Stroboscopic Generation of Topological Protection

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Trapped neutral atoms offer a powerful route to robust simulation of complex quantum systems. We present here a stroboscopic scheme for realization of a Hamiltonian with n-body interactions on a set of neutral atoms trapped in an addressable optical lattice, using only 1- and 2-body physical operations together with a dissipative mechanism that allows thermalization to finite temperature or cooling to the ground state. We demonstrate this scheme with application to the toric code Hamiltonian, ground states of which can be used to robustly store quantum information when coupled to a low temperature reservoir.

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Introduction.—Among the most exciting aspects of quantum simulation is the possibility of generating and studying exotic quantum phases such as those possessing topological order that can be used to robustly store and process quantum information. The Hamiltonians governing these phases frequently require more-than-2-body interactions that are hard or even impossible to realize naturally. This difficulty has spurred much theoretical and experimental effort in the artificial engineering of Hamiltonians, particularly for trapped neutral atoms [1]. Many proposals have been made for the generation of 2-body Hamiltonians using static emulation schemes and some experimental realizations have appeared [2,3]. Specific proposals have appeared for generating n-body interactions [4], but have focused on static emulation.

We present here an alternative, dynamic emulation approach to systematic generation of n-body interactions that is based on sequences of control pulses which individually realize 1- and 2-body operations on internal atomic levels. We show that this stroboscopic realization of the Hamiltonian can be implemented simultaneously with a dissipative thermalization protocol to stabilize the system from the effects of imperfect quantum operations and environmental noise. In the zero temperature limit, this can be viewed as replacing algorithmic error correction in an equivalent quantum circuit model with a dissipative procedure to remove errors [5]. The resource requirements for this thermalization protocol are different from those of algorithmic error correction, and may be more accessible to experiment in the foreseeable future. We illustrate the approach here with stroboscopic generation of the four-body toric code Hamiltonian, which constitutes one of the simplest exactly solvable models with a ground state topological phase [6]:

$$H_0^{\mathrm{TC}} = -J_e \sum_{v} \prod_{j \in v} \sigma_j^z - J_m \sum_{p} \prod_{j \in p} \sigma_j^x, \qquad (1)$$

where σ_j denotes a Pauli operator on the links of a square lattice and $v(p)$ denote the vertex (plaquette) of the lattice. The ground state of this model possesses topological order, and therefore has anyonic quasiparticle excitations and, on a lattice with periodic boundary conditions, an emergent topological degeneracy. Quantum information can be encoded in this ground state degeneracy and manipulated with controlled creation and braiding of anyons [6,7]. In a finite sized system [8,9], the topological order of the ground state and gap to excited states protects against decoherence and loss of quantum information due to noise provided the system is coupled to a low temperature bath. Our analysis below will provide a scheme for generating both H_0^{TC} and an effective low temperature bath, realizing the topological protection characteristic of the toric code.

The physical context for our analysis is a set of $\sim$250 individual ^{133}Cs atoms trapped at the sites of an addressable simple cubic optical lattice [10]. A lattice spacing of 5 μm [10] allows essentially perfect addressability [11]. The orbital degrees of freedom are frozen on the time scales relevant to our analysis and we need consider only internal atomic degrees of freedom. Two hyperfine levels (e.g., $|F, m_F\rangle = |4, 4\rangle$, $|3, 3\rangle$) define a 2-level pseudospin system. We realize H_0^{TC} in the interaction representation defined by the pseudospin energies. Auxiliary internal levels are used to realize 1-spin and 2-spin quantum operations, using optical frequency Raman pulses to generate arbitrary single-spin operations and excitation of one atom to a Rydberg state, e.g., the $n \approx 80$ state, to generate controlled-phase gates, CPHASE [12]. To achieve thermalization or cooling, the Hamiltonian H_0^{TC} is supplemented by coupling the primary system spins to an ancillary set of pseudospins that will be dissipatively controlled to simulate a thermal reservoir. Since the pseudospins are localized at the sites of a cubic lattice, one can choose to either realize H_0^{TC} on a single plane using a surface code [13] or

in a three-dimensional cubic array with toroidal boundary conditions realized by SWAP operations (see Fig. 1).

Effective Hamiltonian evolution.—Given the ability to perform both Rydberg-induced CPHASE gates between atoms in neighboring sites and arbitrary 1-body rotations, $\exp(-i\theta\sigma_j)$, on individual atoms, where θ is a variable phase angle, sequences of these operations can be chosen to generate effective n-body interactions through high-order terms in the Magnus expansion [14], allowing stroboscopic simulation of a broad class of Hamiltonians. Consider the operator sequence $U_n U_{n-1} \cdots U_2 U_1$, where the U_j are the one- or two-body gates described above. Effective interactions are found through

$$H_{\mathrm{eff}}(t) \equiv \frac{i\hbar}{t} \ln(U_n U_{n-1} \cdots U_2 U_1) = \sum_j \frac{i\hbar}{t} \ln U_j - \sum_{j<k} \frac{i\hbar}{2t} [\ln U_j, \ln U_k] + \mathcal{O}(\|\ln U\|^3).$$

Consider now simulation of the four-body interactions in H_0^{TC}. We use the notation $U_j(\phi) \equiv e^{-i\phi\Sigma_j}$ and define $\Sigma_1 = \sigma^z\sigma^y\sigma^0\sigma^0$, $\Sigma_2 = \sigma^0\sigma^x\sigma^y\sigma^0$, and $\Sigma_3 = \sigma^0\sigma^0\sigma^x\sigma^z$, where σ^0 is the identity operator. For simplicity, it is assumed that each $U_j(\alpha)$ takes a time τ to execute. We construct the operator sequence

$$U_{123}(\alpha, \beta, \gamma) = U_{12}(\alpha, \beta) U_3(\gamma) U_{12}^\dagger(\alpha, \beta) U_3^\dagger(\gamma), \quad (2)$$

where $U_{12}(\alpha, \beta) = U_2(\beta) U_1(\alpha) U_2^\dagger(\beta) U_1^\dagger(\alpha)$. This sequence acts over a time 10τ to generate the following effective Hamiltonian at a single vertex v:

$$H_{\mathrm{eff}}^{zzzz} = J_e \prod_{j\in v} \sigma_j^z + \frac{\chi}{\alpha\gamma} (\alpha [\sigma^0\sigma^x\sigma^z\sigma^z]_v + \gamma [\sigma^z\sigma^z\sigma^y\sigma^0]_v) + \chi([\sigma^0\sigma^x\sigma^y\sigma^0]_v - 2\beta/\gamma [\sigma^0\sigma^0\sigma^x\sigma^z]_v) + \mathcal{O}(\phi^6),$$

where $[O]_{v(p)}$ denotes the application of the (up to) four-body operator O to the spins meeting at a vertex v or surrounding a plaquette p, we choose $|\alpha| = |\beta| = |\gamma| \equiv \phi$, and $J_e = \chi(1 - 3\phi^2)/\alpha\gamma + \mathcal{O}(\phi^6)$ with $\chi \equiv \alpha^2\beta\gamma^2(2\hbar/5\tau)$. By repeating the operator sequence a second time with sign reversals $\alpha \to -\alpha$ and $\gamma \to -\gamma$, we cancel the fourth order terms in ϕ, giving $U_{123}(-\alpha, \beta, -\gamma) U_{123}(\alpha, \beta, \gamma)$ that acts for a time 20τ to generate the effective Hamiltonian

$$H_{\mathrm{eff}}^{zzzz} = J_e \prod_{j\in v} \sigma_j^z + \chi [\sigma^0\sigma^x\sigma^y\sigma^0]_v + \mathcal{O}(\phi^6). \quad (3)$$

The sequence $U_{123}(-\alpha, \beta, -\gamma) U_{123}(\alpha, \beta, \gamma)$ is specifically designed to cancel the lowest-order (ϕ^4) perturbation terms without affecting the gap. The remaining ϕ^5 term is a 2-body perturbation to H_0^{TC}. Repeating this sequence with appropriate sign reversals will cancel these higher order terms. However, the ground state subspace of H_0^{TC} is robust to these remaining perturbations (see below). A shorter operator sequence may then be preferable to reduce gate errors. The plaquette operator H_{eff}^{xxxx} can be generated by cyclic permutation of the Pauli operators in the above expressions for Σ_1, Σ_2, and Σ_3.

Simulation of H_0^{TC} then requires application of the pulse sequence to all vertices and plaquettes conforming to a two-dimensional square lattice with periodic boundary conditions. Vertex and plaquette terms may be applied serially as $\exp(-iH_{\mathrm{eff}}^{xxxx} t/\hbar) \exp(-iH_{\mathrm{eff}}^{zzzz} t/\hbar) \approx \exp -i(H_{\mathrm{eff}}^{xxxx} + H_{\mathrm{eff}}^{zzzz}) t/\hbar$. Because only the perturbation terms fail to commute, the truncation error in the above expression occurs at orders larger than ϕ^7. For 18 pseudospins, representing a 3×3 system with toroidal boundary conditions, a completely serial implementation yields a stroboscopic cycle time of 720 μs using the estimate $\tau \sim 500$ ns [15] and the minimal count of one CPHASE and four one-spin gates to realize all $U_j(\alpha)$ [16]. This may be reduced by implementing some operators in parallel.

Simulated thermalization.—The pseudospin subspace of the system defined by the internal states of the trapped atoms will interact with the external environment through the controlled quantum operations in the above pulse sequences and uncontrolled noisy interactions. Noise in the optical lattice system will not drive the simulation subspace to a state that is thermal under the simulated Hamiltonian [17]. Additionally, noise in the above sequence of control gates will add entropy and effectively heat the system. The entropy production ΔS resulting from imperfect gate operation is approximately $\Delta S \sim \mathrm{EPG}$, where EPG is the error per gate [18]. Quantum circuit models are usually supplemented by error correction schemes to effectively remove entropy from the system. We take a different approach here, constructing an effective system-reservoir interaction to control system entropy and relax the system to the ground state or a thermal state.

To maintain the simulated system at a thermal steady state we add an interaction H_{sr} of the system pseudospins with a set of ancillary pseudospins. In the optical lattice system, these ancillary pseudospins, which may be a second species of atom, are trapped in an offset, intercalated optical lattice, such that each ancillary atom is adjacent to a system atom. Consider a Hamiltonian with local n-body interactions of the form $H_0 = -\sum_\nu J_\nu \sum_{\mathcal{N}} h_{\mathcal{N}}^\nu$, where $h_{\mathcal{N}}^\nu$ is an n-body operator involving a neighborhood of pseudospins $\mathcal{N}$, including pseudospin i, with eigenvalues ± 1, ν labels the type of interaction, and J_ν is a constant. Additionally we define the pseudospin flip operator Σ_i^ν such that $\Sigma_i^\nu |h_{\mathcal{N}}^\nu = \pm 1\rangle = |h_{\mathcal{N}}^\nu = \mp 1\rangle$ when $i \in \mathcal{N}$. When all $[h_{\mathcal{N}}^\nu, h_{\mathcal{N}'}^\mu] = 0$, as is the case for Eq. (1), we can define $E_{i,\nu}^\dagger = \frac{1}{4}\Sigma_i^\nu (\mathbb{1} + h_{\mathcal{N}}^\nu)(\mathbb{1} + h_{\mathcal{N}'}^\nu)$, $T_{i,\nu} = \frac{1}{4}\Sigma_i^\nu (\mathbb{1} - h_{\mathcal{N}}^\nu)(\mathbb{1} + h_{\mathcal{N}'}^\nu)$, with $i = \mathcal{N} \cap \mathcal{N}'$. These are $(2n-1)$-body interactions; $E_{i,\nu}^\dagger$ creates a pair of excitations about i and $T_{i,\nu}$ translates an excitation about i. The

energy gap for creation of a pair of excitations is $\Delta_\nu = 4J_\nu$.

A route to guaranteeing the thermal equilibration of this system is for it to evolve under the Lindblad master equation $\dot{\rho} = -i/\hbar[H_0, \rho] + L[\rho]$, where ρ is the density matrix and $L[\rho]$ is the superoperator $L[\rho] = \sum_\omega (2c_\omega \rho c_\omega^\dagger - c_\omega^\dagger c_\omega \rho - \rho c_\omega^\dagger c_\omega)$, with $\{c_\omega, c_\omega^\dagger\}$ the Lindblad operators. With $\{c_{i,\nu}\}$ given by

$$\left\{\sqrt{\frac{1-p}{2}}\lambda^* E_{i,\nu}, \sqrt{\frac{p}{2}\lambda^*} E_{i,\nu}^\dagger, \sqrt{\frac{\gamma^*}{4}} T_{i,\nu}, \sqrt{\frac{\gamma^*}{4}} T_{i,\nu}^\dagger\right\}, \quad (4)$$

the Lindblad master equation describes equilibration with a bath of temperature $T = -\Delta/\ln(p)$. The unique stationary state of the system is then the thermal state under H_0 with temperature T. λ^* and γ^* are relaxation rates, and their values dictate the thermalization time. For simplicity we have set $\Delta_\nu = \Delta$. To generate evolution under such a master equation, we introduce a set of noninteracting ancillary pseudospins that independently undergo strong dissipation. Each local neighborhood of the system interacts locally with a single thermal ancillary pseudospin $\mathbf{T}_{i,\nu}$ and a single maximally mixed ancillary pseudospin $\mathbf{M}_{i,\nu}$ for each ν via $H_{\mathrm{sr}} = g\sum_{\nu,i}(E_{i,\nu}^\dagger \sigma_{\mathbf{T}_{i,\nu}}^- + T_{i,\nu}\sigma_{\mathbf{M}_{i,\nu}}^- + \mathrm{H.c.})$. The master equation of the system and ancilla pseudospins combined is of the above Lindblad form with

$$\{c_{i,\nu}\} = \left\{\sqrt{\frac{1-p}{2}}\lambda\sigma_{\mathbf{T}_{i,\nu}}^-, \sqrt{\frac{p}{2}\lambda}\sigma_{\mathbf{T}_{i,\nu}}^+, \sqrt{\frac{\gamma}{4}}\sigma_{\mathbf{M}_{i,\nu}}^+, \sqrt{\frac{\gamma}{4}}\sigma_{\mathbf{M}_{i,\nu}}^-\right\},$$

where λ and γ define the relaxation rates of the individual ancillary pseudospins. With this choice of H_{sr} it can be shown [18] that for $g \ll \lambda$ the system pseudospins evolve under a renormalized Lindblad master equation with c_ω given by Eq. (4) and $\lambda^* = 4(g/\hbar)^2\lambda$, thus leaving the thermal state of H_0 as the unique stationary state.

For $p = 0$, the effective system-reservoir interaction cools the system towards the ground state, and the Lindblad operators can be reduced to the n-body terms $\{\sqrt{\lambda^*}(E_{i,\nu} + T_{i,\nu}), \sqrt{\lambda^*}(E_{i,\nu} + T_{i,\nu}^\dagger)\}$ [19,20]. In this limit the ancillary pseudospins become an effective low temperature bath with a cooling rate $\Gamma_c \sim \lambda^*$ and heating rate determined by gate errors and any environmental noise. Competition between these rates leads to a minimum reachable temperature for the system $T_{\min} \sim \Delta/\ln(\Gamma_c/\Gamma_e)$, where $\Gamma_e \sim \mathrm{EPG} \times \Omega$, with EPG and Ω the error rate and frequency of application of $U_j(\alpha)$, respectively.

The Lindblad master equation, with operators given by Eq. (4), generates a unitary system-reservior interaction but nonunitary reservoir relaxation. Stroboscopic simulation of H_{sr} is performed in a manner analogous to the H_0^{TC} simulation described above. Phase angles are chosen in the one- and two-body gates to generate an effective static interaction strength g over the time t_{sr} between applications of H_{sr}, such that $gt_{\mathrm{sr}}/\hbar < \pi/2$. Nonunitary evolution of the reservoir is generated by encoding the reservoir as two levels of a Λ-system. The pseudospin states are the ground state $|0\rangle$ and the meta-stable state $|1\rangle$. State $|2\rangle$ is chosen to have fast spontaneous emission to $|0\rangle$, with rate Γ_{20}. This spontaneous emission is the decoherence mechanism required to generate the nonunitary Lindblad evolution. The ancillary pseudospin levels can be placed in a thermal state via the following procedure: (i) π pulse on the $|1\rangle \rightarrow |2\rangle$ transition. (ii) Wait for decay to ground state, $|0\rangle$. (iii) π pulse on $|0\rangle \rightarrow |1\rangle$ transition. (iv) θ pulse on the $|1\rangle \rightarrow |2\rangle$ transition. (v) Wait for decay, which now yields the final pseudospin state, $\rho = \mathrm{diag}\{\sin^2(\theta), \cos^2(\theta)\}$, corresponding to an effective temperature $T_{\mathrm{eff}} = \Delta/(2\ln(\cot\theta))$. The above stroboscopic procedure generates $\lambda^* \approx g^2 t_{\mathrm{sr}}/\hbar^2$ in Eq. (4). The procedure can be simplified in the limit of cooling towards zero temperature by eliminating steps (iii)–(v), when it becomes similar to the optical pumping scheme employed in measurement of qubit states for trapped ions [21]. This thermalization procedure is then repeated and interleaved with the stroboscopic application of H_0.

Thermalization of the Toric code.—H_0^{TC} is in the local form of H_0, with two types of excitations, electric charges, and magnetic vortices ($\nu = e, m$) that reside on vertices and plaquettes, respectively, of the square lattice. The excitation operators are defined with $h_v^e = \prod_{j\in v}\sigma_j^z$, $h_p^m = \prod_{j\in p}\sigma_j^x$, and $\Sigma_i^{e,m} = \sigma_i^{x,z}$. Each link must interact with four ancillary pseudospins in the limit $T \ll \chi$ or $\chi \ll g$ to allow thermalization to the ground state or the thermal state of H_0^{TC}, respectively.

The stroboscopic generation of H_0^{TC} outlined above introduces truncation perturbations in the perturbative expansion, e.g., the second term in Eq. (3), which are distinct from extrinsic errors due to experimental noise and gate inaccuracies. If sufficiently large, such truncation perturbations could drive the system away from the desired ground state phase. We now show on a finite sized system accessible to current experiments [10], that the intrinsic perturbations can be kept sufficiently small. Figure 2(a) plots the gap of $H_{\chi,h_z}^{\mathrm{TC}} = H_0^{\mathrm{TC}} - h_z\sum_i \sigma_i^z + \chi\sum_{\langle i,j\rangle}\sigma_i^x\sigma_j^y$ as a function of the strength of the perturbation for a 3×3 planar lattice with toroidal boundary conditions (Fig. 1). The Zeeman field is added here to fully split the ground state degeneracy and ensure robust characterization of the eigenstates of H even in the presence of small additional perturbations. We define the ground state fidelity as $F_n^{\mathrm{GS}} = |\langle\Psi_n^0|\Psi_n(\chi, h_z)\rangle|$, where the $|\Psi_n^0\rangle$ are the degenerate ground states of H_0^{TC} and $|\Psi_n(\chi, h_z)\rangle$ are the nearly degenerate ground states of $H_{\chi,h_z}^{\mathrm{TC}}$. Figure 2(b) shows the ground state fidelity as a function of χ. This fidelity determines the robustness of topological operations that will be performed via string operators [6] to measure or perform gates on the system. We see that for $|\chi| \lesssim 0.4$ the features of the topological phase persist, including the approximate four-

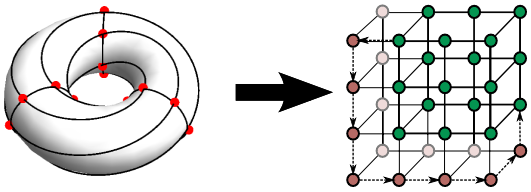

FIG. 1 (color online). An embedding of the toric code into a cubic lattice. Twisted-periodic boundary conditions are imposed by SWAP-gate shuttling along auxiliary sites, indicated by dashed arrows. Bold lines connect logical nearest neighbors [18].

fold degeneracy of the ground state and a finite gap to excitations. This corresponds to a maximum value of $\phi \sim 0.4$, which constrains the gate operations in the pulse sequences, Eq. (2). This robustness should increase with increasing lattice size, and is consistent with known stability of H_0^{TC} to perturbations [22].

Increasing ϕ increases J and therefore the gap of the H_0^{TC}; however, it also increases χ/J, which reduces the gap of $H_{\chi,h_z}^{\mathrm{TC}}$ and topological protection for large χ. We also note that $J \sim 1/N_G$ where N_G is the number of sequential gates used to simulate H. For larger lattices, some degree of parallelization is thus desirable to ensure that the gap does not decrease with the lattice size. Choosing $\chi = 0.2$, the gap achieved by a completely serial implementation is $\Delta \approx 0.6\ \mu\mathrm{K}/N_{\mathrm{sys}}$, where N_{sys} is the number of system atoms used. With the cooling sequence serially interleaved, $\Delta \approx 0.1\ \mu\mathrm{K}/N_{\mathrm{sys}}$ and $\lambda^* \sim 10^4\ \mathrm{s}^{-1}/N_{\mathrm{sys}}$ are achievable [18]. For a minimal system of 18 system atoms, this allows for an effective temperature $T_{\mathrm{eff}} < \Delta$ to be reached with an error rate of EPG $\sim 10^{-4}$ or less [18].

Sources of errors.—This scheme is designed to be robust against errors within the pseudospin subspace. The dominant source of residual error in the implementation discussed here is leakage from the Rydberg levels due to spontaneous emission and black body radiation. The latter may be effectively suppressed by working at low temperatures [23], and spontaneous emission is minimized by utilizing states with larger n. With $n \lesssim 180$, we estimate that spontaneous emission errors can be reduced to $\sim 10^{-6}$ per gate, allowing for up to 10^3 stroboscopic cycles.

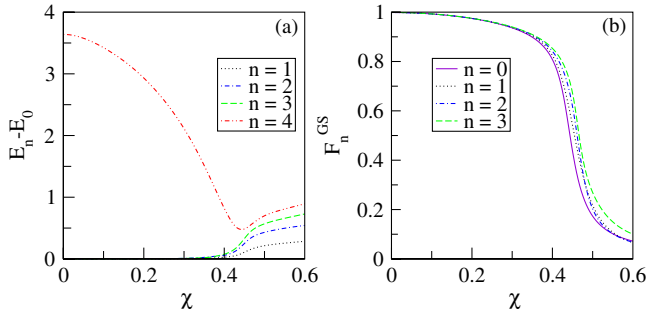

FIG. 2 (color online). (a) Energy spectrum (in units of $J_e = J_m$) and (b) ground state fidelity vs perturbation strength from exact diagonalization of the 18 site toric code with $h_z = 0.05$.

Discussion.—We have developed a formalism for the stroboscopic generation of n-body Hamiltonians using 1- and 2-body quantum operations together with a dissipative thermalization scheme. We have applied this to the toric code Hamiltonian in the context of addressable optical lattice experiments [10]. Our method applies to a wide range of lattice spin models [18] as well as to other experimental setups [24]. The dynamic generation both of a Hamiltonian possessing a topologically ordered ground state and of an effective thermalization mechanism offers the possibility of *robust* simulation of the ground state and of the creation and braiding of anyonic excitations [25]. These are essential components required for the topologically protected storage and manipulation of quantum information.

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