28$. The equation $M^2 + 28 = 2K^2$, with $M^2 < N$, has only one solution, namely $M = 2$ and $K = 4$. That yields the tile $(2, 3, 4)$ and the

triangle ABC is (14,12,16). This is the same shape tile and triangle as in the case $N = 7$, except now the triangle is twice as big in linear dimensions. Its angles can be worked out from Lemma 2, since $s = 1/2 = a/c$ and $b/c = 1 - s^2$. One finds

$$\alpha = 28.955024°$$
$$\beta = 46.567463°$$

For some time I thought that there was no 28-tiling. I could not find one by hand searching, even with the aid of paper tiles. I then wrote a computer program to search for one, expecting to show that none exist. But on October 7, 2011, the program was producing "boundary tilings", placing triangles in many possibly ways around the boundary of ABC, and I saw that one of these can be filled in. See Fig. 6.

FIGURE 6. A 28-tiling
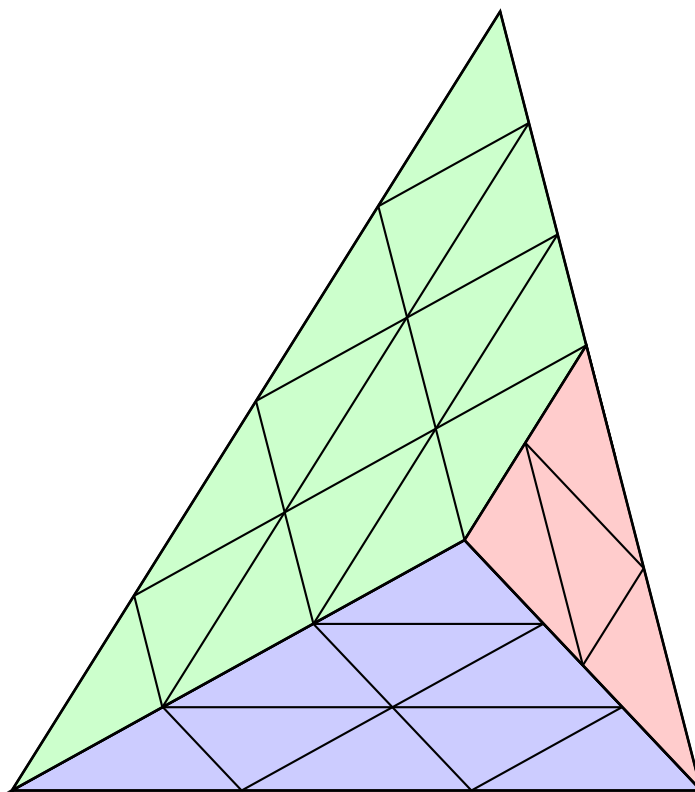
7.5. **Triquadratic tilings.** In this section we generalize the 28-tiling, showing that it is just the simplest member of a new family of tilings.

Theorem 5. *Let N be given. Suppose $M^2 + N = 2K^2$ with $M^2 < N$ and suppose K divides N (or equivalently, K divides M^2). Let ABC be the triangle with sides $BC = MN/K$, $AC = N - K^2$, and $AB = K^2$, and let the triangle T have sides $a = M$, $b = N/K - K$ (which must be a positive integer), and $c = K$. Then there*

is an N-tiling of triangle ABC by tile T. This tile satisfies $3\alpha + 2\beta = \pi$, and ABC has angles $(2\alpha, \beta, \alpha + \beta)$.

Proof. First note that since $M^2 < N$, we have $2K^2 = M^2 + N < 2N$, so $K^2 < N$. Hence $K < N/K$. Hence b, which is defined to be $N/K - K$, is positive. Since K divides N, b is a positive integer.

Define $J = c - b = K - (N/K - K) = 2K - N/K = (2K^2 - N)/K = M^2/K$. Since K divides N, J is an integer. We then have $c = K = M^2/J$ since $J = M^2/K$. Since $a = M$ and $b = c - J$ we have

$$
\begin{aligned}
a &= M \\
c &= \frac{a^2}{J} \\
b &= \frac{a^2}{J} - J
\end{aligned}
$$

Let α, β, and γ be the angles of the tile opposite sides a, b, and c. Then we have $b = c - a^2/c$ since $b = c - J$ and $J = M^2/K = a^2/c$. By Lemma 7 we have $3\alpha + 2\beta = \pi$.

We can construct a tiling as follows. Fix the "center point" Q. Construct three quadratic tilings whose vertex angles meet at Q, one with a^2 tiles, and two with b^2 tiles. Let the two b^2 tilings share a common side AQ. Let the a^2 tiling share a common side CQ with one of the b^2 tilings. This is possible because along CQ there are, on the side with the a^2 tiling, a tiles, each with its b edge on that side, so CQ has length ab; on the side of CQ with the b^2 tiling, there are b tiles, but each tile has its a edge along QC, so the length on that side is also ab. Hence the corners of the a^2 tiling and the b^2 tiling occur at the same point C. There are two α angles at A and at C there are an α and a β angle. Let D be the other vertex of the b^2 tiling that does not share side QC. Let E be the other vertex of the a^2 tiling, where the corner tile has a β angle. Now construct point B as the intersection of the line containing AD and the line containing CE. Since angle DAC is 2α and angle ACE is $\alpha + \beta$, the sum of these two angles is $3\alpha + \beta < \pi$. Hence, by Euclid's fifth postulate, point B does exist and lies on the same side of AC as the tilings.

Now consider the quadrilateral $BDQE$. The interior angle at B is β, since the angles of triangle ABC must sum to π, and the angles at A and C sum to $3\alpha + \beta = \pi - \beta$. The exterior angle QEC is also β, so AB is parallel to QE. The exterior angle ADQ is also β, so BC is parallel to DQ. Hence quadrilateral $BDQE$ is a parallelogram. Its side QD is composed of b edges of tiles, each of length a. We can therefore divide quadrilateral $BDQE$ into b parallelograms by drawing $b - 1$ equally spaced lines parallel to EB. Each of these parallelograms has a side equal to QE. QE is composed of a edges of tiles, with each edge of length a, so the length of QE is a^2. But $a^2 = Jc$. It is therefore possible to break each of the b small parallelograms into J yet smaller parallelograms, with sides c and a, and one angle γ. Each of these smaller parallelograms can be cut into two copies of the tile T. We can thus tile quadilateral $BDQE$ by $2bJ$ tiles. (Note that these tiles extend the b^2 quadratic tiling of QAD, so they would be part of a c^2 quadratic tiling, but the vertex of that larger quadratic tiling would overlap the a^2 tiling.)

It remains to count the tiles and verify that there are N of them. The total number of tiles, which we temporarily call Z until we prove it is equal to N, is

$$
Z = 2b^2 + a^2 + 2bJ
$$

We substitute $a = M$ and $b = N/K - K$, obtaining

$$Z = 2(N/K - K)^2 + M^2 + 2(N/K - K)J$$

Now substitute $N = 2K^2 - M^2$; then $N/K - K = 2K - M^2/K - K = K - J$, since $J = M^2/K$. We obtain

$$
\begin{aligned}
Z &= 2(K - J)^2 + M^2 + 2(K - J)J \\
  &= 2K^2 - 4KJ + 2J^2 + M^2 + 2KJ - 2J^2 \\
  &= 2K^2 - 2KJ + M^2 \\
  &= 2K^2 - 2K(M^2/K) + M^2 \qquad \text{since } J = M^2/K \\
  &= 2K^2 - 2M^2 + M^2 \\
  &= 2K^2 - M^2 \\
  &= N
\end{aligned}
$$

as desired. The number of tiles is N. That completes the proof of the theorem.

Figures 7, 9, and 8 illustrate triquadratic tilings. Note that in Fig. 9, we have $\alpha > \beta$ since $M > K/2$. (One can show using $s = 2\sin(\alpha/2) = M/K$, that $M > K/2$ is equivalent to $\alpha > \beta$.)

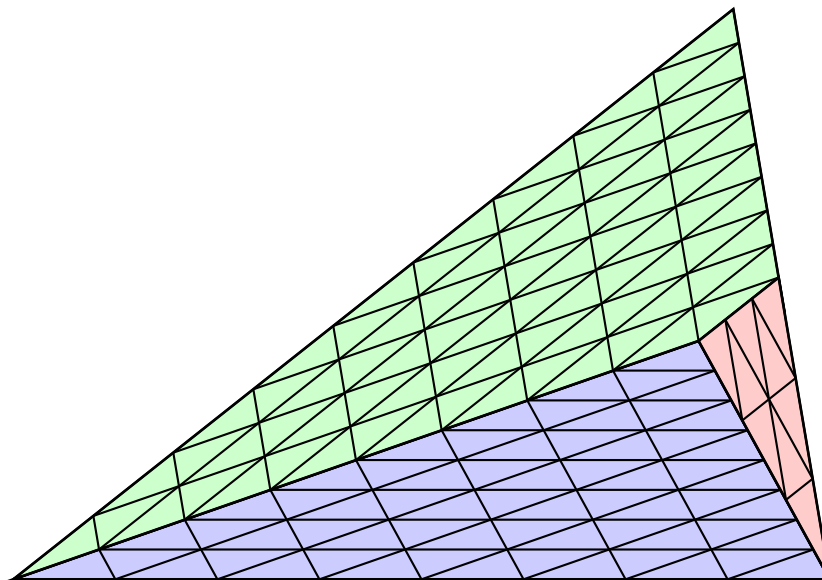FIGURE 7. A triquadratic tiling with $N = 153 = 9 \cdot 17, M = 3, K = 9$

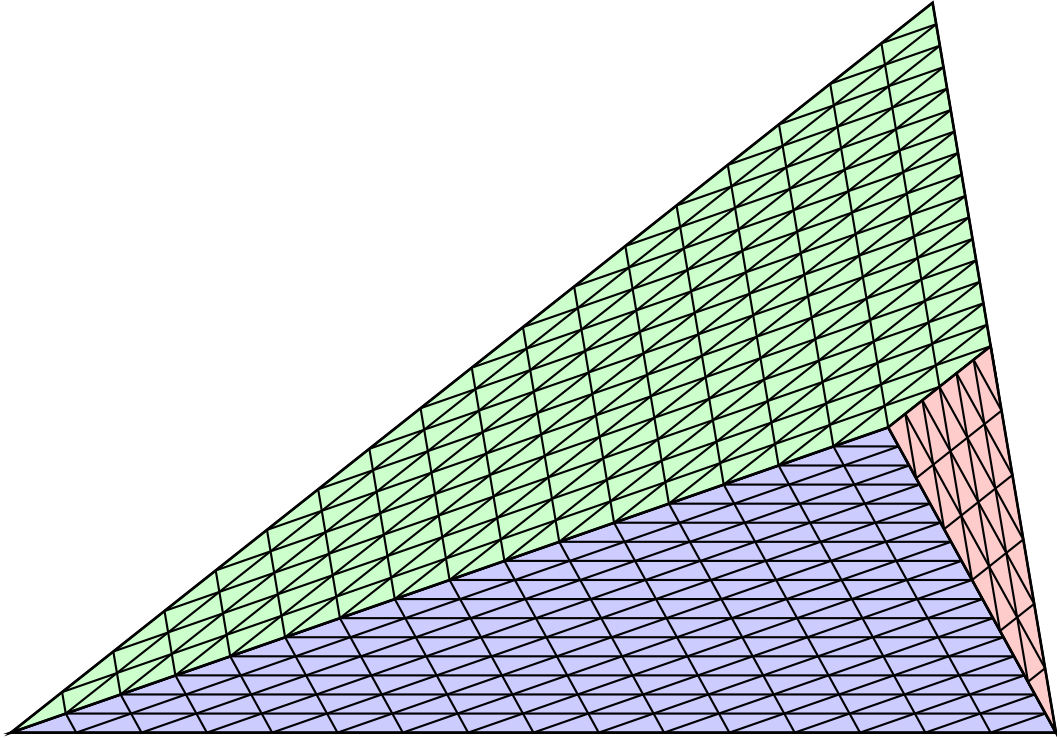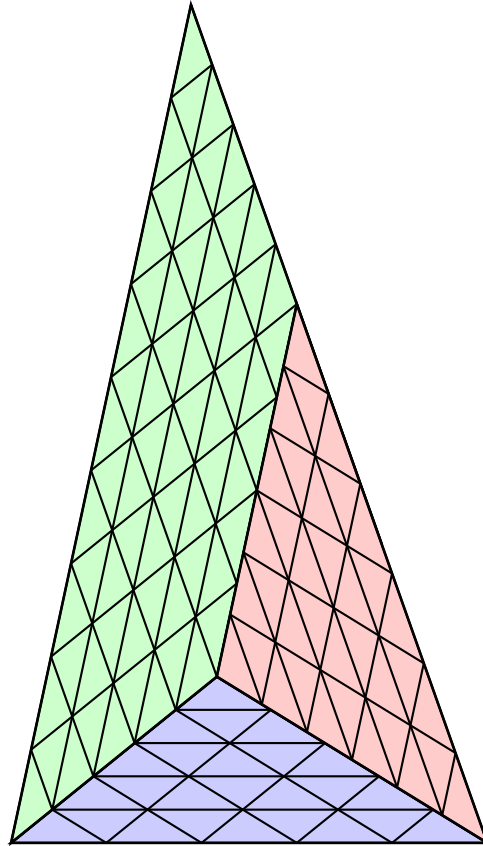FIGURE 8. A triquadratic tiling with $N = 612 = 17 \cdot 6^2, M = 6, K = 18$

FIGURE 9. A triquadratic tiling with $N = 126 = 9 \cdot 14, M = 6, K = 9$.

8. Non-existence of tilings when K does not divide N

In this section, we assume that ABC has angles $(2\alpha, \beta, \alpha + \beta)$, and the tiling equation $N = 2K^2 - M^2$ has a solution, but K does not divide M^2. We wish to show that under those hypotheses, there is no N-tiling of a triangle ABC with angles $(2\alpha, \beta, \alpha + \beta)$, where $a/c = K/M$. This existence theorem implies that the solvability of the tiling equation, together with the divisibility condition that K does not divide M^2, is both necessary and sufficient for the existence of a triquadratic N-tiling.

Recall that $s = 2\sin(\alpha/2) = M/K$. As usual we choose (a, b, c) to be integers with no common factor. That uniquely determines the number λ such that

$$(a, b, c) = \lambda(M, N/K - K, K).$$

8.1. Lower bounds on the lengths of maximal segments.

In the triquadratic tilings, there are maximal interior segments emanating from the center, with b edges on one side and a edges on the other. In this section, we prove that if K does not divide M^2, such segments must be "too long". That is, too long to fit in triangle ABC in a suitable direction.

Lemma 21. *Suppose ABC has angles $(2\alpha, \alpha + \beta, 2\beta)$ and there is an N-tiling of ABC, for which $N = 2K^2 - M^2$ and K does not divide M^2. Suppose $jb = ua + vc$, where u, v, and j are integers. Then K divides j.*

Proof. Let $d = \gcd(M, K)$. Let $e = \gcd(K, M^2)$. As usual we assume that the tile sides (a, b, c) are integers with no common factor. Define

$$\lambda := \frac{K}{ed}.$$

I claim that

(9) $$(a, b, c) = \lambda(M, K - M^2/K, K).$$

In other words, the three numbers $(\lambda M, \lambda(K - M^2/K), \lambda K)$ are integers with no common factor.

We have

$$(K/e)(K - M^2/K) = \frac{K^2 - M^2}{e}.$$

Since $e = \gcd(K, M^2)$, the right side is an integer; hence the left side $(K/e)(K - M^2/K)$ is an integer. Hence $(KM/e, (K/e)(K - M^2/K), K^2/e)$ are all integers.

I say their greatest common divisor is d. Suppose the prime $q \neq 1$ divides all three. Then q divides KM/e and K^2/e. Now $(K/e)(K - M^2/K) = (K^2/e) - M^2/e$. Then q divides M^2/e. By definition of e, K/e and M^2/e are relatively prime. Hence q does not divide K/e. But it does divide KM/e. Therefore q divides M. Since q divides K^2/e but not K/e, q divides K. So q divides both M and K. Therefore q divides $d = \gcd(M, K)$. Conversely, d divides $KM/e = (K/e)M$ since it divides M, and d divides $K^2/e = (K/e)K$, and d divides $(K/e)b = (K/e)K - (M/e)M$. Hence, the greatest common divisor of $(KM/e, (K/e)(K - M^2/K), K^2/e)$ is d, as claimed.

Hence

$$\left(\frac{K}{ed}M, \frac{K}{ed}(K - M^2/K), \frac{K}{ed}K \right)$$

are all integers, and have no common factor. That completes the proof of (9), with $\lambda = K/(ed)$.

Now let

$$g = gcd(a,c) = gcd\left(\frac{KM}{ed}, \frac{K^2}{ed}\right)$$

Then $c = g^2$ by Lemma 8. That is, $g^2 = K^2/(ed)$. Hence ed is a square and $K = g\sqrt{ed}$. We have $g = (K/e)\gcd(M/d, K/d) = K/e$. Then $g^2 = (K/e)^2 = K^2/e^2$ but also $g^2 = K^2/(ed)$, which implies $e = d$. That is, $\gcd(K, M^2) = \gcd(K, M)$. That equivalence is the key step in this proof.

Now suppose $jb = ua + vc$. Dividing by λ we have

$$j(K - M^2/K) \quad = \quad uM + vK$$

Since $d = \gcd(M, K)$, d divides the right side $uM + vK$. Let ℓ be the quotient; then $d\ell = uM + vK$. Then $d\ell = j(K - M^2/K)$. Hence

$$
\begin{aligned}
\frac{jM^2}{K} &= jK - d\ell \\
jM^2 &= K(jK - d\ell) \\
&= jK^2 - dK\ell \\
j(K^2 - M^2) &= dK\ell \\
j(K^2/d - M^2/d) &= K\ell
\end{aligned}
$$

Since $K^2/d = K(K/d)$ and $M^2/d = M(M/d)$, these are integers and we may consider the equation mod K:

$$j\frac{M^2}{d} \quad \equiv \quad 0 \qquad \mod K$$

Since $d = \gcd(M^2, K)$, M^2/d is relatively prime to K. It follows that $j \equiv 0 \mod K$. That completes the proof of the lemma.

Lemma 22. *Suppose ABC has angles $(2\alpha, \alpha + \beta, 2\beta)$ and there is an N-tiling of ABC, for which $N = 2K^2 - M^2$ and K does not divide M^2. Then any essential segment with at least one b edge of a tile on it either has the same number of b edges on each side, or it has length at least Kb.*

Proof. Let E be the essential segment in question. It gives rise to an edge relation $ja + kb + \ell c = pa + qb + rc$, with all coefficients nonnegative. If $k = q$ then E has the same number of b edges on each side, so we are done. Otherwise, we may suppose $k > q$. Then $(k - q)b = (p - j)a + (r - \ell)c$. By Lemma 21, K divides $k - q$. Since $k > q$, we have $k - q \geq K$. Since $q \geq 0$ we have $k > K/d$. Since there are k edges of length b on one side of E, the length of E is at least Kb. That completes the proof.

Lemma 23. *Suppose ABC has angles $(2\alpha, \alpha + \beta, 2\beta)$ and there is an N-tiling of ABC, for which $N = 2K^2 - M^2$ and K does not divide M^2. Then in that tiling, any interior segment with only b edges on one side, and at least one a or c on the other side, has length at least Kb.*

Proof. Suppose L is an interior segment with only b edges on one side; say there are ℓ edges of length b on that side. On the other side there may be a, b, and c edges, so we have

$$\ell b = ua + vb + wc$$

for some nonnegative integers u, v, and w, and by hypothesis not both u and w are zero. Let $j = \ell - w$; then $jb = ua + vb$ and $j \neq 0$. By Lemma 21, we have $j \geq K$. The length of segment L is at least jb, which is at least Kb. That completes the proof of the lemma.

Lemma 24. *Suppose ABC has angles $(2\alpha, \alpha + \beta, 2\beta)$ and there is an N-tiling of ABC, for which $N = 2K^2 - M^2$ and K does not divide M^2. Assume $jc = ua + vb + wc$, where a, b, and c are the lengths of the sides of the tile, and j, u, v, and w are nonnegative integers with $0 < j < M$. Then $v = 0$.*

Proof. Without loss of generality we can assume $w = 0$ (by subtracting wc from both sides and replacing j by $j - w$). According to Lemma 21, which is applicable since $vb = jc - ua$, K divides v. Let λ be the proportionality factor such that $(a, b, c) = \lambda(M, K - M^2/K, K)$. Since $j < M$ and $c = \lambda K$ the left side of $jc = ua + vb$ is less than $\lambda M K$. Hence the right side is less than $\lambda M K$ too; that is, $ua + vb < \lambda M K$. Hence $vb < \lambda M K$. Since $a < b$ we have $va < vb < \lambda M K$; since $a = \lambda M$ we have $va < aK$, and dividing by a we have $v < K$. But since K divides v, we have $v = 0$. That completes the proof of the lemma.

Lemma 25. *Suppose ABC has angles $(2\alpha, \alpha + \beta, 2\beta)$ and there is an N-tiling of ABC, for which $N = 2K^2 - M^2$ and K does not divide M^2. Then an interior segment with only c edges on one side, and fewer than M of them, cannot have any b edges on the other side; and if $(K, M) = 1$ such an interior segment does not exist.*

Proof. Let j be the number of c edges on one side of the interior segment, and apply Lemma 24. That completes the proof.

Remark. If $K = 12, M = 2, u = 1, v = j = 6$, we have $jK = uM + vb$, and K divides vM^2 but not M^2.

8.2. **Number, location, and orientation of b edges on the boundary.** For convenience we recall the notation for the $\mathbf{d}$ matrix:

$$\mathbf{d} = \begin{pmatrix} p & d & e \\ g & m & f \\ h & \ell & r \end{pmatrix}$$

Lemma 26. *Assume triangle ABC has angles $(2\alpha, \beta, \alpha + \beta)$, and there is an N-tiling of ABC, and $N + M^2 = 2K^2$. Then K divides mN. If K does not divide M^2, then $m = \ell = 0$ and $d = M$ i.e. there are no b edges on AC and no b edges on AB, and $d = M$, i.e., there are exactly M edges of length b on BC.*

Remark. As always for this shape of ABC, A is the vertex with angle 2α, B is the vertex with angle β, C is the vertex with angle $\alpha + \beta$; and (X, Y, Z) are the lengths of the sides opposite vertices A, B, and C respectively.

Proof. Suppose K does not divide M^2. The second row of the $\mathbf{d}$ matrix equation is, according to Lemma 15,

$$Y = \mu b = ga + mb + fc.$$

Since $\mu = K$ we have $Kb = ga + mb + fc$. All the terms are nonnegative, so $m \leq K$, and subtracting mb we have $(K - m)b = ga + fc$. We have $m < K$ since if $m = K$ then $g = f = 0$, i.e., there are no c edges on the boundary, contradicting Lemma 6.

Then $K - m > 0$. By Lemma 21, K divides $K - m$; but since $m \geq 0$ and $m < K$, that implies $m = 0$. That is, there are no b edges on AC.

By Lemma 19, the length of BC is $X = M(b + c)$. We have $X = pa + db + ec$. Therefore $X - Mb = Mc = pa + (d - M)b + ec$, or $(d - M)b = (M - e)c - pa$. Therefore by Lemma 21, we have $(d - M)$ congruent to zero mod K. Since $s = M/K < 1$, we have $M < K$, so $d - M > -K$. Hence $d \geq M$. Assume, for proof by contradiction, that $d \neq M$. Then we have $d - M \geq K$. Hence $d \geq K + M$. Let us calculate $(K + M)b$ and show it exceeds X. We have

$$
\begin{aligned}
(K + M)b &= (K + M)\lambda(N/K - K) \\
&= \lambda(N - K^2) + MN/K - MK \\
&= \lambda(N - K^2) + X - \lambda MK \qquad \text{since } X = \lambda MN/K \text{ by Lemma 42} \\
&= Y + X - \lambda MK \qquad \text{since } Y = N - K^2 \text{ by Lemma 42} \\
&= Y + X - Mc \qquad \text{since } c = \lambda K \text{ by Lemma 42}
\end{aligned}
$$

It therefore suffices to show $Y > Mc$, for then the right side will exceed X, so there cannot be as many as $K + M$ b edges on X. Now $Y = \mu b = Kb$, so it suffices to show $Kb > Mc$. We have

$$
\begin{aligned}
Kb &= K\lambda(N/K - K) = \lambda(N - K^2) = \lambda(K^2 + M^2) \\
&> \lambda MK \qquad \text{since } K > M, \text{ which implies } K^2 > KM \\
&= Mc \qquad \text{since } c = \lambda K
\end{aligned}
$$

That completes the proof of the lemma.

Lemma 27. *Suppose ABC has angles $(2\alpha, \alpha + \beta, 2\beta)$ and there is an N-tiling of ABC, for which $N = 2K^2 - M^2$ and K does not divide M^2. Then we have*

$$
\begin{aligned}
(10) \qquad X = |BC| &= pa + Mb + ec \\
Y = |AB| &= ga + fc \\
Z = |AC| &= ha + rc
\end{aligned}
$$

Proof. By Lemma 26, there are no b edges on AB or AC, and exactly M on BC. That completes the proof.

Lemma 28. *Assume triangle ABC has angles $(2\alpha, \beta, \alpha + \beta)$, and there is an N-tiling of ABC, and K does not divide M^2. Then all the tiles on side AC have their β angle on AC with the β angle nearer to C than the other angle.*

Remark. Lemma 26 tells us that the tiles on AC have their a or c edges on AC, not their b edges. This lemma tells us exactly what orientation those tiles have.

Proof. By Lemma 26, we have $m = 0$, which means there are no b edges of tiles on AC. Hence the β angle of each tile with a side on AC occurs on AC. Since the two tiles at A both have their α angle at A, no β angle occurs at A. Hence the number of β angles of tiles on AC is equal to the number of vertices on AC that are not equal to A. By Lemma 9, exactly one β angle occurs at each vertex; so no two of the tiles on AC have their β angles at the same vertex. It follows, proceeding from A to C along AC, that each tile on AC has its β angle at the vertex nearer to C.

Next we have to prove that each tile sharing an edge with side BC has its α angle nearer to C than to B. The tile at B has its β angle at B. Then the number of tiles sharing an edge with side X is the same as the number of vertices on side

X, including C but not B. Since exactly one α angle occurs at each vertex on the interior of side X (by Lemma 9), proceeding from B towards C along X, we see by induction that each tile in turn has an angle at the corner nearest B that is not equal to α, and an α angle at the corner nearest C. (This is consistent with the result of Lemma 28, according to which the last tile has its α angle at C.) That completes the proof of the lemma.

8.3. **Two geometrical lemmas.** We present two lemmas that do not mention tilings, but are just geometrical observations about triangles of a particular shape.

Lemma 29. *Suppose ABC has angles $(2\alpha, \beta, \alpha+\beta)$. Let Q be a point on BC such that AQ bisects angle BAC, so angle CAQ is α. Then triangle AQC is isosceles: AQ has the same length as AC, and angle AQC is $\alpha + \beta$, equal to angle ACQ.*

Proof. The sum of the angles of triangle ACQ is π. Since angle $QAC = \alpha$ and angle $ACQ = ACB = \alpha + \beta$, angle $AQC = \pi - 2\alpha - \beta$. Writing $\pi = 3\alpha + 2\beta$ and simplifying, we have $AQC = \alpha + \beta$. Hence triangle AQC is isosceles, since its base angles ACQ and AQC are equal. Hence the opposite sides AC and AQ are equal. That completes the proof of the lemma.

Lemma 30. *Suppose ABC has angles $(2\alpha, \beta, \alpha+\beta)$. Let Q be the point on BC such that angle QAC is α, and let R be a point on AB such that angle BQR is α. Then*

(i) $RQ \leq AC$, and

(ii) triangle ARQ is isosceles, with the angles are R and Q both equal to $\alpha + \beta$, and

(iii) any segment in Direction C contained in triangle ARQ has length strictly less than Kb, except RQ, which has length Kb.

Proof. Ad (i). By Lemma 29, triangle AQC is isosceles, with $AQ = AC$. Therefore it suffices to show $RQ < AQ$. Consider triangle AQR. It has angle α at A, by construction, and by Lemma 29, angle $AQC = \alpha + \beta$. Since angle $RQB = \alpha$, and the sum of angles is π, we have

$$
\begin{aligned}
\pi &= AQC + AQR + RQB \\
&= (\alpha + \beta) + AQR + \alpha \\
AQR &= \pi - (2\alpha + \beta) \\
&= (3\alpha + 2\beta) - (2\alpha + \beta) \\
&= \alpha + \beta
\end{aligned}
$$

Now angle AQR has angle α at A and angle $\alpha + \beta$ at Q. Since $3\alpha + 2\beta = \pi$, the third angle (at R) is $\alpha + \beta$. According to Euclid, in a triangle the greater side is opposite the greater angle. In triangle AQR, the angle opposite RQ is α and the angle opposite AQ is $\alpha + \beta$. Therefore $RQ < AQ = AC$. That completes the proof of part (i).

Ad (ii). As shown above, angle AQR and angle ARQ are both equal to $\alpha + \beta$. Therefore triangle ARQ is isosceles.

Ad (iii). Triangle ABQ is composed of the two triangle ARQ and BRQ. In each of those two triangles, RQ is evidently longer than any other segment in Direction C. By part (ii), the length of RQ is the same as that of AQ, and by

Lemma 29 that has length the same as AB, which has length Kb. That completes the proof of the lemma.

8.4. **No essential segment in Direction A.**

Lemma 31. *Let ABC have angles $(2\alpha, \beta, \alpha + \beta)$, and let integers N, K, and M solve the tiling equation $N = 2K^2 - M^2$, and K does not divide M^2. Suppose there is an N-tiling of ABC by a tile with angles (α, β, γ) and sides (a, b, c), and $a/c = M/K$. Then there is no essential segment in Direction A.*

Proof. Let the triangle ABC, the tile (a, b, c), and (K, M) be as specified in the theorem. By Lemma 23, any essential segment has length at least Kb. (That is where we use the hypothesis that K does not divide M^2.) Assume, for proof by contradiction, that there is an essential segment in Direction A. Then that essential segment is AQ, since by Lemma 30, that is the only segment in Direction A that is contained in triangle ABC and of length at least Kb. Since the length of AQ is exactly Kb, and according to Lemma 23, K divides the j such that $jb = ua + vb$ is the relation of the essential segment, it must be so that on one side of AQ all the tiles supported by AQ have their b edges on AQ, and on the other, there are zero tiles with their b edges on AQ. Now consider the two tiles at A. Each has their α angle at A. By Lemma 26, there are no b edges of tiles on AB or BC. Hence both those tiles have their b edges on AQ. Thus, neither side of AQ has zero b edges. That contradiction completes the proof of the lemma.

Lemma 32. *With the same hypotheses as in Lemma 31, and the point Q defined as in Lemma 30, there is no essential segment in Direction C contained in the triangle ABQ.*

Proof. According to Lemma 30, any line in Direction C contained in triangle ABQ is shorter than Kb. According to Lemma 23, any essential segment has length at least Kb. That completes the proof of the lemma.

8.5. **No tilings unless K divides M^2.** In this section, "essential segment" is short for "essential segment with associated relation $jb = ua + vc$."

We begin by defining a fancier version of the set Ω defined in Definition 11. We call this set Δ to avoid confusion with Ω.

Definition 14. *An **a/c segment** is a segment pq that is part of the tiling and supports a tile on each side having a vertex at p (or at q) and on one side, all the tiles supported by pq have their a or c edges on pq.*

*pq is an **in-sync border** if pq is a segment in the tiling, and on (at least) one side of pq all the tiles supported by pq are of Type I or Type II, and one of the following conditions holds:*

(i) pq is in Direction A or C and every vertex on pq is a vertex of a tile on both sides of pq. (All of those tiles will have their b edges on pq, and the edges "line up" by having common vertices.)

(ii) pq is part of (contained in) an a/c segment parallel to AB or BC.

*The **legal direction** to cross an in-sync border is from north to south for a border in Direction A, Direction C. For a border in direction BC, the legal direction could be described as east to west, or as north to south. For a border in direction AB, it is east to west*

Definition 15. *We define Δ to be the set of all Type I or Type II tiles that can be connected to the top tile by a southerly path not passing through any vertex and crossing only in-sync borders in their respective legal directions.*

Remark. Thus Δ is a subset of the region Ω defined in Definition 11.

Lemma 33. *If segment pq on the boundary of Δ is in Direction A or Direction C, then*

   (i) Δ is on the north of pq (not the south), and
   (ii) the tiles in Δ supported by pq are black.

Proof. Ad (i). Suppose, to the contrary, that Δ is on the south of pq. Then all the tiles supported by pq on the south of pq have their b edges on pq and their other two edges parallel to AB and BC. Hence, they can be entered legally only across pq. According to the definition of Δ, such tiles can be in Δ only if a tile to their north shares their b edge and is also in Δ. That is not the case here since tiles north of pq are not in Δ, because pq is on the boundary of Δ. That completes the proof of part (i).

Ad (ii). Tiles in Δ are all of Type I or Type II. They are black if their β angles are to the north, and white if their β angles are to the south. Those supported by the line pq on the north side of pq have their β angles to the north, and hence are black. That completes the proof of the lemma.

Now we consider a segment pq of the boundary of Δ that lies parallel to BC or AB, with Δ on the east. Then the tiles supported by pq on the east (that is, the Δ side) have their a or c edges on pq. These edges will be black if pq is in direction AB and white if pq is in direction BC. Similarly, if pq is parallel to BC or AB, with Δ on the west, the tiles supported by pq on the west (that is, the Δ side) are white if pq is in direction AB and black if it is in direction BC.

Let F be the southernmost point on BC such that BF supports only tiles with their a or c edges on BF. Then the next tile south of F on BC has its b edge on BC. There must be such a tile (i.e., F cannot be C) since, by Lemma 26, there are exactly M tiles with their b edges on BC. Hence F lies on AQ, where Q is the point on BC that is Mb from C.

Remark. We need to refer to the two oriented directions associated with each of the four directions of edges of tiles of Type I or Type II. We use the directional words *south, west, southwest, northeast*, etc. For example we could refer to "Direction A southwest", which would be the same as "Direction A south" or "Direction A west." But care must be taken with direction AB north, which will be northeast or northwest according as $\alpha < \pi/4$ or $\alpha > \pi/4$. Therefore we will use neither word, but only "direction AB north." For this reason, we had to use "east" in describing the legal direction to cross a boundary in direction AB.

Definition 16. *We define the directed graph Γ'. The nodes are vertices of the boundary of Δ. pq is a link in Γ' if pq is a (directed) boundary segment of Δ, with Δ on the right, and pq is terminated at p, and p lies in triangle RAQ, where Q is the point mentioned in Lemma 30, and one of the following holds:*

   (i) pq is Direction C northwest and is an arrow with tail at q, or is unterminated at q.

(ii) pq is in Direction A southwest and is unterminated at q, or

(iii) pq is in Direction BC north or AB north.

Lemma 34. *Let Q be the point mentioned in Lemma 30. If PR is a link in Γ' then R lies in triangle ABQ.*

Proof. All the links in Γ' are in one of the four directions A or C west, AB or BC north. Each of those directions moves away from the line AQ. Hence, R is farther from AQ than P is. But P is in triangle ABQ by the definition of Γ'. Hence R is also. That completes the proof of the lemma.

Lemma 35. *Assume ABC is N-tiled. If pq is a link in Γ', with p in triangle AQR, then the next boundary segment qr is also a link in Γ'.*

Proof. By Lemma 34, since pq is a link in Γ', q is in triangle ABQ. Hence that condition for qr to be a link in Γ' is automatically satisfied and does not need to be checked case by case. We check the other conditions by cases.

Case 1: pq is in Direction C northwest and is an arrow with tail at q. If qr is in Direction A southwest, it has a b edge on the north at q and an a or c edge on the south at q. Since there are no essential segments in Direction A, qr is unterminated at r and hence is a link in Γ', by case (ii) of the definition.

qr cannot be in Direction A northeast, by Lemma 33. Since the tile south of Q has its γ angle at q, the only other two possible directions for qr are AB and BC north.

Suppose qr is in direction BC north or AB north. Then it is by definition a link in Γ'. That completes Case 1.

Case 2: pq is in Direction C northwest and is unterminated at q. Then all the tiles supported by pq on the north have their b edges on pq, and one of them has a vertex at p, since pq is terminated at p. Suppose that pq is unterminated on its north side; then q lies on the interior of an edge of a tile north of pq. Then the next tile west on the south side of pq, with a vertex at q, is in Δ, but that is impossible, since its northern edge on pq is not in sync with the vertices of the tiles north of pq. Therefore, pq is unterminated on its south side, not its north side. Then the next boundary segment qr must turn right. By Lemma 33, qr cannot go in Direction A northeast or Direction C southeast. The only possible directions are AB north or BC north. In those cases, qr is a link in Γ' by definition.

Case 3: pq is in Direction A southwest and is unterminated at q. As in the previous case, it must be unterminated on the south side, rather than the north side. Then the next boundary segment qr must turn right, so the possible directions are Direction C northwest, or directions AB and BC north. The latter two are links in Γ' by definition; so assume qr is in Direction C northwest. Then the angle PQR is γ, and the tiles supported by qr on the north side have their b edges on qr and their γ angles to the northwest. If qr is terminated at R, then qr is either an essential segment, or an arrow with tail at Q (since if the tiles south of qr were oriented the other way, they would be in Δ). Assume, for proof by contradiction, that qr is an essential segment. By Lemma 23, its length is at least Kb. Since all the points p, q, and r lie in triangle AQR, and pq is in Direction A west, q is not equal to Q. Then by Lemma 30, the length of qr is strictly less than Kb. That is a contradiction, since the length is at least Kb. That contradiction shows that

qr cannot be an essential segment. Therefore qr is an arrow with a tail at Q, and hence is a link in Γ'. That completes Case 3.

Case 4: pq is in direction AB north and is an essential segment. Let qr be the next segment of the boundary of Δ after pq. We will show that qr is a link in Γ'. To start, I say that qr cannot turn right from pq. Suppose, to the contrary, that qr does turn right from pq. Let Tile 1 be the tile supported by pq with a vertex at q. (There is such a tile, both because pq is an essential segment, and because qr turns right.) Tile 1 is in Δ since pq is on the boundary of Δ with Δ on the east. By the definition of Δ, the tile north of Tile 1 is also in Δ. Therefore qr is not in direction BC southeast. Suppose qr is in Direction C southeast. Then angle pqr is $\alpha + \beta$. Let Tile 2 be the tile north of Tile 1 at q. Then qr is the northern boundary of Tile 2, which has its b edge on qr. But since Tile 2 is in Δ, the tile across qr must also be in Δ, contradicting the fact that qr is on the boundary of Δ. Hence qr is not in Direction C southeast. Suppose qr is in direction A northeast. Then angle pqr is $\beta + \gamma$. Since this angle lies in Δ, it is filled with two tiles, Tile 1 and Tile 2, with vertices at G. Tile 1 has its a or c side on pq and its β angle at Q. Tile 2 has its γ angle at q and its b edge on qr. Tile 2 is in Δ, which is only possible if the tile across qr shares its b edge with Tile 2 and is in Δ. But that contradicts the fact that qr is on the boundary of Δ. The only possible directions in which qr could turn right are Directions A and C, and direction BC southeast, all of which have been ruled out. That completes the proof that qr does not turn right from pq.

There are two possible directions for qr now: Directions A or C west. In either case, the tiles north of qr supported by qr lie in Δ and hence have their b edges on qr. I say that qr is terminated at q. Suppose, to the contrary, that qr is not terminated at qr. Then RQ can be extended east of q to a termination point J. Let Tile 2 be the tile on the south side of qJ, supported by qJ and having a vertex at q. Then Tile 2 is in Δ and has its b edge on qJ. Therefore the tile north of Tile 2 shares its b edge with Tile 2, and hence has a vertex at q. Hence qr is terminated at q.

All the tiles supported by qr on the north side of qr are in Δ, so they have their b edges on qr. Since q lies in triangle ABQ, and by Lemma 34 r does also, the entire segment qr lies in that triangle. Since q is the endpoint of pr, q does not lie on the line AQ.

Assume that qr is in Direction A. Then it is not an essential segment, since by Lemma 30 it has length strictly less than Kb, but by Lemma 23, an essential segment must be longer than that. Therefore qr is either an arrow or unterminated at q. If it is unterminated at H, we are done. Therefore we may assume qr is an arrow. Since qr is in Direction A, angle pqr is equal to α. The tiles north of GH have their γ angles to the east, so the tail of the arrow is at q. But then angle pqr is at least γ, contradiction.

Therefore qr is in Direction C. Then the tiles north of qr have their γ angles to the west, so the tail of the arrow is at q. Then qr is a link in Γ'. That completes Case 4.

Case 5: pq is in direction BC north and is an essential segment. First I say that qr cannot turn right. If it does turn right, it must be to Direction A or C east. But then, the tiles supported by qr to the south must be in Δ, so they have their b edges on qr and the tiles north of them must also be in Δ, contradiction. Hence

qr does not turn right. Then qr turns left, either to Direction A or C west, or direction AB north or south.

Suppose qr turns left to Direction A or C. If qr is not terminated at q, then the tile east of q supported by qr and on the south side of qr cannot be in Δ, since it cannot share its b edge with the tile to the north. That is a contradiction; hence qr is terminated at q. If it is unterminated at r, we are done. If it is terminated at r, it must be an arrow, since there are no segments of length Kb or greater in Direction A or C that lie inside triangle ABQ. If it is in Direction C then the tail of the arrow is at r and we are done. If it is in Direction A then the tail of the arrow is at q. Hence angle rqp is at least γ. But in fact angle rqp is $\alpha + \beta$, which is less than γ since $\gamma > \pi/2$. Hence qr cannot be an arrow in Direction C. So we are done in case qr is in Direction A or C west.

We may therefore assume qr is in direction AB north or south. All the tiles on the west of pq supported by pq have their b edges on pq, since if any has an a or c edge on pq, they would be in Δ, since pq is an in-sync border because it is an essential segment. In particular the top tile on the west of pq, having a vertex at q, cannot have its β angle at q. Hence the next segment qr is not in the direction AB south. Also, if the next segment is in the direction AB north, it is terminated at q, and therefore is a link in Γ'. That completes Case 5.

Case 6: pq is in direction BC north and not an essential segment. In that case, I say that pq has all b edges on the non-Δ (west) side. For if any tile, say Tile 1, on the west of pq has an a or c edge on pq, it can be entered from the Δ side of pq, which is the legal direction to cross pq. And pq is an in-sync boundary, since pq is terminated at p. (Note here that the definition of $a/csegment$ only requires common vertices at *one* endpoint.) Therefore Tile 1 belongs to Δ, contradicting the fact that pq lies on the boundary of Δ. Therefore, as claimed, the tiles west of pq have their b edges on pq. If pq is terminated at q, then pq is an essential segment, contrary to assumption. Therefore pq is unterminated at q. Therefore qr is terminated at q. Assume, for proof by contradiction, that qr is in direction AB south. Then Δ is on the west of qr. Since pq is unterminated at q, there is a tile supported by qr west of qr with a vertex at q. That tile, say Tile 1, is in Δ, but that would only be possible if the tile to its east across rq were in Δ, which it is not. Hence qr cannot be in direction AB south.

By Lemma 33, qr cannot be in Direction A east or Direction C east, as that would put Δ on the north of qr. If qr is in direction AB north, we are done, since qr is terminated at q as already mentioned. There remain only two possibilities: qr must turn left from pq into Direction A or Direction C west.

Then since qr is contained in triangle ABQ, and q is in the interior of that triangle, qr has length less than Kb, by Lemma 30. Then by Lemma 23, qr is not an essential segment. Since qr is terminated at q, it is not terminated at r. Hence qr is a link in Γ'. That completes Case 6.

Case 7: pq is in direction AB north and not an essential segment. This case is treated like the previous case; since pq is terminated at P, pq is an in-sync boundary, so membership in Δ can propagate across pq to tiles of Type I or Type II. Therefore all the tiles west of pq supported by pq have their b edges on pq. Since pq is not an essential segment, pq does not terminate at q. Therefore the next boundary segment qr is terminated at q. Segment qr cannot go in Direction A east or Direction C east, because that would put Δ on the north, contradicting Lemma 33. So it must turn

left. If qr goes in direction BC north, we are done. If qr goes in Direction A west or Direction C west, then as in the previous case it is shorter than Kb by Lemma 30, and hence not an essential segment by Lemma 23. Hence it is unterminated at r. Hence it is a link in Γ'. That completes Case 7.

That completes the proof of the lemma.

Theorem 6. *Let ABC have angles $(2\alpha, \beta, \alpha + \beta)$, and let integers N, K, and M solve the tiling equation $N = 2K^2 - M^2$. Suppose there is an N-tiling of ABC by a tile with angles (α, β, γ) and sides (a, b, c), and $a/c = M/K$. Then K divides M^2.*

Proof. Suppose, for proof by contradiction, that there is such a tiling in which K does not divide M^2. Let Γ' be the graph defined in Definition 16. Let F be the lowest point on BC such that BF supports only tiles with their a or c edges on BF. Let Q be the point defined in Lemma 29. Then F lies on BQ, since according to Lemma 26, there are exactly M tiles supported by BC with their b edges on BC, and BQ has length Mb.

As in the proof of Theorem 2, we argue that by Lemma 35, the in-degree of every node P in the graph Γ' is less than or equal to its out-degree; and therefore, since it is a finite graph, the in-degree is equal to the out-degree.

We claim that there is a link F with F on BC. Let F be the northernmost point on BC such that BF supports only tiles with their a or c edges on BF. Then there is a boundary segment FE of Δ starting from F, in Direction A west or Direction C west. By Lemma 26, there are M tiles edges of length b on BC, so FC has length at least Mb. But Mb is the length of QC, where Q is the point mentioned in Lemma 30. Therefore FE has length less than Kb, unless $F = Q$ and $E = R$ or $E = Q$. By Lemma 31, AQ is not an essential segment. Suppose $EF = AQ$. Then every tile supported by EF on the north has its b edge on EF. Since AQ is not an essential segment, all the tiles supported by EF on the south have their b edges on EF. The ones on the north have their γ angles to the east; so the ones on the south, which are not in Δ, have their γ angles to the east also. But that is impossible for the tile south of EF at Q, since angle AQF is $\alpha + \beta < \gamma$. The same argument applies if $EF = RQ$, appealing to Lemma 32 for the proof that RQ is not an essential segment. Hence the boundary segment EF is not an essential segment. Hence it is unterminated at E. Hence FE is a link in Γ', as claimed.

But no link pq can have q on BC or AC. That is, the in-degree of F is zero, but its out-degree is non-zero. Therefore we have reached a contradiction. That completes the proof of the theorem.

8.6. Solution of the tiling problem when ABC has angles $(2\alpha, \beta, \alpha + \beta)$. Combining Theorem 6 with the existence of the triquadratic tilings, we have the complete solution of the tiling problem when ABC has angles $(2\alpha, \beta, \alpha + \beta)$.

Theorem 7. *Let ABC have angles $(2\alpha, \beta, \alpha + \beta)$, and let N be a positive integer. Then there exists an N-tiling of ABC if and only if the tiling equation $N = 2K^2 - M^2$ has a solution in positive integers (M, K) such that K does not divide M^2 (or equivalently, K does not divide N).*

Proof. First we note that the tiling equation implies K divides M^2 if and only if K divides N, since if $N = 2K^2 - M^2$ then N is congruent to M^2 mod K.

By Theorem 5, the stated condition implies the existence of an N-tiling. The non-existence when the tiling equation has no solution is Theorem 4. The case when

the tiling equation does have a solution, but K does not divide N, is Theorem 6. That completes the proof.

Corollary 1. *For $N \leq 500$, the only possible N-tilings of any triangle ABC with angles $(2\alpha, \beta, \alpha + \beta)$ are those listed in Table 1.*

TABLE 1. All $N \leq 500$ permitting a tiling of ABC of shape $(2\alpha, \beta, \alpha + \beta)$

N	M	(a, b, c)
28	2	(2, 3, 4)
112	4	(2, 3, 4)
126	6	(6, 5, 9)
153	3	(3, 8, 9)
252	6	(2, 3, 4)
368	12	(12, 7, 16)
448	8	(2, 3, 4)
496	4	(4, 15, 16)

Proof. We note that the tiling equation for these N is solvable with $M = 1$. But what we need in order to apply the theorem is that the tiling equation has no solutions (K, M) with K dividing N. Because of the possibility that the tiling equation might have more than one solution, it does not suffice that it have a solution with $M = 1$. We must check all solutions. But that is easily done by a simple computer program. For example, the SageMath code in Fig. 10 works, although for simplicity that code does not format the table. Therefore, by Theorem 7, there is no N-tiling for $N \leq 500$ not occurring in the table. That completes the proof.

Remark. SageMath code runs as interpreted Python, so it is comparatively slow. Similar code running in C++ runs in a few seconds up to ten million. That is how we found the values of N with two solutions mentioned above.

FIGURE 10. SageMath code to solve the tiling equation

```
def solveTilingEquationUpTo(N):
    for nn in range(1,N+1):
        for M in range(1,sqrt(nn)+1):
            K = sqrt((nn+M*M)/2)
            if K <= M:
                continue
            if K in NN and nn % K == 0:
                s = M/K
                (a,b,c) = getABC(s)  # See Fig. 11
                print (nn,M,a,b,c)
```

FIGURE 11. SageMath code to compute (a, b, c) from s

```
def getABC(s):
   a = int(s.numerator())
   c = int(s.denominator())
   if not a^2 % c == 0:
      a = a*c
      c = c*c
   b = c - a^2/c
   g = gcd(b,gcd(a,c))
   if not g==1:
      a = a/g
      b = b/g
      c = c/g
   return [a,b,c]
```

To summarize our results: we have give a complete solution of the tiling problem for triangles ABC of the shape $(2\alpha, \beta, \alpha + \beta)$, as follows. First, given ABC and N, the previous theorem gives a necessary and sufficient condition for it to be N-tiled. Second, if we start not with ABC and N, but just with N and a solution (K, M) of the tiling equation such that K divides M^2, then $s = M/K$ determines the angle α via $s = 2\sin(\alpha/2)$, and β is determined by $3\alpha + 2\beta = \pi$. Then $K - M^2/K$ is an integer, since K divides M^2, and (a, b, c) are chosen so that (a, b, c) is proportional to $(M, K - M^2/K, K)$, and (a, b, c) have no common factor. Then ABC is determined by the requirement that it must have area N times the area of the tile (a, b, c). Since for a given N, there are at most $\sqrt{N}$ possible values of M, there are a finite number of possible triangles ABC of this form that could be N-tiled, and for each M, just one possible tile. And by solving the tiling equation and testing whether K divides N or not, we have an algorithm for deciding if there is any N-tiling of any ABC of the form $(2\alpha, \beta, \alpha + \beta)$.

8.7. **Number theory of the tiling equation.** Here we show that the tiling equation always has a solution when N has the right divisibility properties. Whether that solution corresponds to a tiling depends on whether K divides M^2 or not.

Lemma 36. *If the tiling equation $N = 2K^2 - M^2$ is solvable, then N is a square times a product of distinct primes, each of which is either 2 or is of the form $8n \pm 1$. If N is not a square or twice a square, then there is a solution with $0 < M < K$.*

Proof. Without loss of generality we can assume N is square free. If N is odd, then M is odd, so mod 8 the right side $2K^2 - M^2 = \pm 1$. Hence every odd prime dividing N is congruent to ± 1 mod 8, as claimed. If N is even then M is also even, so $N/2 = K^2 - 2(M/2)^2$ is congruent to ± 1 mod 8, and again every odd prime dividing N is congruent to ± 1. That completes the proof.

Lemma 37. *Suppose N is a square times a product of distinct primes, each of which is either 2 or is of the form $8n \pm 1$. Then the tiling equation $N + M^2 = 2K^2$ has a solution in positive integers M and K with $M < K$.*

Proof.[1] Suppose N satisfies the stated divisibility conditions. We first prove that there exists an integer solution (M, K). After that we will prove there is one with $0 < M < K$. The tiling equation asks for an integer in the field $\mathbb{Z}[\sqrt{2}]$ whose norm is $-N$. In detail, the integers of $\mathbb{Z}[\sqrt{2}]$ have the form $M + K\sqrt{2}$ and the norm of such an integer is $M^2 - 2K^2$. The standard theory of factorization in $\mathbb{Z}[\sqrt{2}]$ tells us that when N has the form given in the hypothesis, there is an integer solution. Specifically, because the norm is multiplicative, we can assume without loss of generality that N is an odd prime congruent to ± 1 mod 8. (When $N = 2$ the equation is solvable with $M = 4$ and $K = 3$, and when $N = -1$ it is solvable with $M = 1$ and $K = 0$.) The form $x^2 - y^2$ has discriminant 8, so by Theorem 4.23 on p. 74 of [2], it integrally represents every prime congruent to ± 1 mod 8. Hence the tiling equation does have some integer solution (M, K).

It remains to prove that there is a solution with $M < K$. (Now we no longer assume N is prime.) One can verify by computation that if (M, K) is a solution of the tiling equation, then $(3M \pm 4K, 3K \pm 2M)$ is also a solution. One can verify it without computation by observing that this is the product of (M, K) with the unit $3 - 2\sqrt{2}$. Now choose a solution with M as small as possible but still positive. We may assume $K \geq 0$ since if (M, K) is a solution, so is $(M, -K)$. Since M is as small as possible, we have $3M - 4K \leq 0$. If $M < K$ we are done, so we may assume $K \leq M$. Then consider $M' = -(3M - 4K) = 4K - 3M$ and $K' = 3K + 2M$. Then (M', K') is also a solution, and $M' \geq 0$ and $K' > 0$. Then

$$\begin{aligned} K' - M' &= (3K + 2M) - (4K - 3M) \\ &= M - K \\ &\geq 0 \qquad \text{since } K \leq M \end{aligned}$$

Hence $M' \leq K'$. But if $M' = K'$ then N is a square. If $M' = 0$ then $4K = 3M$, so $N = 2K^2 - M^2 = 2(3/4)^2 M^2 - M^2 = M^2/8 = 2(M/2)^2$, That completes the proof.

Remark. For large N, it will be easier to solve the tiling equation than to factor N.

8.8. N is not prime when ABC has angles $(2\alpha, \beta, \alpha + \beta)$.

Theorem 8. *Suppose ABC with angles $(2\alpha, \beta, \alpha + \beta)$ is N-tiled by a tile with angles (α, β, γ). Then N is not prime.*

Proof. By Theorem 7, the tiling equation $N = 2K^2 - M^2$ has a solution with $K | M^2$, or equivalently, $K | N$. Suppose, for proof by contradiction, that N is prime. Then $K = 1$ or $K = N$. We have $K < M$ since $a/c = K/M$, and $M < N$ since M is the number of black tiles minus the number of white tiles. Hence $K < N$. Therefore $K = 1$. Then $N - 2$ is a square. By Lemma 36, N is congruent to 1 mod 8. Hence $N - 2$ is congruent to 7 mod 8. That is a contradiction, since squares cannot be congruent to 7 mod 8. That completes the proof.

9. THE CASE WHEN ABC HAS ANGLES $(2\alpha, \alpha, 2\beta)$

We begin by observing that for ABC with angles $(2\alpha, \alpha, 2\beta)$, there do exist some previously unknown tilings, discovered in July, 2018. Examples are given in Figs. 12,13, and 14. It should be mentioned again that Laczkovich [4] studied

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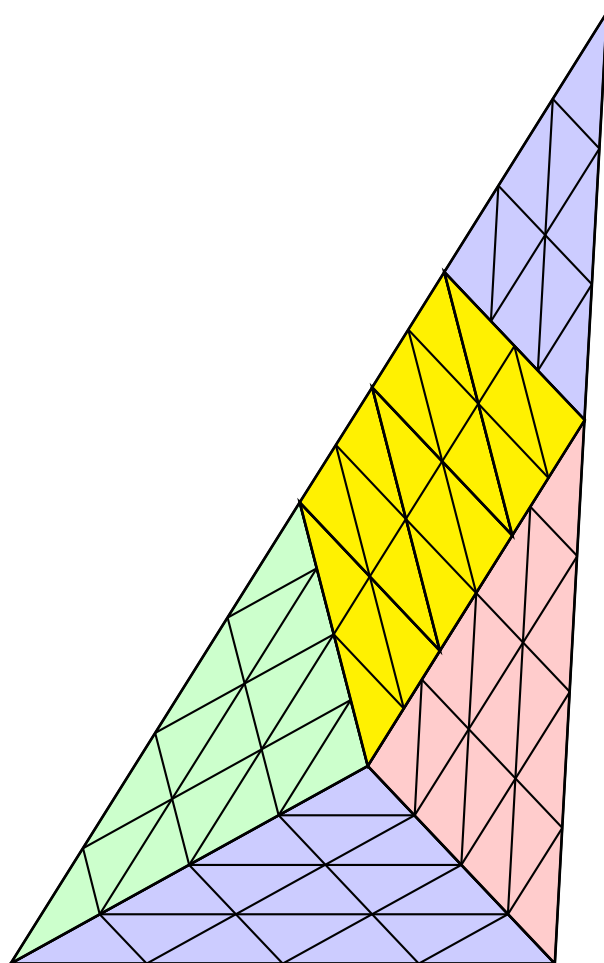[1]The idea of this proof was given to me on MathOverflow by Noam Elkies and Will Jagy.

FIGURE 12. $N = 77$. The tile (a, b, c) is $(2, 3, 4)$.

decompositions of these and other triangles into similar triangles, and knew that
if these triangles were rational, then finer decompositions into some number of
congruent triangles would be possible. Here we have proved that the tiles must
be rational, and focused attention on N as well as ABC; some of our tilings have
much smaller N than would arise by Laczkovich's method.

9.1. **The second tiling equation.** In this section we derive the "second tiling
equation", which provides a necessary condition for the existence of an N-tiling
of a triangle ABC with angles $(2\alpha, \alpha, 2\beta)$. Let the sides opposite those angles be
(X, Y, Z) respectively. If ABC is N-tiled, let the sides of the tile be a, b, and c,
which are proportional to $\sin\alpha$, $\sin\beta$, and $\sin\gamma$, respectively. By Theorem 3, the
tile is rational.
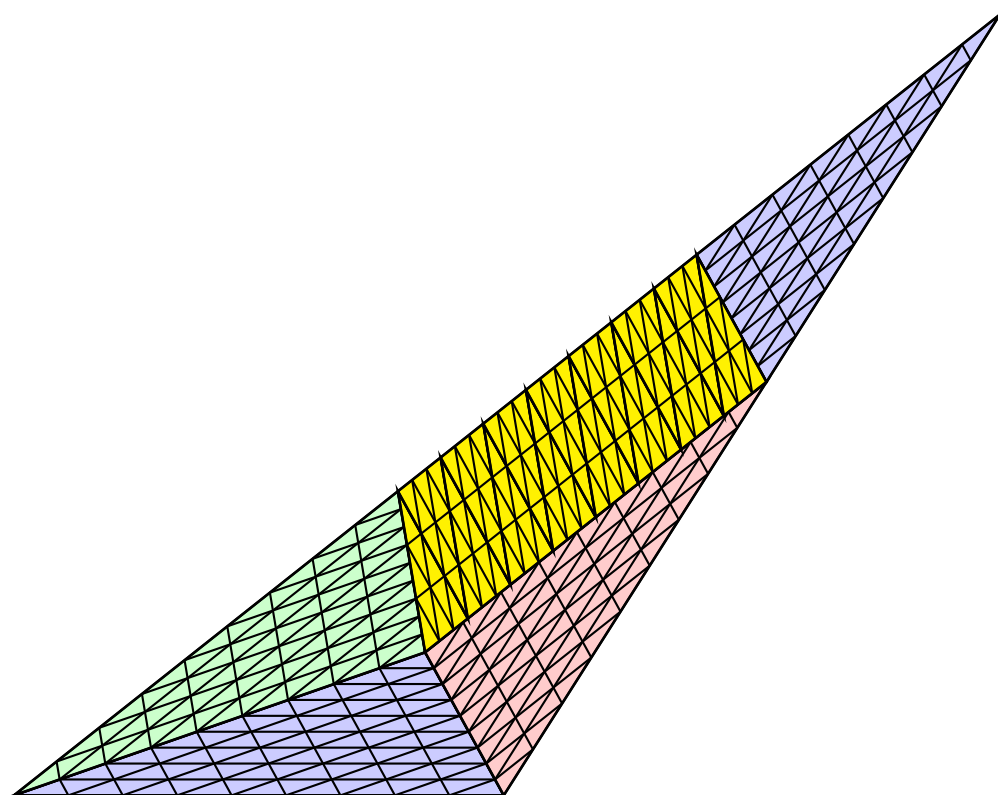
FIGURE 13. $N = 442$. The tile (a, b, c) is $(3, 8, 9)$.

Lemma 38. *Suppose $3\alpha + 2\beta = \pi$. Let $s = 2\sin\alpha/2$. Suppose the triangle with sides (a, b, c) and angles (α, β, γ) is scaled by κ, so that*

$$(a, b, c) = \kappa(\sin\alpha, \sin\beta, \sin\gamma).$$

Then

$$\frac{\kappa \sin 2\alpha}{c} = s(2 - s^2)$$
$$\sin 2\alpha = (2 - s^2)\sin\alpha$$
$$\frac{\kappa \sin 2\beta}{c} = \frac{\kappa \sin 3\alpha}{c} = s(1 + s)(1 - s)(3 - s^2)$$

Proof. In Lemma 2 we proved

$$\sin\gamma = \cos\frac{\alpha}{2}$$
$$\frac{a}{c} = s$$
$$\frac{b}{c} = 1 - s^2$$

Using these formulas, we argue as follows.

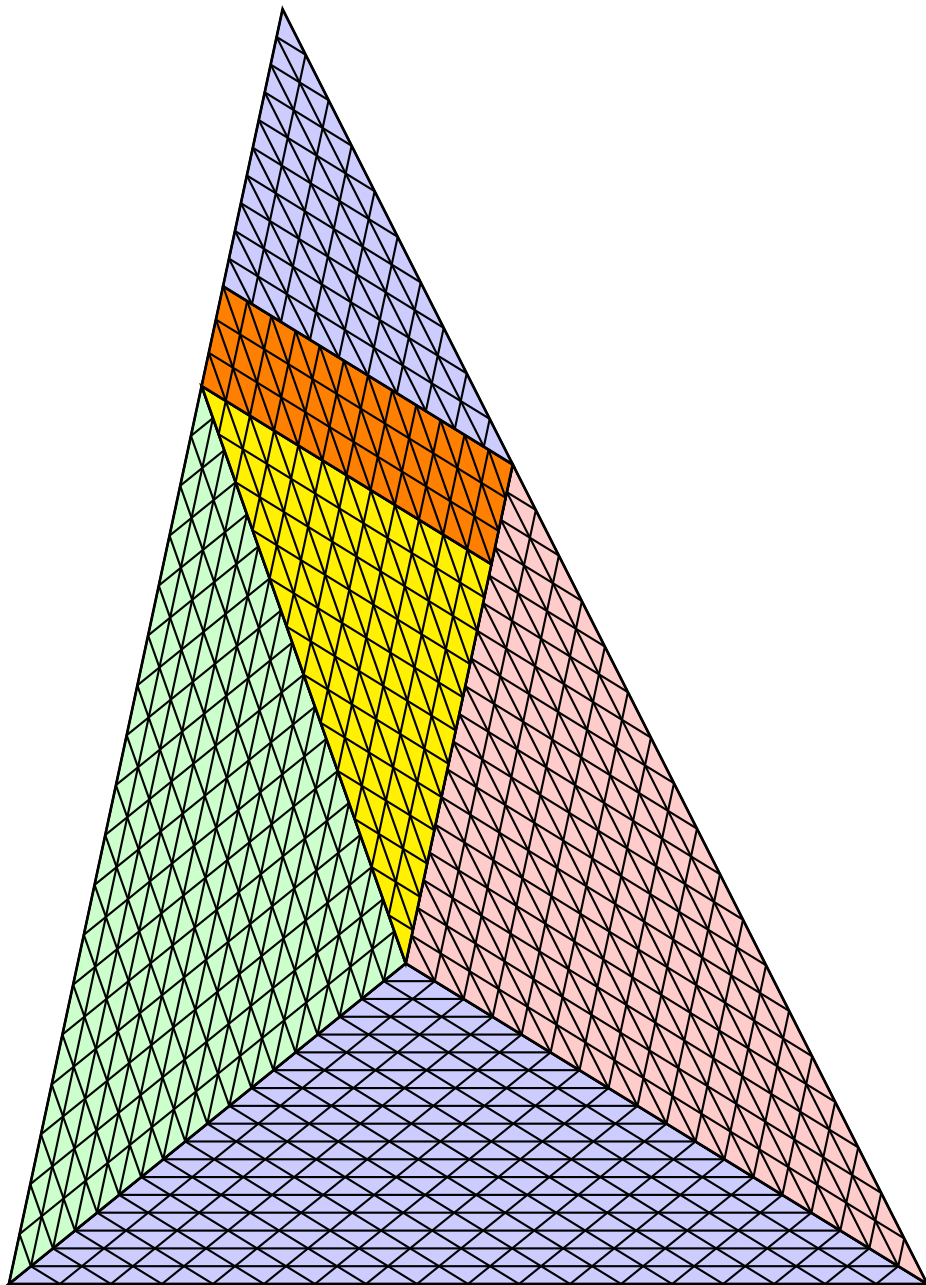FIGURE 14. $N = 3 \cdot 18^2 + 12^2 + 10^2 + 3 \cdot 2 \cdot 12 = 1288$. The tile is $(a, b, c) = (6, 5, 9)$. Note, here $a > b$. The tile is not $(5, 6, 9)$ but $(6, 5, 9)$.

$$\begin{aligned}
\frac{\sin 2\alpha}{\sin \gamma} &= \frac{2\sin \alpha \cos \alpha}{\sin \gamma} \\
&= 2\frac{a}{c}\cos \alpha \\
&= 2s\cos \alpha \\
&= 2s(1 - 2\sin^2 \frac{\alpha}{2}) \\
&= s(2 - s^2)
\end{aligned}$$

Multiplying numerator and denominator on the left by κ, we obtain the first formula of the lemma.

To prove the second formula:

$$\sin 2\alpha = s(2 - s^2)\sin \gamma = s(2 - s^2)c/\kappa = (2 - s^2)a/\kappa = (2 - s^2)\sin \alpha.$$

Next we work on the third formula.

$$\begin{aligned}
\frac{\kappa \sin 2\beta}{c} &= \frac{\kappa \sin 3\alpha}{c} \qquad \text{since } 2\alpha + 3\beta = \pi \\
&= \frac{\kappa \sin \alpha \cos 2\alpha + \kappa \sin 2\alpha \cos \alpha}{c} \\
&= \frac{a}{c}\cos 2\alpha + \frac{\kappa \sin 2\alpha}{c}\cos \alpha \\
&= s\cos 2\alpha + s(2 - s^2)\cos \alpha \\
&= s(2\cos^2 \alpha - 1) + s(2 - s^2)(1 - 2\sin^2 \frac{\alpha}{2}) \\
&= s(2(1 - 2\sin^2 \frac{\alpha}{2})^2 - 1) + s(2 - s^2)(1 - \frac{1}{2}s^2) \\
&= s(2(1 - \frac{1}{2}s^2)^2 - 1) + s(2 - s^2)(1 - \frac{1}{2}s^2) \\
&= s(1 + s)(1 - s)(3 - s^2)
\end{aligned}$$

That completes the proof of the lemma.

Theorem 9. *Let ABC have angles* $(2\alpha, \alpha, 2\beta)$ *and suppose ABC is N-tiled, and let the tiles be colored alternately black and white with a black tile at B, and let M be the number of black tiles minus the number of white tiles. Let* $s = a/c$. *Then s is rational and the "second tiling equation" is satisfied:*

$$\frac{N}{M^2} = \frac{(2 - s^2)(3 - s^2)}{(1 - s)^2(2 + s)^2}$$

Example 1. Consider the 77-tiling from Fig. 12. Then $N = 77$, $M = 5$, $s = \frac{1}{2}$, $(a, b, c) = (2, 3, 4)$. Both sides evaluate to $77/25$.

Example 2. Consider the 1288-tiling from Fig. 14. Then $M = 16$, as is verified using the observation that in a quadratic tiling with m^2 tiles, there is an excess of m black over white tiles, if the top tile is black. Therefore M for this example is given by $M = 10 - 12 + 0 + 18 + 18 - 18 = 16$. So now, let's check that the second

tiling equation is satisfied in the example:

$$
\begin{aligned}
\frac{N}{M^2} &= \frac{(2-s^2)(3-s^2)}{(1-s)^2(2+s)^2} \\
N &= 16^2\frac{(14/9)(23/9)}{(1/3)^2(8/3)^2} \\
N &= 16^2\frac{7\cdot 23}{32} \\
N &= 8\cdot 7\cdot 23 \\
N &= 1288
\end{aligned}
$$

So the second tiling equation correctly predicts the number of tiles from M and s.

Proof of the Theorem. By Theorem 3, s is rational. By the law of sines we have, for some $\lambda > 0$,

$$
\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \lambda \begin{pmatrix} \sin 2\alpha \\ \sin \alpha \\ \sin 2\beta \end{pmatrix}
$$

By Theorem 1 we have $M(a+b+c) = X - Y + Z$. Hence

$$
M(a+b+c) = \lambda(\sin 2\alpha - \sin \alpha + \sin 2\beta)
$$

Now we consider the size and scaling of the tile. For some $\kappa > 0$, we have

$$
(a,b,c) = \kappa(\sin \alpha, \sin \beta, \sin \gamma).
$$

Dividing and multiplying the right side of the previous equation by κ, we have

$$
M(a+b+c) = \frac{\lambda}{\kappa}(\kappa\sin 2\alpha - \kappa\sin \alpha + \kappa\sin 2\beta)
$$

Dividing by c and then using Lemmas 2 and 38, and $\kappa\sin \alpha = a$, to express everything in terms of $s = 2\sin(\alpha/2)$, we have

$$
\begin{aligned}
M(\frac{a}{c} + \frac{b}{c} + 1) &= \frac{\lambda}{\kappa}(s(2-s^2) - s + s(1+s)(1-s)(3-s^2)) \\
M(s + (1-s^2) + 1) &= \frac{\lambda}{\kappa}s(2+s)(1+s)(1-s)(2-s) \\
M(2-s)(1+s) &= \frac{\lambda}{\kappa}s(2+s)(1+s)(1-s)(2-s) \\
\text{(11)} \qquad M &= \frac{\lambda}{\kappa}s(1-s)(2+s)
\end{aligned}
$$

The area of one copy of the tile is $bc\sin \alpha$. Equating the area of N tiles to the area of ABC, we have

$$
Nbc\sin \alpha = XZ\sin \alpha
$$

since the angle opposite Y is α. Dividing by $c^2 \sin \alpha$ we have $Nb/c = XZ/c^2$. Expressing this in terms of s we have

$$
\begin{aligned}
N(1 - s^2) &= \lambda^2 \frac{\sin 2\alpha}{c} \frac{\sin 2\beta}{c} \\
&= \left(\frac{\lambda}{\kappa}\right)^2 \frac{\kappa \sin 2\alpha}{c} \frac{\kappa \sin 2\beta}{c} \\
&= \left(\frac{\lambda}{\kappa}\right)^2 s(2 - s^2)s(1 + s)(1 - s)(3 - s^2) \\
N &= \left(\frac{\lambda}{\kappa}\right)^2 s^2(2 - s^2)(3 - s^2)
\end{aligned}
$$
(12)

Solving for $(\lambda/\kappa)^2$ we have

$$
(13) \qquad \left(\frac{\lambda}{\kappa}\right)^2 = \frac{N}{s^2(2 - s^2)(3 - s^2)}
$$

Squaring (11) we have

$$
\begin{aligned}
M^2 &= \left(\frac{\lambda}{\kappa}\right)^2 s^2(1 - s)^2(2 + s)^2 \\
\left(\frac{\lambda}{\kappa}\right)^2 &= \frac{M^2}{s^2(1 - s)^2(2 + s)^2}
\end{aligned}
$$

Putting this expression into (12), we have

$$
\begin{aligned}
\frac{N}{M^2} &= \frac{s^2(2 - s^2)(3 - s^2)}{s^2(1 - s)^2(2 + s)^2} \\
&= \frac{(2 - s^2)(3 - s^2)}{(1 - s)^2(2 + s)^2}
\end{aligned}
$$

That is the "second tiling equation". That completes the proof of the theorem.

Lemma 39. *If the second tiling equation has a solution then $M^2 < N$.*

Proof. It suffices to show that the right side of the second tiling equation is greater than 1. That is, it suffices to show

$$
(1 - s)^2(2 + s)^2 < (2 - s^2)(3 - s^2).
$$

To prove that, consider the difference:

$$
(2 - s^2)(3 - s^2) - (1 - s)^2(2 + s)^2 = -2s^3 - 2s^2 + 4s + 2
$$

which is positive at 0 and 1; its derivative is $-6s^2 - 4s + 4$, which has one zero $s = 0.548\ldots$ in the interval $[0, 1]$ and is positive when $s = 0$ and negative when $s = 1$. Hence the displayed expression is positive on $[0, 1]$. That completes the proof of the lemma.

Lemma 40. *If triangle ABC is N-tiled, and we multiply each side of the tile (a, b, c) by a positive number κ, and rescale triangle ABC by the same factor κ, then we still have a tiling, with the same N, M, and $s = a/c$. If s is rational, after a suitable scaling we can assume the tile has integer sides.*

Proof. Let s be rational, say $s = a/c$ with a and c integers, not necessarily in lowest terms. Suppose the angles of the tile with sides (a, b, c) satisfy $3\alpha + 2\beta = \pi$. Then, by Lemma 7, $b = c - a^2/c$. Perhaps b is not an integer. For example with $s = 2/3$ we could have $(a, b, c) = (2, 5/3, 3)$. In that example, we could consider instead $(6, 5, 9)$. In general, if b is not an integer, we could scale (a, b, c) by multiplying by c. Then the new (a, b, c) is $(ac, c^2 - a^2, c^2)$, and all sides of this triangle are integers. That completes the proof of the lemma.

Remarks. This scaling does not change s; it only changes the marks on the ruler that we use to measure lengths. But therefore, it does change not only (a, b, c), but also (X, Y, Z), as the lengths of the sides of ABC remain the same when measured in terms of a, b, and c, but they change in absolute units. Thus, for example, if the sides a, b, and c are tripled, the sides of ABC also would triple, but N would remain the same. The expansion factor λ such that $Y = \lambda \sin \alpha$, would triple. But λ/κ would not change, and M, which is a function of s times λ/κ, would also not change. That is good, since M is the excess number of black tiles over white tiles, and the tiles do not change when we change the scale.

Example. With $s = 2/3$, $N = 1288$, $M = 16$, the second tiling equation is solved. Here $s = 2/3$; if we write $s = 6/9$ then c divides a^2, and the tile $(6, 5, 9)$ has integer sides, and there is an N-tiling of triangle ABC as shown in Fig. 14. But if we write s as $2/3$ then c does not divide a^2 and $b = c - a^2/c = 5/3$ is not an integer. Nevertheless there is an N-tiling of a triangle similar to ABC, but one-third the size, by the tile $(2, 5/3, 3)$, which is one-third the size of $(6, 5, 9)$.

Theorem 10. *Let the tile (a, b, c) have c divides a^2, where a and c are integers with $a < c$. Let $b = c - a^2/c$. (So $3\alpha + 2\beta = \pi$ and b is an integer.) Then some triangle ABC with angles $(2\alpha, \alpha, 2\beta)$ can be N-tiled by the tile (a, b, c). The number N of tiles in the tiling is given by the second tiling equation, with $s = a/c$, ℓ the least common multiple of a and c, and*

$$M = \ell(2 - s^2 - s),$$

which is an integer.

Proof. Let (a, b, c) be as in the first part of the theorem, so all three are integers. We imitate and generalize the example of the 1288-tiling shown in Fig. 14. Let $\ell = \mathrm{lcm}(a, c)$. Then $m = \ell/a$ is an integer, and $j = \ell/c$ is an integer, and we have $ma = jc = \ell$.

In the example we have

$$
\begin{aligned}
(a, b, c) &= (6, 5, 9) \\
s &= 6/9 = 2/3 \\
N &= 1288 \\
\ell &= 18 \\
m &= 18/6 = 3 \\
j &= 18/9 = 2 \\
b &= c - a^2/c = 5 \\
3a &= 2c \\
18a &= 12c \qquad \text{at the yellow-green boundary} \\
18b &= 15a \qquad \text{at the yellow-pink and red-pink boundary} \\
10a &= 12b \qquad \text{at the blue-red boundary}
\end{aligned}
$$

We have generalized $3a = 2c$ to $ma = jc$. The equation $18a = 12c$ becomes $\ell a = ma^2 = (ja)c$. We now want to generalize $18b = 15a$. That becomes $\ell b = ka$. Here $k = \ell b/a$ is defined to make that equation true, and the point to be proved is that k is an integer. To prove that,

$$
\begin{aligned}
k &= \frac{\ell b}{a} \\
&= mb \qquad \text{since } m = \ell/a
\end{aligned}
$$

This is an integer, since $b = c - a^2/c$ and c divides a^2 by hypothesis. In the example, $k = 15$.

Finally we want to generalize $10a = 12b$. The $12b$ becomes $(ja)b$. The $10a$ becomes $(jb)a$. Then, as in the example, we can construct a tiling, following the pattern of Fig. 14. Let us count the number of tiles required. There are three quadratic tilings, each ℓ by ℓ, touching points A and C and contributing $3\ell^2$. There is another quadratic tiling touching point B at the top of the figure. That one contributes $(jb)^2$ tiles. The yellow quadratic tiling in the middle of the figure contributes $(ja)^2$ tiles. Finally the red parallelogram consists of $k - ja$ rows ($15 - 12 = 3$ in the example) and ja columns (12 in the example), each place containing two tiles, so the total number of tiles in the red parallelogram is $2(k - ja)(ja)$. Adding these numbers, we compute the number of tiles in this tiling. This should come out to be N, but that is yet to be proved, so for now we use another letter:

$$
\begin{aligned}
Q &= 3\ell^2 + (jb)^2 + (ja)^2 + 2(k - ja)(ja) \\
&= 3\ell^2 + (\ell b/c)^2 + (\ell a/c)^2 + 2(\ell b/a - ja)(ja) \qquad \text{since } k = \ell b/a \\
&= 3\ell^2 + \ell^2(1 - s^2)^2 + \ell^2 s^2 + 2(\ell b/a - \ell a/c)(\ell a/c) \qquad \text{since } ja = \ell a/c \\
&= 3\ell^2 + \ell^2(1 - s^2)^2 + \ell^2 s^2 + 2\left(\frac{\ell(1 - s^2)}{s} - \ell s)(\ell s)\right) \\
&= \ell^2(3 + (1 - s^2)^2 + s^2 + 2((1 - s^2) - s^2)) \\
&= \ell^2((2 - s^2)^2 + 2 - s^2) \\
(14) \quad &= \ell^2(2 - s^2)(3 - s^2)
\end{aligned}
$$

As a check, in the example we have $\ell = 18$ and $s = 2/3$, yielding $Q = 1288$ as expected.

In the general case, M will be $jb - ja + \ell$, which is an integer. We have $M = jb - ja + \ell = \ell b/c - \ell a/c + \ell = \ell(2 - s^2 - s)$, as claimed in the theorem. Putting this value into the second tiling equation we find a value for the number of tiles N:

$$
\begin{aligned}
N &= M^2 \frac{(2 - s^2)(3 - s^2)}{(1 - s)^2(2 + s)^2} \\
&= \ell^2(2 - s^2 - s)^2 \frac{(2 - s^2)(3 - s^2)}{(1 - s)^2(2 + s)^2} \\
&= \ell^2(2 + s)^2(1 - s)^2 \frac{(2 - s^2)(3 - s^2)}{(1 - s)^2(2 + s)^2} \\
&= \ell^2(2 - s^2)(3 - s^2)
\end{aligned}
$$

which is exactly the value obtained for the number of tiles in (14). That is, the constructed tiling does indeed have N tiles. That completes the proof of the first assertion of the theorem.

Theorem 11. *The following are equivalent:*

(i) Some triangle ABC with angles $(2\alpha, \alpha, 2\beta)$ can be N-tiled by a tile with sides (a, b, c) and angles (α, β, γ) such that $3\alpha + 2\beta = \pi$.

(ii) The second tiling equation has a solution (M, s) with s rational and $s = a/c$.

If there is such a tiling, we can scale it so that (a, b, c) have no common divisor and c divides a^2. Then with $\ell = \mathrm{lcm}(a, c)$ we have

$$M = \ell(2 - s^2 - s).$$

Proof. (i) implies (ii) by Theorem 9. It remains to prove (ii) implies (i). Suppose that ABC and N are given, and the second tiling equation has a solution (M, s) with s rational. Write s as a/c with a and c integers such that c divides a^2. (That can always be done, since if $s = e/f$, we can take $a = ef$ and $c = f^2$.) Then by Theorem 10, there is an N-tiling of some triangle Δ with angles $(2\alpha, \alpha, 2\beta)$ by the tile (a, b, c), where $b = c - a^2/c$. This triangle Δ is similar to ABC, since it has the same angles $(2\alpha, \alpha, 2\beta)$. Hence ABC can be tiled by a tile similar to (a, b, c). This tiling will have the same $s = a/c$ and N as the tiling of Δ. That completes the proof that (i) and (ii) are equivalent.

If c does not divide a^2, we multiply (a, b, c) by c and then divide by the gcd of (a, b, c); after that rescaling we have c divides a^2 and the gcd is 1. In Theorem 10, we proved that some tiling exists using the given N, M, and tile, and in that tiling we have $M = \ell(s - s^2 - 2)$, where $\ell = \mathrm{lcm}(a, b, c)$. Even though that may be a different tiling from the one at hand, (a, b, c) and M are the same; the equation involves the tiling only through M and (a, b, c). That completes the proof.

9.2. N is not prime when ABC has angles $(2\alpha, \alpha, 2\beta)$.

Theorem 12. *Suppose $3\alpha + 2\beta = \pi$. Let triangle ABC with angles $(2\alpha, \alpha, 2\beta)$ be N-tiled by a tile with angles (α, β, γ). Then N is not prime.*

Proof. Suppose ABC is N-tiled as in the statement of the lemma. Let the tile have sides (a, b, c) and define $s = a/c$. We may assume that (a, b, c) are integers with no

common factor and that c divides a^2. According to Theorem 9,

$$\frac{N}{M^2} \;=\; \frac{(2 - s^2)(3 - s^2)}{(1 - s)^2(2 + s)^2}$$

Replacing s by a/c and simplifying, we have

$$N(c - a)^2(2c + a)^2 \;=\; M^2(2c^2 - a^2)(3c^2 - a^2)$$

Now suppose, for proof by contradiction, that N is prime.

I say that M divides $(c - a)(2c + a)$. To prove this, suppose p is a prime and $p^e | M$. Then

$$p^{2e} | M^2$$
$$p \nmid N \qquad \text{since } N \text{ is prime}$$
$$p^{2e} | (c - a)^2(2c + a)^2 \qquad \text{(remaining term on the left)}$$
$$p^e | (c - a)(2c + a)$$

Since that applies to every prime power p^e dividing M, we have $M | (c - a)(2c + a)$, as claimed.

According to Theorem 11,

$$\begin{aligned} M &= \ell(2 - s^2 - s) \\ &= \ell(2 + s)(1 - s) \\ Mc^2 &= \ell(2c + a)(c - a) \qquad \text{since } s = a/c \\ \frac{M}{(2c + a)(c - a)} &= \frac{\ell}{c^2} \end{aligned}$$

The left side is an integer, as proved in the preceding paragraph. Hence c^2 divides ℓ. Let $g = \gcd(a, c)$. Then $\ell = \mathrm{lcm}(a, c) = ac/g)$, so $\ell/c^2 = (a/g)/c$. Since ℓ/c^2 and a/g are integers, c divides a/g. By the definition of greatest common divisor, c must be 1. But (a, b, c) are integers, and $a < c$ (since $\gamma > \pi/2$). We have reached a contradiction. That completes the proof of the theorem.

9.3. Solving the second tiling equation.

Theorem 13. *There is an algorithm that proceeds from input N to termination, and at termination either produces a solution (M, s) of the second tiling equation with s rational, or reports that no such solution exists.*

Proof. Recall the second tiling equation

$$\frac{N}{M^2} \;=\; \frac{(2 - s^2)(3 - s^2)}{(1 - s)^2(2 + s)^2}$$

By Lemma 39, if the equation has a solution, then it has one with $M^2 < N$. It therefore suffices, in principle, to show that there is an algorithm for solving the equation and deciding if the solutions are rational.

In polynomial form we have

$$\begin{aligned} 0 &= N(1 - s)^2(2 + s)^2 - M^2(2 - s^2)(3 - s^2) \\ &= (N - M^2)s^4 + 2Ns^3 + (5M^2 - 3N)s^2 - 4Ns + (4N - 6M^2) \end{aligned}$$

This is a polynomial of degree four. Since formulas for the solutions of a quartic are known, we can determine whether the equation has a rational solution by applying those formulas. This, in principle, supplies the required algorithm.

Using SageMath, we were able to actually implement this algorithm in a few lines of code. There are two crucial steps that SageMath supplies. First, the `solve` command can solve a quartic equation, and second, SageMath can test whether the solution (expressed using some complicated expressions inside square roots) is or is not rational. Fig. 15 shows the function `checkN`, which checks whether there is a solution (M, s) of the second tiling equation for a given N, and if so, prints out M and s.

FIGURE 15. SageMath code to solve the second tiling equation

```
var('N','M','s');
f = N*(s + 2)^2*(s - 1)^2 - (s^2 - 2)*(s^2 - 3)*M^2;
def checkN(n):
    for m in range(1,sqrt(n)+1):
        g = f.substitute(M=m,N=n);
        u = solve(g,s);
        for i in range(0,len(u)):
            v = u[i].operands()[1];
            if v in QQ and 0 < v and v < 1:
                print("M=%d" % m)
                print(v)
```

We used the function defined in Fig. 15 to determine all solutions of the second tiling equation for $N \leq 2000$. The results are shown in Table 2. The table also shows the smallest integer-sided tile (a, b, c) with $a/c = s$ and $b = c - a^2/c$.

TABLE 2. All solutions of the second tiling equation for $N \leq 2000$.
These solutions necessarily correspond to tilings.

N	M	s	(a, b, c)
77	5	1/2	(2,3,4)
308	10	1/2	(2,3,4)
322	8	2/3	(6,5,9)
442	14	1/3	(3,8,9)
693	15	1/2	(2,3,4)
897	11	3/4	(12,7,16)
1232	20	1/2	(2,3,4)
1288	16	2/3	(6,5,9)
1457	27	1/4	(4,15,16)
1768	28	1/3	(3,8,9)
1925	25	1/2	(2,3,4)

If the second tiling equation is solvable for N, then it is solvable for every square multiple of N, since if N is multiplied by λ^2 and M by λ, the equation is still satisfied (with the same s). Thus some of the entries in Table 2 were predictable, e.g., $308 = 4 \cdot 77$ and $693 = 9 \cdot 77$. In fact, only the first occurrence of each value of s in the table has a square-free N.

In Table 2, for each N there is either no solution, or exactly one solution. We do not know if that is an accident of this small data set, or is always true. (The code

seems slow, but remember it is checking whether solutions of quartics are rational, so it would not be simple to rewrite this in C to speed it up.)

10. The case when ABC is isosceles with base angles β

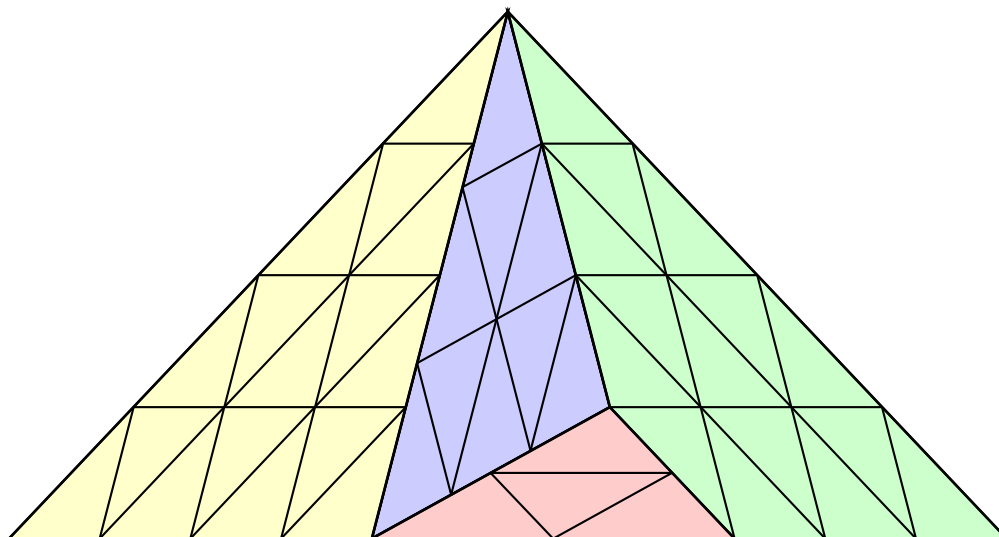In Figs. 16 and 17, we give examples of tilings of triangles of this form.

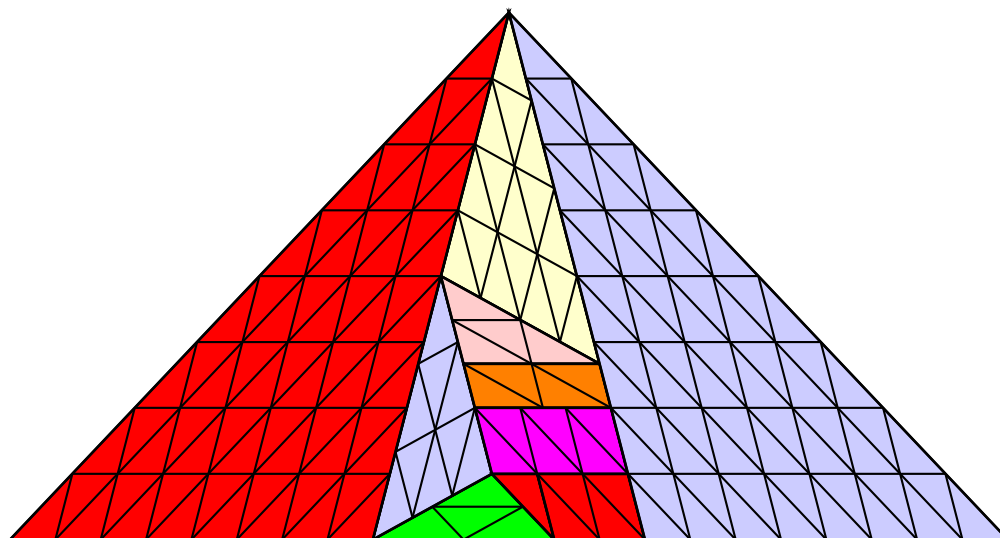FIGURE 16. $N = 44$, $(a, b, c) = (2, 3, 4)$, ABC isosceles with base angles β.

FIGURE 17. $N = 176$, $(a, b, c) = (2, 3, 4)$, ABC isosceles with base angles β.

Let ABC have angles β at A and C. Then the vertex angle at B is 3α, since $3\alpha + 2\beta = \pi$. Let X be the length of AB and BC; let Y be the length of AC. Let μ be the "scale factor", defined by $X = \mu b$. As for other shapes of ABC, we have the "area equation" expressing that the area of ABC is equal to the area of N tiles, and the "coloring equation" from Theorem 1, namely

$$M(a + b + c) \;\; = \;\; 2X + Y$$

Combining these, as we did for other shapes of ABC, will result in a "tiling equation" giving a necessary condition for the existence of an N-tiling of ABC. Then the question is whether the solvability of that equation is also a sufficient condition. It was, in case of the shape $(2\alpha, \alpha, \alpha + \beta)$, but was not in the case of $(\alpha + \beta, \alpha, \alpha + \beta)$. We will show below that it is necessary and sufficient in this case.

10.1. **The tiling equation for ABC isosceles with base angles β.** In this section we show how, given N, to determine a finite set of possible tiles (a, b, c) (each of which then determines the triangle ABC) such that, if any triangle ABC of the form considered here can be N-tiled, then one of these specific triangles ABC can be N-tiled with the corresponding specific tile. That is implied by the "tiling equation" given in the following theorem.

Theorem 14. *Suppose $3\alpha + 2\beta = \pi$, and triangle ABC is isosceles with base angles β, and ABC is N-tiled by a tile with angles (α, β, γ) and sides (a, b, c) of integer lengths with no common factor. Let M be the coloring number of the tiling, $s = a/c$, and $g = \gcd(a, c)$. Then g divides M, and $0 < N/3 < M^2 < 2N$, and*

$$\frac{N}{M^2} \;\; = \;\; \frac{(3 - s^2)}{(1 + s)^2}$$

and consequently

$$s = \frac{\sqrt{3M^2 + 2N}M - N}{M^2 + N}.$$

Remarks. Then $3M^2 + 2N$ is a square, since s is rational. The condition that g divides M is a very important part of the theorem. It depends on the fact that g has to be squarefree, which we proved in Lemma 8. The displayed equation only depends on a/c. The condition that g divides M brings the scaling of the sides of the tile into the equation.

Proof. Let κ be defined by $b = \kappa \sin \beta$. Let μ be defined by $X = \mu b$. We have by the law of sines,

$$\frac{Y}{X} \;\; = \;\; \frac{\sin 3\alpha}{\sin \beta}$$

By Lemma 38 we have

$$\frac{\kappa \sin 3\alpha}{c} \;\; = \;\; s(1 + s)(1 - s)(3 - s^2)$$

Since $b = \kappa \sin \beta$ and $b/c = 1 - s^2$, we have

$$\frac{Y}{X} \;\; = \;\; \frac{s(1 + s)(1 - s)(3 - s^2)}{1 - s^2}$$

Simplifying,

(15) $$\frac{Y}{X} \;\; = \;\; s(3 - s^2)$$

By Theorem 1, the coloring equation is

$$\begin{aligned} M(a+b+c) &= 2X + Y \\ &= X(2 + Y/X) \\ &= \mu b(2 + s(3 - s^2)) \end{aligned}$$

Dividing by c and expressing everything in terms of s,

$$\begin{aligned} M(s + (1 - s^2) + 1) &= \mu(1 - s^2)(2 + 3s - s^3) \\ \frac{\mu}{M} &= \frac{2 + s - s^2}{(1 - s^2)(2 + 3s - s^3)} \\ &= \frac{(1 + s)(2 - s)}{(1 - s^2)(2 - s)(1 + s)^2} \\ &= \frac{1}{(1 - s)(1 + s)^2} \\ &= \frac{1}{(1 - s^2)(1 + s)} \end{aligned}$$

Multiplying by M and squaring, we have

$$(16) \qquad \mu^2 = \frac{M^2}{(1 - s^2)^2(1 + s)^2}$$

The area equation is

$$X^2 \sin 3\alpha = Nbc \sin \alpha$$

Multiplying by κ/c and using Lemma 38 again, as well as $\kappa \sin \alpha = a$, we have

$$(17) \qquad X^2 s(1 + s)(1 - s)(3 - s^2) = Nbc\frac{a}{c} = Nab$$

Putting $X = \mu b$ we have

$$\begin{aligned} \mu^2 b^2 s(1 + s)(1 - s)(3 - s^2) &= Nab \\ \mu^2 bs(1 + s)(1 - s)(3 - s^2) &= Na \end{aligned}$$

Expressing everything in terms of s, and dividing by cs,

$$N = \mu^2(1 - s^2)^2(3 - s^2)$$

Substituting the value of μ^2 from (16),

$$N = \frac{M^2(1 - s^2)^2(3 - s^2)}{(1 - s^2)^2(1 + s)^2}$$

$$(18) \qquad N = M^2\frac{(3 - s^2)}{(1 + s)^2}$$

This is the "tiling equation" of the theorem; the formula for s follows by solving this equation, which is quadratic in s. (The other solution is negative and is therefore irrelevant.)

It remains to show that $g|M$. Recalling (17) we have

$$X^2 s(1 + s)(1 - s)(3 - s^2) = Nab$$

Substituting the value of N from the tiling equation (18), we have

$$X^2 s(1 + s)(1 - s)(3 - s^2) = abM^2\frac{3 - s^2}{(1 + s)^2}$$

Multiplying both sides by the denominator, using $(1 + s)(1 - s) = 1 - s^2$, and canceling $(3 - s^2)$, we have

$$X^2 s (1 - s^2)(1 + s)^2 \quad = \quad abM^2$$

Multiplying both sides by c^5 and using $s = a/c$, we have

(19) $$X^2 a (c^2 - a^2)(a + c)^2 \quad = \quad abc^5 M^2$$

Suppose, for proof by contradiction, that g does not divide M. Then g^6 divides the right side but g^7 does not divide the right side, since g does not divide b. On the left, g^5 divides the factor $a(a^2 - c^2)(a + c)^2$, but g^6 does not. Hence g divides X^2, but g^2 does not. By Lemma 8, g is squarefree. Therefore $g | X^2$ implies $g | X$. Therefore $g^2 | X^2$, contradiction. That contradiction shows that g does divide M.

It remains to prove the inequality $0 < M^2 < 2N$. As s varies from 0 to 1, the value of the right side of the tiling equation ranges from 3 to $1/2$, as can be shown by calculus, or by the SageMath command `plot((3-s^2)/(1+s)^2,0,1)`. Hence the inequality is a consequence of the tiling equation.

That completes the proof of the theorem.

10.2. N is not prime when ABC is isosceles with base angles β.

Lemma 41. *Suppose (a, b, c) are integers with no common factor, and are the sides of a triangle with angles (α, β, γ) and $3\alpha + 2\beta = \pi$. Let $g = gcd(a, c)$. Then*

(i) g^3 does not divide $a^2 + c^2$, and

(ii) g^4 does not divide $(c - a)(2c + a)^2$

Proof. Ad (i). Assume, for proof by contradiction, that g^3 does divide $a^2 + c^2$. According to Lemma 8, we have $c = g^2$. Let $\hat{a}$ be defined by $a = g\hat{a}$. Then g is relatively prime to $\hat{a}$, since $g = gcd(a, c)$. Now $a^2 + c^2 = g^2(\hat{a}^2 + g^2)$, and $\hat{a}^2 + g^2$ is relatively prime to g since $\hat{a}$ is. Therefore g^3 does not divide $a^2 + c^2$. That proves (i) of the lemma.

Ad (ii). $(c - a)(2c + a)^2 = g^3(g - \hat{a})(2g + \hat{a})^2$. Since g is relatively prime to the last two factors, g^4 does not divide the product. That completes the proof of the lemma.

Theorem 15. *Suppose $3\alpha + 2\beta = \pi$, and triangle ABC is isosceles with base angles β, and ABC is N-tiled by a tile with angles (α, β, γ). Then N is not a prime number.*

Proof. Let M be the coloring number of the tiling, and let (a, b, c) be the sides of the tile, scaled so that they are integers with no common factor. According to Theorem 14, we have

$$\frac{N}{M^2} \quad = \quad \frac{(3 - s^2)}{(1 + s)^2}$$

and $g | M$, where $g = gcd(a, c)$. Using $s = a/c$ and clearing denominators, we have

(20) $$N(a^2 + c^2) \quad = \quad M^2(3c^2 - a^2)$$

Since $g | M$, the right side is divisible by g^4. Evidently g^2 divides $a^2 + c^2$, but by Lemma 41, g^3 does not divide $a^2 + c^2$.

Since g^4 divides the right side of (20), it must also divide the left side $N(a^2 + c^2)$. But only g^2 divides $(a^2 + c^2)$. Therefore g^2 divides N. Since N is prime that implies

$g = 1$. But $c = g^2$, so $c = 1$. But $c > a$, and a is an integer. We have reached a contradiction. That completes the proof of the theorem.

10.3. Construction of tilings when ABC is isosceles with base angles β.
Let ABC be isosceles with base angles β. Let the vertex angle at B be trisected into three α angles, and let the trisecting lines meet the base AC at D and E, in the order $ADEC$. Then triangle BDE is isosceles and triangle BDC has angles $(2\alpha, \alpha + \beta, \beta)$ (at B, D, and C respectively). Therefore there are two obvious approaches to constructing tilings:

(i) Tile the triangle BDC with a triquadratic tiling, and tack on a quadratic tiling of ABD. This is illustrated in Fig. 16.

(ii) Tile the isosceles triangle BDE, which has base angles $\alpha + \beta$, and tack on two quadratic tilings. This is illustrated in Fig. 17.

We were able to find a necessary and sufficient condition for the existence of tilings of isosceles ABC with base angles β via method (i). We attempt to tile ABC by constructing a triquadratic tiling of BDC. What happens, in summary, is that if we assume there is a solution for ABC of the third tiling equation given in Theorem 14, then we can verify that the (triquadratic) tiling equation for BCD is solvable, and the extra condition that g divides M implies the additional divisibility condition required for the existence of a triquadratic tiling of BDC.

Theorem 16. *Suppose the tiling equation in Theorem 14 is solvable by (M, s), with $0 < s < 1$. Let (a, b, c) be integers with no common factor such that $s = a/c$ and c divides a^2 (so the angles of (a, b, c) satisfy $3\alpha + 2\beta = \pi$), and let ABC be an isosceles triangle with base angles β and scaled so that its area is equal to N times the area of the tile (a, b, c). Suppose in addition that $a + c$ divides M, or equivalently that $g = \gcd(a, c)$ divides M. Then there exists an N-tiling of ABC by (a, b, c).*

Proof. What fraction of the area of ABC is the area of ABD? Let X be the length of AB and BC, and let Z be the length of BD and BE. Then $X/Z = c/b$. Twice the area of ABD is $XZ \sin \alpha$; that is also twice the area of CBE. The area of DBE is $Z^2 \sin \alpha$. So the ratio of the area of ABD to the area of ABC is

$$\frac{XZ \sin \alpha}{2XZ \sin \alpha + Z^2 \sin \alpha} = \frac{X}{2X + Z}$$
$$= \frac{1}{2 + (b/c)}$$
$$= \frac{1}{3 - s^2}$$

So the ratio of DBC to ABC is

$$1 - \frac{1}{3 - s^2} = \frac{2 - s^2}{3 - s^2}$$

Now suppose there is an N-tiling of ABC. Then the number of tiles needed to tile DBC would be

$$N' := N \frac{2 - s^2}{3 - s^2}$$

By the tiling equation in Theorem 14,

$$N = M^2 \frac{(3 - s^2)}{(1 + s)^2}$$

Substituting this in the previous equation we have

$$
\begin{aligned}
N' &= M^2\left(\frac{(3-s^2)}{(1+s)^2}\right)\left(\frac{2-s^2}{3-s^2}\right)\\
&= M^2\frac{2-s^2}{(1+s)^2}
\end{aligned}
$$

Then

(21) $$N' = 2\left(\frac{M}{1+s}\right)^2 - \left(\frac{Ms}{1+s}\right)^2$$

We claim that the fractions $M/(1+s)$ and $Ms/(1+s)$ are integers. Since $s = a/c$ that is the same as claiming that $Mc/(a+c)$ and $Ma/(a+c)$ are integers. Let $g = \gcd(a,c)$ and $\hat{a} = a/g$, $\hat{c} = c/g$. Then it suffices to show $\hat{a} + \hat{c}$ divides M. By Theorem 14,

$$N = M^2\frac{2-s^2}{(1+s)^2} = M^2\frac{2c^2-a^2}{(a+c)^2}.$$

Since N is an integer, it suffices to show that $\hat{a} + \hat{c}$ is relatively prime to $2\hat{c}^2 - \hat{a}^2$. Suppose p divides them both. Then it divides

$$\hat{c}^2 = 2\hat{c}^2 - \hat{a}^2 - (\hat{c}-\hat{a})(\hat{a}+\hat{c})$$

Then it divides $\hat{c}$ and hence also $\hat{a}$, contradiction, since $\hat{a}$ and $\hat{c}$ are relatively prime. That completes the proof that $M/(1+s)$ and $Ms/(1+s)$ are integers. Define

$$
\begin{aligned}
K &:= \frac{M}{1+s}\\
J &:= \frac{Ms}{1+s}
\end{aligned}
$$

Then K and J are integers, and by (21) we have

$$N' = 2K^2 - J^2$$

Hence N' is an integer, and the triquadratic tiling equation is satisfied.

In order to show that a triquadratic tiling exists, we must also show that K divides N'. That is not true without an additional hypothesis, as the example $N = 11$, $M = 3$, $K = 2$, $J = 1$, $N' = 7$ shows. Our additional hypothesis is that $a + c$ divides M, or equivalently that $g = \gcd(a,c)$ divides M. The two conditions are equivalent because $a + c = g(\hat{a} + \hat{c})$, and g divides $\hat{c}$ because $a^2/c = c - b$ is an integer, but g is relatively prime to $\hat{a} + \hat{c}$.

We now show that K divides N'. Since $N' = 2K^2 - J^2$, it suffices to show K divides J^2. Consider the quotient

$$
\begin{aligned}
\frac{J^2}{K} &= \frac{(Ms/(1+s))^2}{M/(1+s)}\\
&= \frac{Ms^2}{1+s}\\
&= \left(\frac{M}{a+c}\right)\left(\frac{a^2}{c}\right)
\end{aligned}
$$

Now a^2/c is an integer, as remarked above, and $M/(a+c)$ is an integer, by hypothesis. Hence a triquadratic tiling of BCD by (a,b,c) exists.

It remains to show that ABD can be quadratically tiled by (a, b, c). The number of tiles needed to tile ABD is $N - N'$. We claim that this is a square. We have

$$\begin{aligned} N - N' &= M^2 \frac{(3 - s^2)}{(1 + s)^2} - M^2 \frac{2 - s^2}{(1 + s)^2} \\ &= \left(\frac{M}{1 + s}\right)^2 (3 - s^2 - (2 - s^2)) \\ &= \left(\frac{M}{1 + s}\right)^2 \end{aligned}$$

and since we proved above that $M/(1+s)$ is an integer, this is a square, as claimed. Since ABD is similar to the tile, and its area is equal to that of a square number of tiles, when we start a quadratic tiling at B, the last row of tiles that fits into ABD will fit exactly, completing a quadratic tiling. That completes the proof of the theorem.

Corollary 2. *If $N \leq 1000$, then there is an N-tiling of some isosceles triangle with base angles β if and only if N occurs in Table 3, and the possible values of the coloring number and tile for these N, are as given in that table.*

TABLE 3. Solutions of the isosceles-β tiling equation for $N \leq 1000$

N	M	side	base	(a, b, c)
44	6	16	22	(2, 3, 4)
176	12	32	44	(2, 3, 4)
207	15	81	138	(6, 5, 9)
234	12	81	78	(3, 8, 9)
396	18	48	66	(2, 3, 4)
624	28	256	468	(12, 7, 16)
704	24	64	88	(2, 3, 4)
752	20	256	188	(4, 15, 16)
828	30	162	276	(6, 5, 9)
936	24	162	156	(3, 8, 9)

Proof. A simple program computes the values in Table 3 by checking for each N in the specified range whether the conditions in Theorem 14 are satisfied. Here is the function that is called for each N to be checked. (The print commands format rows of the table for TeX. For simplicity the lines that compute the side and base are not shown here.)

```
def IsoscelesBeta(N):
    for M in range(sqrt(N/3),sqrt(2*N+1)):
        if is_square(3*M^2+2*N):
            s = (sqrt(3*M^2+2*N)*M - N)/(M^2+N)
            if(s >= 1 or s <= 0):
                continue;
            (a,b,c) = getABC(s);    # See Fig. 11
            g = gcd(a,c)
            if M % g == 0:
                print("%d & %d & (%d, %d, %d) \\\\" % (N,M,a,b,c))
```

Remarks. There are many values of N, including some that are prime numbers, that satisfy the equation of Theorem 14, but not the condition that g divides M. Before it was noticed that the condition g divides M is necessary, as well as sufficient, several special-purpose proofs were constructed of the impossibility of tilings, for example, for $N = 26$, 39, and 47. Now they are all unnecessary.

11. THE CASE WHEN ABC IS ISOSCELES WITH BASE ANGLES $\alpha + \beta$

Given a tile (a, b, c) with angles (α, β, γ) (such that $3\alpha + 2\beta = \pi$), one can use the diagrams given by Laczkovich [4] to construct an N-tiling of some isosceles triangle with base angles $\alpha + \beta$. In general N may be large.

In July 2018, we discovered the tilings shown in Figures 18, 19,20, and 21. These figures show N-tilings with N decreasing. The particular values of N were obtained from an equation to be explained below. Like the first and second tiling equations, this equation does two things: First, if it is not solvable, there is no N-tiling of any ABC (that is isosceles of the form considered in this section). Second, if the equation *is* solvable, each of the (finitely many) solutions determines the shape of the tile (α, β, γ). That reduces the tiling problem to testing whether particular triangles ABC can be tiled by particular tiles. But if the equation is solvable, at present we have no efficient way to determine whether there actually is a corresponding tiling. That has to be checked on a case-by-case basis. In this section we will present these theoretical and empirical results.

11.1. **A tiling equation for isosceles ABC with base angles $\alpha + \beta$.** In this section we present necessary conditions for an N-tiling of an isosceles triangle ABC with base angles $\alpha + \beta$ to exist.

Definition 17. *We assume that the tile sides (a, b, c) are integers with no common factor. Define the "scaling factors"*

$$\begin{aligned} \lambda &= X/\sin\alpha \\ \kappa &= a/\sin\alpha \\ \mu &= \lambda/\kappa \end{aligned}$$

Lemma 42. *Let triangle ABC be isosceles with base angles $\alpha + \beta$, and $3\alpha + 2\beta = \pi$, and ABC is N-tiled by the integer-sided tile (a, b, c) with angles (α, β, γ) and $gcd(a, b, c) = 1$. Let X be the length of AB and Y the length of AC. Then*

$$\begin{aligned} X &= \mu c \\ Y &= \mu a \end{aligned}$$

Proof. Let λ, κ, and μ be given by Definition 17. By the law of sines,

$$\begin{aligned} X &= \lambda \sin(\alpha + \beta) \\ Y &= \lambda \sin\alpha \\ a &= \kappa \sin\alpha \\ c &= \kappa \sin\gamma \end{aligned}$$
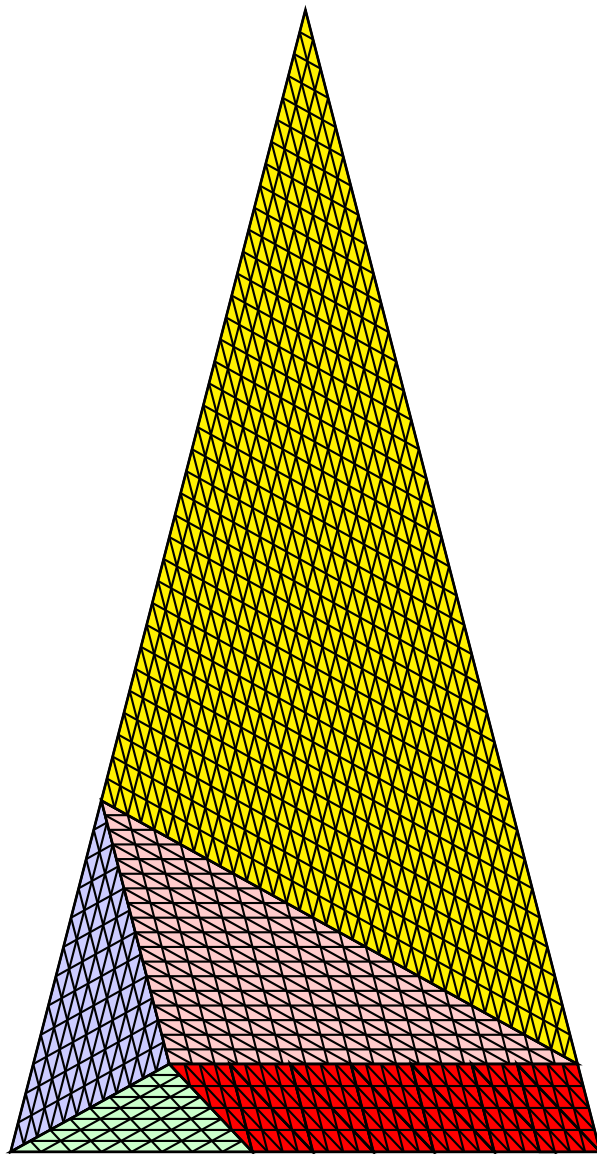
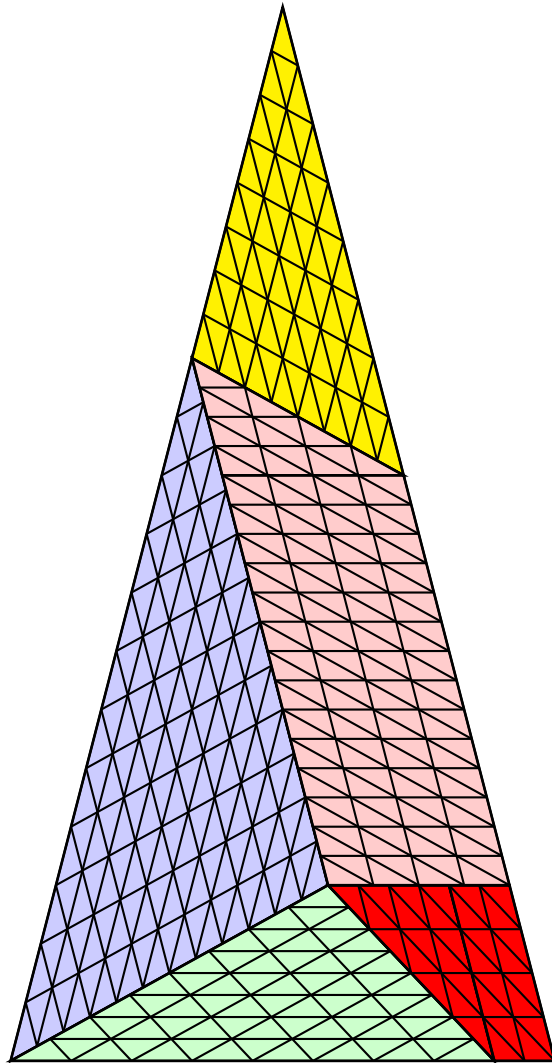FIGURE 18. $N = 2028$. ABC is isosceles. The tile is $(2, 3, 4)$.

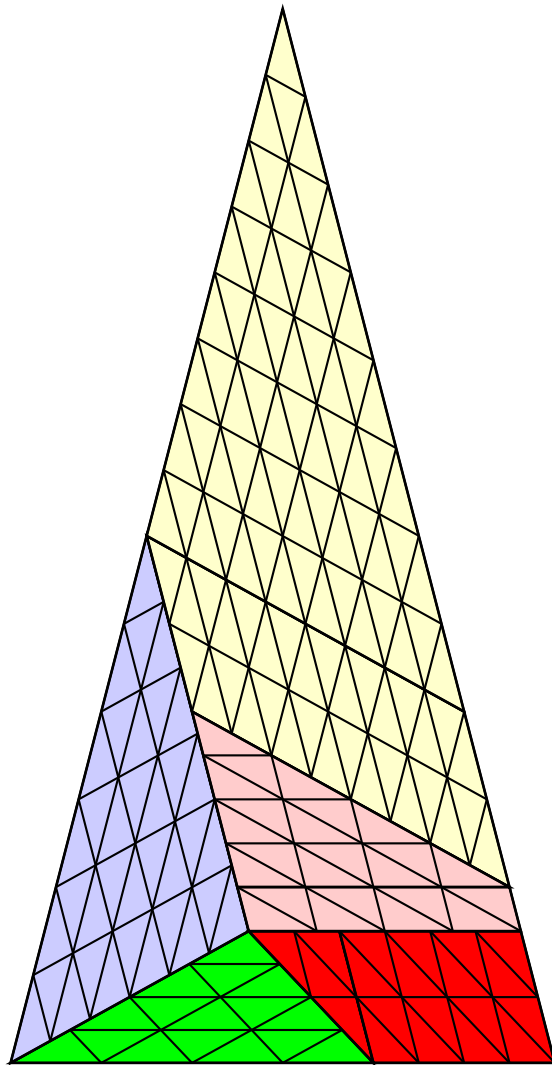FIGURE 19. $N = 432$. ABC is isosceles. The tile is $(2, 3, 4)$.

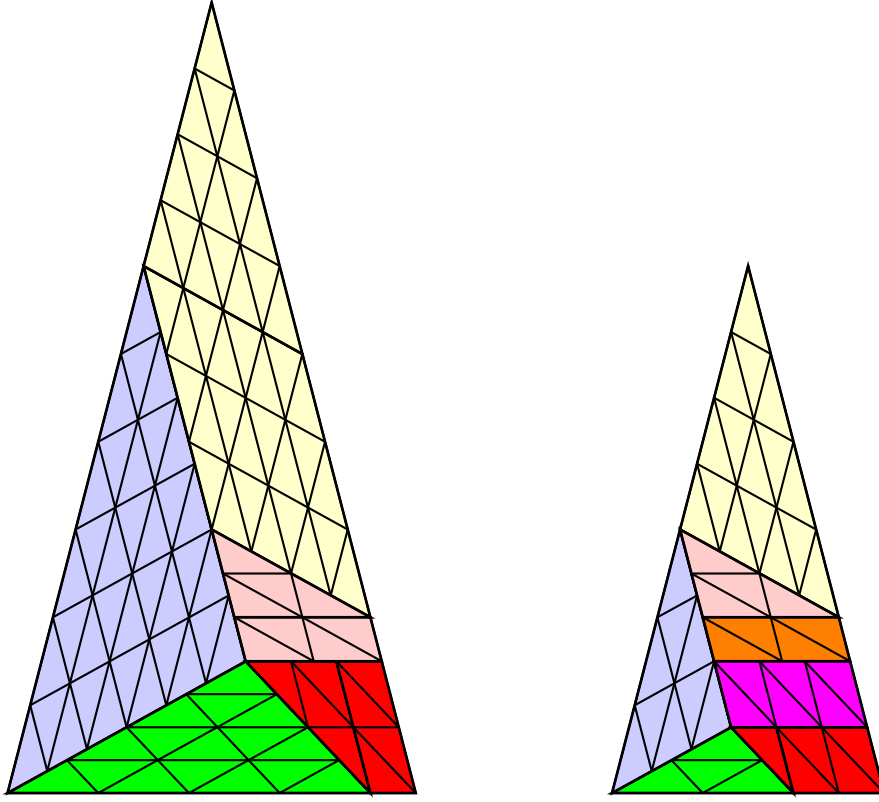FIGURE 20. $N = 192$. ABC is isosceles. The tile is $(2, 3, 4)$.

FIGURE 21. $N = 108$ and $N = 48$. ABC is isosceles. The tile is $(2, 3, 4)$.

Since $\gamma = \pi - (\alpha + \beta)$, the preceding equations imply

$$
\begin{aligned}
X &= \left(\frac{\lambda}{\kappa}\right) c = \mu c \\
Y &= \left(\frac{\lambda}{\kappa}\right) a = \mu a
\end{aligned}
$$

That completes the proof of the lemma.

Theorem 17. *Suppose triangle ABC is N-tiled by the tile (a, b, c) with angles (α, β, γ), where $3\alpha + 2\beta = \pi$. Suppose triangle ABC is isosceles with base angles $\alpha + \beta$, and $\alpha \neq \pi/8$. Let $s = a/c$. Then the following Diophantine equation is satisfied, where M is the coloring number of the tiling:*

$$
\frac{N}{M^2} = \frac{1 + s}{1 - s}
$$

An algebraically equivalent form of the equation is

$$
s = \frac{N - M^2}{N + M^2}.
$$

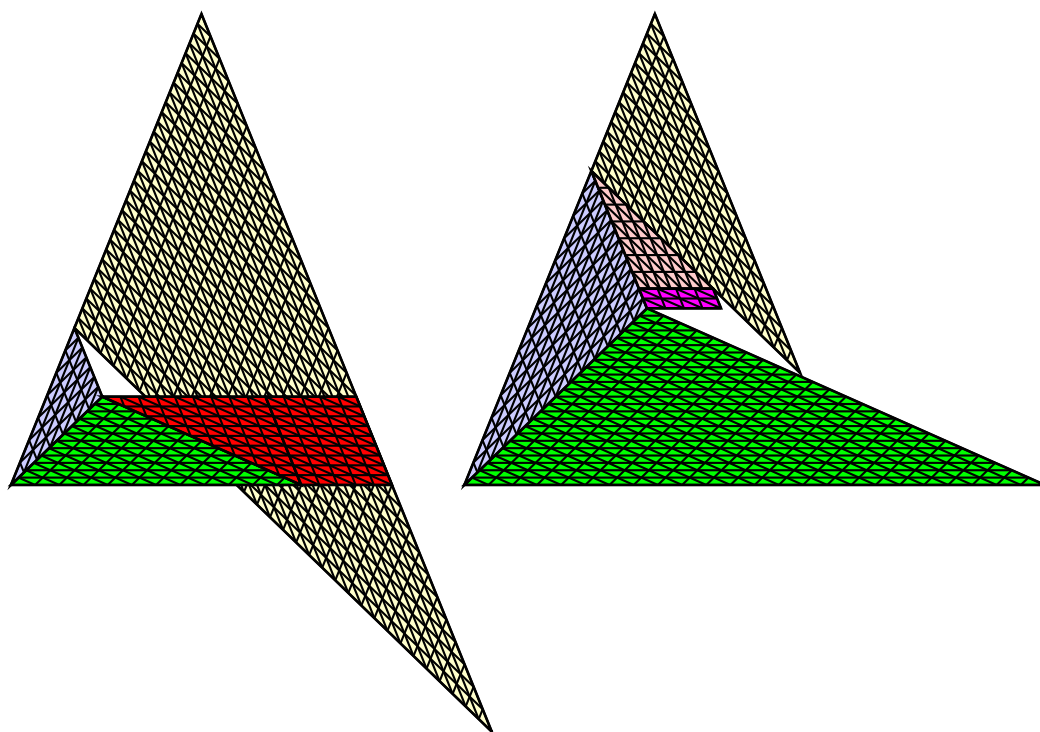*The scale factor μ is given by*

$$
\mu = M(1 + s).
$$

FIGURE 22. $N = 1008$. ABC is isosceles. The tile is $(12, 7, 16)$. The tiling method used for previous tilings fails; $p = 12$ is too small and $p = 24$ is too large.

Example. For the tiling in Fig. 18, we have $N = 2028$, and $s = a/c = 1/2$. The coloring number M is 26, as one can see by counting m for each quadratic subtiling with m^2 tiles, with a sign positive or negative according to the color of the top tile of that subtiling. The red trapezoid contributes 4, since parallelograms contribute 0 and the red trapezoid consists of a 4^2 quadratic tiling plus a parallelogram. Thus $M = 26$. Then both sides of the equation evaluate to 3.

Proof. The tile is necessarily rational by Theorem 3. The vertex angle A of ABC is $\alpha = \pi - 2(\alpha + \beta)$, since $3\alpha + 2\beta = \pi$. As usual let the sides of the tile be (a, b, c) with $c = \sin\gamma$, $a = \sin\alpha$, and $b = \sin\beta$. Let X be the length of the two equal sides of ABC, namely AB and AC, and Y the length of the base BC. Then twice the area of ABC is $X^2 \sin\alpha$, and on the other hand it is N times twice the area of the tile, which is $Nbc \sin\alpha$. Equating this two expressions we have the "area equation"
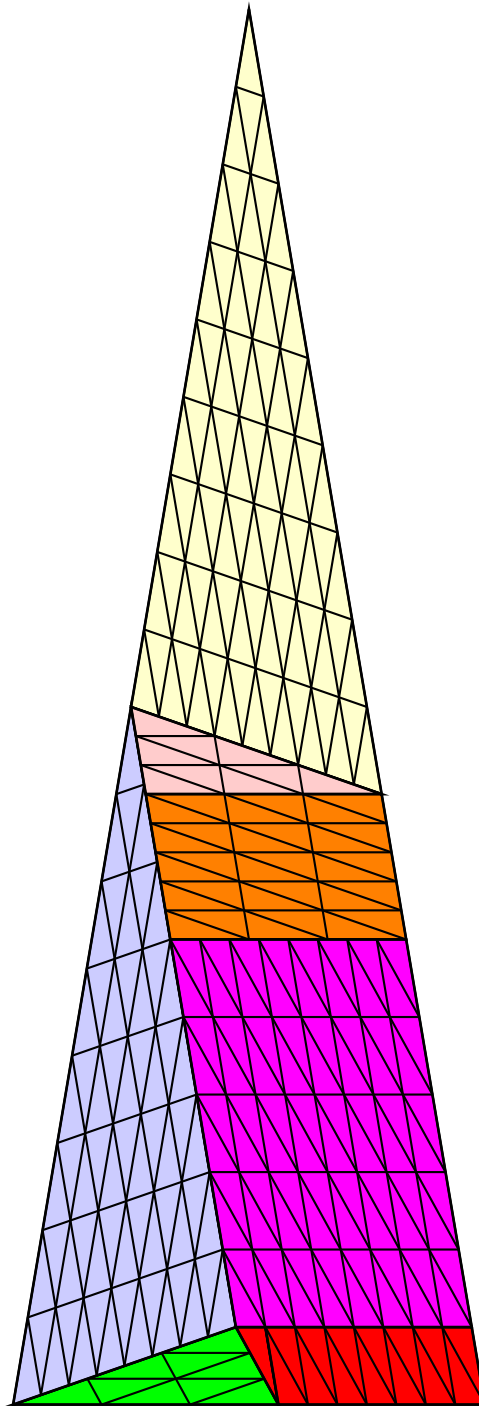
$$X^2 \sin\alpha = Nbc \sin\alpha$$
(22)
$$X^2 = Nbc$$

FIGURE 23. $N = 288$. ABC is isosceles. The tile is $(3, 8, 9)$.
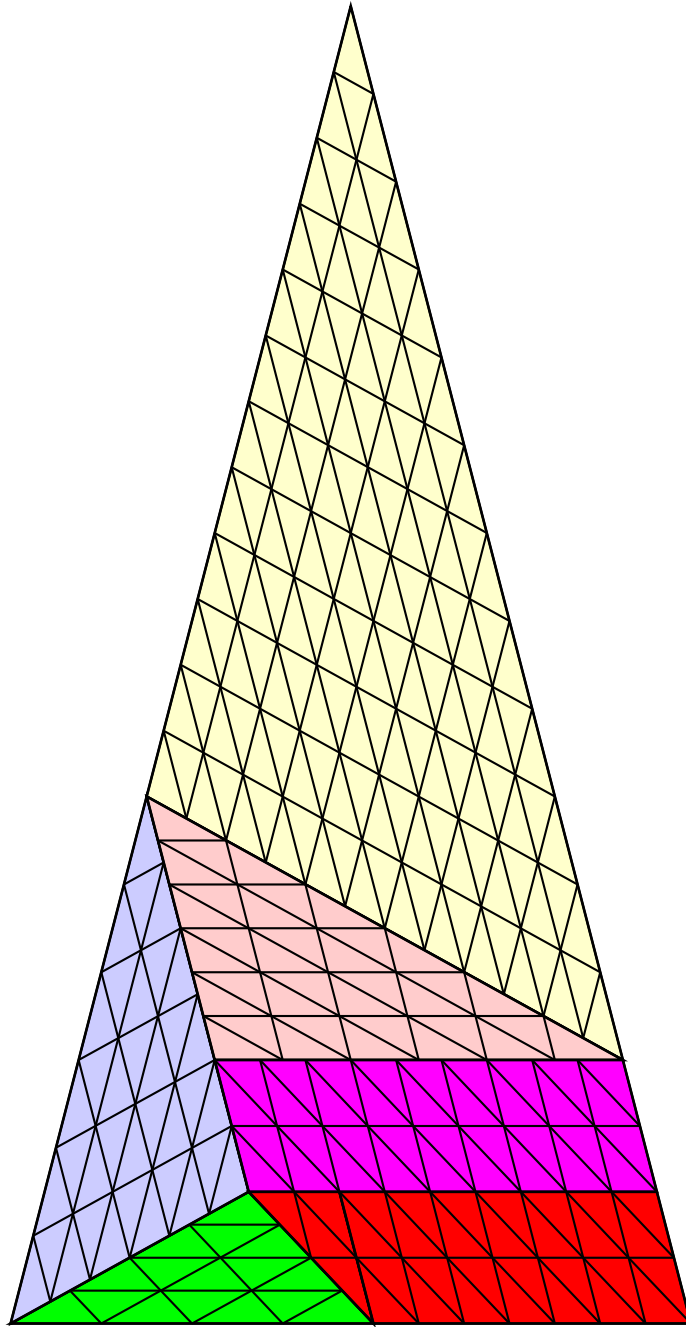
FIGURE 24. $N = 300$. ABC is isosceles. The tile is $(2, 3, 4)$.

On the other hand by Theorem 1, with M the coloring number (excess of black over white tiles) of the tiling, we have

$$(23) \qquad\qquad M(a + b + c) \;\;=\;\; 2X - Y$$

Since ABC is a triangle, the sum of any two sides is greater than the third side, so $2X > Y$. It follows that

$$(24) \qquad\qquad\qquad\qquad M \;\;>\;\; 0$$

By Lemma 42, we have

$$X \;\;=\;\; \mu c$$
$$Y \;\;=\;\; \mu a$$

Putting these into the area equation (22) and the coloring equation (23), respectively, we have

$$(25) \qquad\qquad\qquad \mu^2 c^2 \;\;=\;\; Nbc$$
$$M(a + b + c) \;\;=\;\; \mu(2c - a)$$

Dividing by c we have

$$M\left(\frac{a}{c} + \frac{b}{c} + 1\right) \;\;=\;\; \mu\left(2 - \frac{a}{c}\right)$$

Expressing everything in terms of s by Lemma 2, we have

$$M(s + (1 - s^2) + 1) \;\;=\;\; \mu(2 - s)$$
$$M(2 + s - s^2) \;\;=\;\; \mu(2 - s)$$
$$M(2 - s)(1 + s) \;\;=\;\; \mu(2 - s)$$

Cancelling $(2 - s)$ we have

$$\mu \;\;=\;\; M(1 + s),$$

which is the last assertion in the theorem. Putting this expression into (25) we find

$$c^2 M^2 (1 + s)^2 \;\;=\;\; Nbc$$

Dividing by c^2 and using Lemma 2, we have

$$M^2(1 + s)^2 \;\;=\;\; N(1 - s^2)$$
$$\;\;=\;\; N(1 - s)(1 + s)$$

Therefore

$$\frac{N}{M^2} \;\;=\;\; \frac{1 + s}{1 - s}.$$

That completes the proof of the theorem.

Corollary 3. *Let N and ABC be given, and suppose ABC is isosceles with vertex angle α at B and base angles $\alpha + \beta$. Let Y be the length of the base AC and let X be the length of the equal sides AB and BC. Then the only possible value of $s = a/c$ for a tile that could be used to N-tile ABC is $s = Y/X$, and the only possible value for the coloring number M is given by*

$$M^2 = N\frac{1 - s}{1 + s} = N\frac{X - Y}{X + Y}$$

The number on the right must be a square.

Proof. By Lemma 42 we have $s = Y/X$. By Theorem 17 we have the first equation of the theorem. By Lemma 42 we have

$$\frac{X-Y}{X+Y} = \frac{c-a}{c+a}$$

Dividing numerator and denominator by c and using $s = a/c$ we have

$$\frac{c-a}{c+a} = \frac{1-s}{1+s}.$$

That completes the proof of the corollary.

Although we did not (yet) find an "isosceles tiling equation" that is necessary and sufficient for the existence of an N-tiling of an isosceles triangle of the shape considered here, we now are in a position to show, in some cases, given N, that no N-tiling of such a triangle exists for any isosceles triangle ABC with $3\alpha + 2\beta = \pi$ and vertex angle α.

11.2. N is not prime when ABC is isosceles with base angles $\alpha + \beta$.

Theorem 18. *Let N and ABC be given, and suppose ABC is isosceles with base angles $\alpha + \beta$. Suppose ABC is N-tiled. Then N is not a prime number.*

Proof. Let X be the length of the equal sides AB and BC. By Theorem 3, the tile is rational. Let (a, b, c) be the lengths of the edges of the tiles; we will fix the scaling later, so we do not assume that b is an integer, but we do assume that a and c are integers. By Lemma 7, $b = c - a^2/c$, so bc is an integer, even if b is not necessarily an integer. The area equation is

$$X^2 = Nbc.$$

Since there is a tiling, we have $X = pa + qb + rc$ for some integers (p, q, r). Hence Xc is an integer. Then

$$(Xc)^2 = c^2 Nbc.$$

Assume, for proof by contradiction, that N is prime. Then, since Xc and bc are integers, N divides $(Xc)^2$ to an even power. Hence N divides bc^3. Hence N divides b or N divides c. Hence N divides bc.

According to Theorem 17, we have

$$s = \frac{a}{c} = \frac{N-M^2}{N+M^2}$$

We have not assumed that (a, b, c) have no common factor, so we are free to choose the scaling factor, as long as a and c are integers. We may therefore assume without loss of generality that

(26) $$a = N - M^2$$

(27) $$c = N + M^2$$

Then we have

$$
\begin{aligned}
c^2 b &= c^2(c - a^2/c) \\
&= (N+M^2)^2 \left((N+M^2) - \frac{N-M^2}{N+M^2} \right) \\
&= (N+M^2)^3 - (N+M^2)(N-M^2)
\end{aligned}
$$

Both sides are integers, and N divides the left side since N divides bc. Hence, mod N we have

$$
\begin{aligned}
0 &\equiv M^6 + M^4 && \mod N \\
&\equiv M^4(M^2 + 1) && \mod N
\end{aligned}
$$

Since $0 < M^2 < N$, N does not divide M. Since N is prime, we have

$$
0 \equiv M^2 + 1 \qquad \mod N
$$

Since $0 < M^2 < N$, this is possible if and only if $M^2 = N - 1$. Hence $M^2 = N - 1$. Then by (26) and (27), $a = 1$ and $c = 2N - 1$. Hence

$$
\begin{aligned}
b &= c - a^2/c \\
&= (2N - 1) - \frac{1}{2N - 1}
\end{aligned}
$$

That $N = M^2 + 1$ is not immediately contradictory, as there are plenty of primes of the form $M^2 + 1$.

If we rescale (a, b, c) so that they are integers with no common factor, we have

$$
\begin{aligned}
a &= 2N - 1 \\
b &= (2N - 1)^2 - 1 \\
c &= (2N - 1)^2 \\
g &= \gcd(a, c) = 2N - 1
\end{aligned}
$$

According to Theorem 17, we have

$$
\frac{N}{M^2} = \frac{c + a}{c - a}
$$

Since $c = g^2$ and $a = g\hat{a}$ we can write this as

$$
N(g - \hat{a}) = M^2(g + \hat{a})
$$

Since $\hat{a}$ is relatively prime to g, this equation mod g becomes

$$
N \equiv M^2 \mod g
$$

That is, $N - M^2$ is divisible by g. Since $N = M^2 + 1$, that implies $g = 1$. By Lemma 3, $\gamma > \pi/2$, which implies $c > a$. Since (a, b, c) are integers, that implies $c > 1$. But $c = g^2 = 1$. We have reached a contradiction from the assumption that N is prime. That completes the proof of the theorem.

11.3. The scale factor μ.

Lemma 43. *Let N and ABC be given, and suppose ABC is isosceles with vertex angle α at B and base angles $\alpha + \beta$. Suppose ABC is N-tiled by the tile with angles (α, β, γ) and integer sides (a, b, c) with no common factor. Let μ be as in Definition 17. Then $\mu = M(1 + s)$ and*

$$
\mu^2 = \frac{Nb}{c}.
$$

Proof. We have by Lemma 42

$$
\begin{aligned}
X &= \mu c \\
Y &= \mu a
\end{aligned}
$$

Since twice the area of ABC is $XY \sin(\alpha + \beta)$, and also $Nab \sin\gamma$, we have

$$XY \sin(\alpha + \beta) \quad = \quad Nab \sin\gamma$$

Since $\sin(\alpha + \beta) = \sin\gamma$, we have

$$Nab \quad = \quad XY$$
$$Nab \quad = \quad \mu^2 ac$$
$$Nb \quad = \quad \mu^2 c$$
$$\mu^2 \quad = \quad Nb/c$$

That completes the proof of the lemma.

Lemma 44. *Let N and ABC be given, and suppose ABC is isosceles with vertex angle α at B and base angles $\alpha + \beta$. Let (a, b, c) be the tile with angles (α, β, γ) such that (a, b, c) are relatively prime integers. Then Nbc is a square; equivalently, the square-free parts of N and bc are equal.*

Remark. Since b and c are relatively prime, the square-free part of bc is the product of the square-free parts of b and c.

Proof. By Lemma 42 we have $\mu c = X$ for some integer X. Then by Lemma 43 we have

$$\left(\frac{X}{c}\right)^2 \quad = \quad \frac{Nb}{c}$$

Multiplying by c^2 we have $X^2 = Nbc$, which is the first claim of the lemma. Therefore, the square-free part of Nbc is 1. Therefore the square-free parts of N and bc are equal. That completes the proof of the lemma.

Lemma 45. *Let N and ABC be given, and suppose ABC is isosceles with vertex angle α at B and base angles $\alpha + \beta$ with $3\alpha + 2\beta = \pi$. Suppose ABC is N-tiled by a tile with angles (α, β, γ), and integer sides (a, b, c) with no common factor. Let $g = \gcd(a, c)$. Then*

(i) The number of b edges of tiles on side AB is divisible by g, and the same for side BC.

(ii) μg is an integer, and μ is an integer if and only if $g|M$.

(iii) Let m be the number of b edges of tiles on the base AC, and M the coloring number of the tiling. Then $m \equiv -M \mod g$.

Proof. Ad (i). By Lemma 42, the length X of side AB is equal to μc, and the length Y of side AC is μa. Since AB and AC are exactly matched by the edges of tiles in the tiling, and the tile has integer sides, X and Y are integers. That is, both μc and μa are integers. Hence the denominator of μ divides both a and c. Hence the denominator of μ divides $g = \gcd(a, c)$. Hence μg is an integer.

In the tiling, AB is made up of edges of tiles, so for some integers p, d, and e, we have

$$\mu c \quad = \quad pa + db + ec$$
$$\mu g^2 \quad = \quad pa + db + ec \qquad \text{since } c = g^2 \text{ by Lemma 8}$$

Since μg is an integer, μg^2 is an integer divisible by g. Since a and c are also divisible by g, it follows that db is divisible by g. Since b and c are relatively prime, also b and g are relatively prime. Hence d is divisible by g. But d is the number

of b edges on side AB. That proves the first claim of the theorem. Since ABC is isosceles, the same argument applies to side BC. That proves claim (i) of the theorem.

Ad (ii). From $\mu = M(1+s)$ we have $\mu c = M(c+a)$ and hence $\mu g^2 = M(c+a) = M(g^2 + a)$. Dividing by g^2 we have

$$\mu = M + \left(\frac{M}{g}\right)\hat{a}.$$

Hence μg is an integer, namely $M(g + \hat{a})$, and μ is an integer if and only g divides M.

Ad (iii). The area equation is $X^2 = Nbc$. Hence $c = g^2$ divides X^2. Hence $g|X$. By the coloring equation (Theorem 1), we have $M(a+b+c) = 2X - Y$. Mod g we have $X \equiv 0$ and $a \equiv 0$ and $c \equiv 0$. Let m be the number of b edges on AC. Then $Y \equiv mb$ mod g. Then $M(a+b+c) = 2X - Y$ becomes, mod g,

$$Mb \quad \equiv \quad -mb$$

Since (a, b, c) have no common factor, b is relatively prime to g. Therefore

(28) $$M \quad \equiv \quad -m \text{ mod } g$$

Remark. Consider the case $N = 27$, $M = 3$, $\mu = 9/2$, $(a, b, c) = (2, 3, 4)$. These values of (N, M) satisfy all the equations above. In particular $g = 2$ and $2M = 6 \equiv 0 \text{ mod } 2$ but not $M \equiv 0$, and μ is not an integer. However, there does not exist any tiling corresponding to these values, as will be discussed below.

Lemma 45 is less than perfectly satisfactory, since it does not provide a necessary and sufficient condition for the existence of a tiling in the case of isosceles ABC with base angles $\alpha + \beta$, while we did find necessary and sufficient conditions when ABC has either the angles $(2\alpha, \beta, \alpha + \beta)$ or $(2\alpha, \alpha, 2\beta)$.

Lemma 46. *Let N and ABC be given, and suppose ABC is isosceles with vertex angle α at B and base angles $\alpha + \beta$ with $3\alpha + 2\beta = \pi$. Suppose ABC is N-tiled by a tile with angles (α, β, γ), and integer sides (a, b, c) with no common factor. Let $g = gcd(a, c)$ and let μ be as in Lemma 45. If b is squarefree and relatively prime to $c - a$, then g divides M and μ is an integer.*

Proof. Suppose b is squarefree. By the tiling equation of Theorem 17,

$$N(1 - s) \quad = \quad M^2(1 + s)$$

Multiplying both sides by c and using $s = a/c$ we have

$$N(c - a) \quad = \quad M^2(c + a)$$

(29) $$Nbc(c - a) \quad = \quad M^2(c + a)bc$$

Let X be the length of AB. From the area equation we have $X^2 = Nbc$. From the definition of μ we have $X = \mu c$. Hence $(\mu c)^2 = Nbc$. Substituting this value on the left of (29), we have

$$(\mu c)^2(c - a) \quad = \quad M^2(c + a)bc$$

Since b is relatively prime to $c - a$, it divides $(\mu c)^2$. Since b is squarefree, it divides μc. Let ℓ be the integer $\mu c/b$. Then $X = \mu c = \ell b$.

Now we construct a triangle DBE with the same vertex B as ABC, and whose base DE contains AC, such that DA and EC both have length ℓa. Then triangle

DBA has side DA of length ℓa and side AB of length ℓb, and angle γ at A. Hence it is similar to the tile. Hence it has angle β at D. Similarly, angle E is equal to β, so triangle DBE is isosceles with base angles β. We tile ABD and BCE with quadratic tilings, using the same tile as in the tiling of ABC. See Fig. 17. That is possible since $X = \ell b$. The number $g = gcd(a, c)$ is the same for both tilings. The coloring number of the new tiling is $2\ell + M$, since each of the two quadratic tilings has coloring number ℓ. By Theorem 14, g divides the coloring number of the new tiling. Hence $g | (2\ell + M)$. Let j be an integer such that $gj = 2\ell + M$. We have

$$
\begin{aligned}
gj &= 2\ell + M \\
&= \frac{2\mu c}{b} + M \text{qquad since } \ell = \mu c/b \\
gjb &= 2\mu c + Mb \\
&= 2M(a + c) + Mb \qquad \text{since } \mu = M(s + 1) \\
&= M(2a + b + c)
\end{aligned}
$$

Now g divides $2a + c$ but is relatively prime to b. Hence g does not divide $2a + b + c$. Hence g divides M. That completes the proof of the lemma.

Remark. The lemma permits us to reject some values of N, when the computed (a, b, c) does not satisfy the conditions of the lemma. But if the conditions are satisfied, that does not imply the existence of a tiling. It does imply that the larger triangle DBE can be tiled, but that tiling may not include a tiling of ABC.

11.4. **Results based on computation.** Since we lack a general existence theorem to complement the necessary conditions in the previous section, we resort to computation. Luckily, the theoretical considerations above are enough to reduce the question of existence of an N-tiling of some isosceles triangle with base angles $\alpha + \beta$ to computation. Namely, given N, there are only finitely many possibilities for the coloring number M, and each of these determines a unique tile (a, b, c) and a particular isosceles triangle ABC. Our theoretical considerations show that the question whether some ABC can be N-tiled by some tile is equivalent to the question whether this *particular ABC* can be tiled by this particular tile.

It is a computationally decidable question whether a particular tile (a, b, c) can be used to N-tile a particular triangle ABC. So, for each N, we can computationally determine whether there is *any* N-tiling of *any* ABC. That program is difficult to write and for large N it is slow. We therefore develop a computational "filter" that checks various necessary conditions. Then only the N that pass these tests are serious candidates for further consideration.

Here we summarize the necessary conditions to be checked.

Starting with N, there are finitely many squares M^2 less than N. For each of these, the number s is determined by

$$
s = \frac{N - M^2}{N + M^2}.
$$

Then we determine the tile (a, b, c) by the requirements that (a, b, c) are integers, $a/c = s$, and (a, b, c) have no common factor. We note that b is an integer if and only if c divides a^2, by Lemma 7. By Lemma 44, we can reject N if Nbc is not a square. Assuming Nbc is a square, we can determine μ by the equation

$$
\mu^2 = Nb/c.
$$

Then we can determine the lengths X and Y of the sides of ABC by $X = \mu c$ and $Y = \mu a$, according to Lemma 42. X is an integer since $\mu c = \sqrt{Nbc}$. We have $\mu^2 a^2 = Nba^2/c$, which is an integer since c divides a^2. Then μa is the rational square root of an integer, so $Y = \mu a$ is an integer.

Next, we recall that in an N-tiling of ABC,

(i) There are at least two c edges on each of the three sides of ABC, by Lemma 6.

(ii) Since the angle at the top vertex B is α, either side AB or side BC has at least one b edge.

(iii) All the conditions listed in Lemmas 45 and 46 are satisfied. In particular $g\mu$ is an integer.

Having determined X, Y, and (a, b, c), it is now a computational matter to check whether all the above conditions are met.

Figs. 26 and 25 exhibit a SageMath program that prints a list of all values of N that pass the above tests, up to a specified maximum value of N. This code runs for N up to 10000 in a few minutes (though it made my laptop use all its resources and turn on its fan).

Corollary 4 (to Lemma 45). *For $N \leq 108$, if there are N-tilings of an isosceles triangle with vertex angle α, then N, M, μ, and (a, b, c) occur in Table 4.*

TABLE 4. Possible tilings of isosceles ABC with base angles $\alpha + \beta$

N	M	μ	(a, b, c)	tiling exists
45	3	5	(6,5,9)	?
48	4	6	(2,3,4)	yes
64	4	32/5	(15,16,25)	?
72	6	8	(3,8,9)	?
81	3	27/5	(20,9,25)	?
90	6	60/7	(21,40,49)	?
96	4	48/7	(35,24,49)	?
100	5	8	(15,16,25)	?
108	6	9	(2,3,4)	yes

Remarks. The number of questions marks is disappointing. The experimental evidence supports the conjecture that μ is always an integer, but we could not prove it. At least the code (when run far beyond the values shown in the table) provides experimental confirmation of Theorem 18: there are no prime values of N in the output.

Proof. The table is generated by running the SageMath code in Figs. 25 and 26. The entries of "yes" are inserted manually after actually constructing a tiling (see the figures).

FIGURE 25. Code to find candidates for an N-tiling of isosceles ABC

```
def check_edges(X,a,b,c):
# can X be written as pa+qb+rc with q > 0 and p,q,r >= 0 and g divides q ?
  g = gcd(a,c)
  for p in range(0,X/a+1):
     for q in range(g,X/b+1,g):
         for r in range(0,X/c+1):
             if X == p*a + q*b + r*c:
                 return true
   return false

def check_base(Y,a,b,c,M):
# can Y be written as pa+qb+rc with  p,q,r >= 0 and q congruent to -M mod g
    g = gcd(a,c)
    start = int(M/g)* g - M;
    if start < 0:
        start += g
    for p in range(0,Y/a+1):
        for q in range(start,Y/b+1,g):
            for r in range(0,Y/c+1):
                if Y == p*a + q*b + r*c:
                    return true
    return false
```

FIGURE 26. Code to find candidates for an N-tiling of isosceles ABC

```
def notImpossibleIsosceles(Nmax):
# list possibilities for N up to Nmax
   for N in range(1,Nmax+1):
      for M in range(1, int(sqrt(N))):
          a = N-M^2
          c = N+M^2
          b = c - a^2/c
          if not b in ZZ:
             a = a*c
             c = c^2
             b = c - a^2/c
          g = gcd(a,gcd(b,c))
          if not g==1:
             a = a/g
             b = b/g
             c = c/g
          s = a/c
          g = gcd(a,c)
          if not (c == g^2):
             continue
          mu = M*(1+s)
          if not g*mu in ZZ:
             continue
          if not is_square(N*b*c):
             continue
          if is_squarefree(b) and gcd(c-a,b) == 1:
             if not mu in ZZ:
                 continue    # no such tiling can exist
          X = mu*c
          Y = mu*a
          if check_edges(X-2*c,a,b,c) and check_base(Y-2*c,a,b,c,M):
             print([N,M,mu,a,b,c])
```

11.5. **Construction of N-tilings of isosceles ABC.** We first remark that some of the diagrams in [4] imply the existence of tilings. Those diagrams imply that, in case the diagrams as shown actually exist, and the figures in them are rational, then for large enough N, there are N-tilings of some triangle with the specified shape. We tried to construct tilings from Laczkovich's diagram for the case of isosceles ABC with base angles $\alpha + \beta$, but satisfying the many least-common-multiple conditions on the interior edges requires extremely large values of N. These diagrams did not result in any tilings small enough to draw. We found the examples given above independently, more or less by trial and error, and initially were hopeful that a similar construction could be proved to work in general (when the necessary conditions above are satisfied). That was not the case; the construction that worked so well for a few small examples fails for other cases we tried.

In the program that implements our method, certain conditions must be satisfied for it to work. It is possible to formulate these conditions and state a theorem, but the list of conditions is complicated and we do not get a necessary-and-sufficient condition for the existence of tilings. Moreover, the program is easy to reconstruct by looking at the tilings above. We therefore give here only a brief discussion of the conditions needed for it to succeed, and a proof that when it succeeds, the tiling uses exactly N tiles.

We suppose that all the necessary conditions in Corollary 4 are satisfied. In addition, we assume $\mu \geq c$.

Example. With $N = 112$, $M = 4$, $\mu = 7$, and $(a, b, c) = (12, 7, 16)$, the condition $\mu \geq c$ is $7 \geq 16$, which is false (and the condition $\mu \geq c(a/b)$ is also false), although the first two conditions hold. And indeed the construction used in the tilings we have exhibited fails too, because the line proceeding northeast from A to the center at angle α to the horizontal will meet the other side before the b edges on one side and the a edges on the other side have a common vertex. At present we do not know if isosceles triangle ABC with base 84 and sides 112 can be tiled by this tile.

Our method starts with N and M. Define the scaling factor

$$\mu = M(1 + s).$$

We generalize the construction of the 48-tiling in Fig. 21, which will require certain number-theoretic relations between a, b, and c in order that the boundaries of the colored regions in the figure line up properly.

We start with a positive integer p; the green quadratic tiling will have p^2 tiles. In order that the boundary between green and blue be possible, we must choose p to be a multiple of a. Since p is a multiple of a, and by hypothesis c divides a^2, the following conditions are satisfied:

(i) a divides pb and pc
(ii) c divides pa

We have assumed that $c \leq \mu$. Therefore it is possible to choose p to be a multiple of a such that

(iii) $pc \leq \mu a$

In case $a > b$ we also will need

(iv) $p > a^2/b$

(In practice, we try all the values of p allowed by these constraints.) The tiling is made of quadratic tilings and parallelograms. The number of tiles required is computed by the code in Fig. 11.5, which also suggests how the figure is constructed. We start with the green quadratic tiling, then draw the adjacent light blue tiling; the red tiling east of the green; then the yellow quadratic tiling at the top.

That is where we encounter a problem: the yellow quadratic tiling sometimes overlaps the red, ruining the tiling.

Example. The least example of this turns out to be $N = 1008$ with the tile $(12, 7, 16)$, as shown in Fig. 22. We have $\mu = 21$, so the hypothesis $\mu > c$ is satisfied.

We count the tiles using the SageMath code shown in Fig. 11.5. We use Q for the number of tiles, until we prove it is equal to N. When we run `countTiles(p,M,s)`, it returns $M^2(1+s)/(1-s)$, which is N, according to Theorem 17.[2] It is remarkable that SageMath can perform the algebra without worrying whether the yellow overlaps the red. The algebra works out correctly even in that case; some of the areas get counted with a negative sign to compensate. So the area comes out right, if we allow negative areas and double coverings! The code does not need to worry whether the final parallelogram can actually be tiled.

```
def countTiles(p,M,s):
    q = p*(1-s^2)/s
    mu = M*(1+s)
    r = mu - p/s
    t = p*s
    u = (mu-q)/(1-s^2)
    m = (t+r)*s/(1-s^2)
    green = p^2
    blue = q^2
    red = t^2 + 2*t*r
    pink = m^2
    yellow = u^2
    Q = green+blue+red+pink+yellow
    # print([q,mu,r,t,u,m])
    # print([green,blue,red,pink,yellow])
    # add in orange plus violet
    Q = Q + 2*q*m*(1-s^2)/s - 2*m^2
    Q=factor(expand(Q))
    return Q
```

12. THE CASE WHEN ABC IS ISOSCELES WITH BASE ANGLES α

Theorem 19. *Suppose $3\alpha + 2\beta = \pi$. Let ABC be isosceles with base angles α. Suppose ABC is N-tiled by the tile whose integer sides (a, b, c) have no common factor, and whose angles are (α, β, γ). Let $s = a/c$ and $g = \gcd(a, c)$, and let M*

---

[2]It won't work just to cut and paste this code, because after that you will have to ensure that what looks like spaces at the beginnings of the lines are tabs. Look up the Unix programs `expand` and `unexpand` if you want to run this code, or any other Python code you cut and pasted.
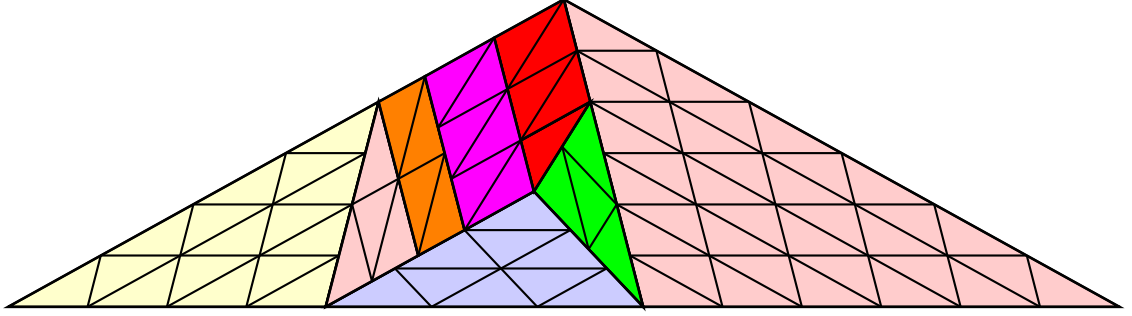
FIGURE 27. $N = 84$. ABC is isosceles with base angles α. $(a, b, c) = (2, 3, 4)$

be the coloring number of the tiling. Then g divides M, $M^2 \leq 2N$, and (M, s) is a solution of the "isosceles-α tiling equation"

$$N = M^2 \frac{(1+s)(2-s^2)}{(1-s)(2+s)^2}$$

Proof. Since the fraction in the equation is a monotonically increasing function of s on the interval $(0, 1)$, and takes on every value greater than or equal to $1/2$, there is one and only one solution for each (N, M) such that $N \geq M^2/2$, and no solution when $N < M^2/2$. Hence it suffices to show that if there is an N-tiling with coloring number M, then the equation is satisfied, and g divides M.

Consider tilings of a triangle ABC with base angles α. Then the vertex angle is $\alpha + 2\beta$. We have

$$
\begin{aligned}
\sin(\alpha + 2\beta) &= \sin(\pi - 2\alpha) \\
&= \sin 2\alpha \\
&= (2 - s^2)\sin\alpha \qquad \text{by Lemma 38}
\end{aligned}
$$

Twice the area of ABC is $X^2 \sin(\alpha + 2\beta)$. Hence

$$
\begin{aligned}
(30) \qquad Nbc\sin\alpha &= X^2\sin(\alpha + 2\beta) \\
&= X^2(2 - s^2)\sin\alpha \\
(31) \qquad Nbc &= X^2(2 - s^2)
\end{aligned}
$$

We have $X = \lambda\sin\beta = \mu b$ (that defines μ), so

$$
\begin{aligned}
Nbc &= \mu^2 b^2 (2 - s^2) \\
N &= \mu^2 \frac{b}{c}(2 - s^2) \\
(32) \qquad N &= \mu^2(1 - s^2)(2 - s^2)
\end{aligned}
$$

Since $X = \mu b$, by the law of sines we have

$$
\begin{aligned}
Y &= X\frac{\sin(\alpha + 2\beta)}{\sin\alpha} \\
&= X\frac{(2 - s^2)\sin\alpha}{\sin\alpha} \\
&= X(2 - s^2) \\
&= \mu b(2 - s^2)
\end{aligned}
$$

The coloring equation is

$$
\begin{aligned}
M(a + b + c) &= 2X + Y \\
&= X(2 + 2 - s^2) \\
&= X(4 - s^2) \\
&= \mu b(4 - s^2)
\end{aligned}
$$

Dividing by c and expressing everything in terms of s,

$$
\begin{aligned}
M(s + (1 - s^2) + 1) &= \mu(1 - s^2)(4 - s^2) \\
M(2 - s)(1 + s) &= \mu(1 - s)(1 + s)(2 - s)(2 + s) \\
M &= \mu(1 - s)(2 + s)
\end{aligned}
$$

Hence

$$
\mu = \frac{M}{(1 - s)(2 + s)}
$$

We can square that equation and substitute for μ^2 in (32) to obtain the area equation for this shape of ABC.

$$
\begin{aligned}
N &= \frac{M^2}{(1 - s)^2(2 + s)^2}(1 - s^2)(2 - s^2) \\
\text{(33)} \qquad N &= M^2\frac{(1 + s)(2 - s^2)}{(1 - s)(2 + s)^2}
\end{aligned}
$$

That completes the proof of the tiling equation in the theorem.

It remains to prove that g divides M. Recall (31):

$$
Nbc = X^2(2 - s^2)
$$

We substitute the value of N from (33):

$$
M^2\frac{(1 + s)(2 - s^2)}{(1 - s)(2 + s)^2} = X^2(2 - s^2)
$$

Cancel $(2 - s^2)$, multiply both sides by the denominator and by c^3, and use $s = a/c$; we obtain

$$
\text{(34)} \qquad M^2 bc^3(a + c) = (c - a)(2c + a)^2 X^2
$$

Assume, for proof by contradiction, that g does not divide M. Then the left side is divisible by g^4. Therefore the right side is also divisible by g^4. The factor $(c - a)(2c + a)^2$ is plainly divisible by g^3. By Lemma **??**, it is not divisible by g^4. Therefore, g divides X^2. By Lemma 8, g is squarefree. Hence g divides X. Then the right side of (34) is divisible by g^5. According to Lemma 8, $c = g^2$, and a is not divisible by c since $a < c$; so $a + c$ is not divisible by g^2. Hence $bc^3(a + c)$ is not divisible by g^5. But $M^2 bc^3(a + c)$, which is the left side of (34), must be divisible by g^5 since the right side is. Hence g divides M^2. But g is squarefree, so g divides M. That completes the proof of the theorem.

12.1. N is not prime when ABC is isosceles with base angles α.

Theorem 20. *Suppose $3\alpha + 2\beta = \pi$. Let ABC be isosceles with base angles α. Suppose ABC is N-tiled by a tile whose angles are (α, β, γ). Then N is not a prime number.*

Proof. By Theorem 3, the tile is rational. Therefore we may assume without loss of generality that the tile has integer sides (a, b, c) with no common factor. Let $g = \gcd(a, b, c)$ and $s = a/c$. Let M be the coloring number of the tiling. According to Theorem 19,

$$N = M^2 \frac{(1+s)(2-s^2)}{(1-s)(2+s)^2}$$

Multiplying by the denominator and using $s = a/c$, we have

$$N(c-a)(2c+a)^2 = M^2(a+c)(2c^2-a^2)$$

According to Theorem 19, g divides M. Therefore g^5 divides the right side. Therefore g^5 divides the left side $N(c-a)(2c+a)^2$. By Lemma 41, g^4 does not divide $(c-a)(2c+a)^2$. But g^5 divides $N(c-a)(2c-a)^2$ since that is the left side of (35). Therefore g^2 divides N. Now $g \neq 1$, since if $g = 1$ then $c = g^2 = 1$, but $a < c$ and a is a positive integer, contradiction. Hence N, being divisible by g^2 with $g \neq 1$, is not a prime number. That completes the proof of the theorem.

12.2. Computational results.

Corollary 5 (to Theorem 14). *All possible N-tilings of isosceles triangles with base angles α and $3\alpha + 2\beta = \pi$, and $N \leq 200$, are listed in Table 5.*

TABLE 5. Solutions of the isosceles-α tiling equation for $N \leq 500$

N	M	(a, b, c)	tiling exists
21	5	(2, 3, 4)	no
34	7	(3, 8, 9)	no
70	8	(6, 5, 9)	?
84	10	(2, 3, 4)	yes
136	14	(3, 8, 9)	?
161	11	(12, 7, 16)	?
164	13	(15, 16, 25)	?
189	15	(2, 3, 4)	?
280	16	(6, 5, 9)	?
294	22	(5, 24, 25)	?
306	14	(20, 9, 25)	?
306	21	(3, 8, 9)	?
336	20	(2, 3, 4)	yes
438	19	(35, 24, 49)	?
465	27	(4, 15, 16)	?

Proof. Theorem 14 is implemented in the SageMath program shown in Fig. 28, which (embedded in a top-level loop from 1 to 200, not shown) printed the table. The line with `solve` finds the solutions of the cubic equation in the theorem, and the line with `s in QQ` tests whether a solution s is rational. Just one line is required

FIGURE 28. SageMath code to solve the isosceles-α tiling equation

```
def IsoscelesAlpha(N):
   for M in range(1, sqrt(2*N)+1):
      slist = solve(N*(1-x)*(2+x)^2 == M^2*(1+x)*(2-x^2),x)
      ell = len(slist)
      for i in range(0,ell):
         s = slist[i].rhs()
         if s <= 0 or s >= 1:
            continue
         if not s in QQ:
            continue
         (a,b,c) = getABC(s) # See Fig. 11
         print("%d & %d & (%d, %d, %d) & ? \\\\" % (N,M,a,b,c))
```

for each of those steps in SageMath. "No" entries in the table represent cases where a computer search for boundary tilings showed there are none.

13. Searching for tilings

In preceding sections we have given necessary equations, that must be solvable if there is an N-tiling, one equation for each possible shape of ABC. In two of the cases we were able to give matching sufficient conditions that gave a complete solution of the question as to which ABC have an N-tiling. In other cases we could not find such sufficient conditions, thus leaving the question unsolved for N such that the equations are satisfied but no tiling is known.

Our equations, however, do show that if N is given, then the possible shapes of ABC and the tile are restricted to a finite set, easily computable from N. Therefore the question, for a particular N, whether an N-tiling of some ABC by some tile exists, reduces to the question whether an N-tiling of a particular ABC by a particular tile exists. That question is in turn, in principle, decidable by a trial-and-error search; it amounts to assembling N tiles as a jigsaw puzzle into a frame with the shape of ABC.

This is a standard search problem, a staple of undergraduate computer science, and we wrote a standard program to solve it, using C++ for efficiency. We also used some custom touches for efficiency: we searched by first placing tiles along the boundary (a standard technique in human jigsaw-puzzle solving), and if a tile touched a previously-laid tile, we computed the area of the "hole" created, rejecting the tiling if the hole has an area that is not a multiple of the tile's area.

The question naturally arises whether this program is actually correct. To address that question, the program was written with a "verbose" option, which causes it to draw pictures of each stage of the search. In this way we produced LaTeX documents with several thousand pages, each containing a picture of a rejected partial tiling. Somewhat to our amazement, LaTeX and associated software were able to display these large files, and the program appeared to be searching as designed. There might still be an error in the program causing us to miss a tiling, just as any human-written proof might contain an error.

As an example, we mention some of the results searching for tilings of isosceles ABC with base angles $\alpha + \beta$. For the value $N = 12$ it draws 228 pictures, showing

each "blocked" partial boundary tiling that cannot be continued. $N = 12$ and $N = 18$ were ruled out, which at the time was new information, since Lemma 6 asserting that there are at least two c edges on each side of ABC had not yet been proved, so there were four more entries at the beginning of Table 4. For isosceles ABC with base angles α, there still are two small values of N that can be eliminated by exhaustive search for a boundary tiling. The largest value of N for which we could complete an exhaustive search (for any shape of ABC) was $N = 23$, although the successful search in the case $N = 28$ led to the discovery of the triquadratic tilings. For $N = 45$, 72, and 75, we found thousands of boundary tilings, but they could not be completed to tilings of ABC. We ran this program for $N = 99$ for more than 48 hours. Given these disappointing results, we did not extend our program to search for a completion of a boundary tiling to a full tiling.

14. Conclusions

In this paper we assumed $3\alpha + 2\beta = \pi$ and studied N-tilings of triangles ABC by a tile with angles (α, β, γ). It was already known that (except for the case $\alpha = \pi/8$) the angles cannot be rational multiples of π, so there are only five possible shapes of ABC, given (α, β, γ). We showed that the tile must be rational, and by means of the area equation and coloring equation, we derived a "tiling equation" for each of the five possible shapes of ABC. By means of these tiling equations, we reduced the existentially quantified question whether there is an N-tiling of *some ABC* by *some* tile, to the specific question of whether there is an N-tiling of a specific, particular ABC by a specific, particular tile (or tiles) depending on N. The tiling equations supply necessary conditions by which we can rule out a great many values of N. In particular, we were able to rule out prime values of N:

Theorem 21. *Let ABC be N-tiled by a tile with angles (α, β, γ) such that $3\alpha + \beta = \pi$, and ABC is not similar to the tile. Then N is not prime.*

Proof. By [4] (as detailed in Lemma 1) ABC has one of the five shapes considered in this paper. For each of those shapes, we have proved that N is not prime, in Theorems 8, 12, 18, 15, and 20. That completes the proof.

We list the five shapes and the corresponding necessary tiling equations in Table 6. In each case, there are finitely many possible values of s, determined by N and the coloring number M, which has to be less than N, so that for each particular N it can be checked by computation, and very efficiently, whether the equations have a solution or not. For three of the five shapes we could supply sufficient conditions that exactly match the necessary conditions, which are also given in the table.

Table 7 lists the N-tilings that we discovered, for $N \leq 100$. Those tilings, and others for larger N, are illustrated elsewhere in this paper. In that table there is one tiling for each of the five shapes of ABC.

Using the equations in Table 6, we can list some values of N for which the equations have solutions, and therefore the existence of an N-tiling is not ruled out. There are still 12 values of $N \leq 100$ for which we do not know if an N-tiling exists. See Tables 5 and 4.

For each N, these equations do restrict the possible shape of the tile to a finite number (usually just one) of specific tiles. Therefore the existence or non-existence of a tiling can in principle be verified by computation.

TABLE 6. Tiling equations when $3\alpha + 2\beta = \pi$, with $s = a/c$

ABC	Necessary	Sufficient
$(2\alpha, \beta, \alpha + \beta)$	$\dfrac{N}{M^2} = 2s^2 - 1$	$s = K/M$ and $K \mid M^2$
$(2\alpha, \alpha, 2\beta)$	$\dfrac{N}{M^2} = \dfrac{(2 - s^2)(3 - s^2)}{(1 - s)^2(2 + s)^2}$	$c \mid a^2$
$(\beta, \beta, 3\alpha)$	$\dfrac{N}{M^2} = \dfrac{3 - s^2}{(1 + s)^2}$	$g = \gcd(a, c) \mid M$
$(\alpha + \beta, \alpha + \beta, \alpha)$	$\dfrac{N}{M^2} = \dfrac{1 + s}{1 - s}$	?
$(\alpha, \alpha, \alpha + 2\beta)$	$\dfrac{N}{M^2} = \dfrac{(1 + s)(2 - s^2)}{(1 - s)(2 + s)^2}$	?

TABLE 7. N-tilings with $3\alpha + 2\beta = \pi$ for $N \leq 100$

N	M	(a, b, c)	(A, B, C)
28	2	(2, 3, 4)	triquadratic
44	6	(2, 3, 4)	isosceles-β
48	4	(2, 3, 4)	isosceles-$\alpha + \beta$
77	5	(2, 3, 4)	$(\alpha, 2\alpha, 2\beta)$
84	10	(2, 3, 4)	isosceles-α

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