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Research Article

The Novel Bivariate Distribution: Statistical Properties and Real Data Applications

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This article proposes a novel class of bivariate distributions that are completely defined by stating their conditionals as Poisson exponential distributions. Numerous statistical properties of this distribution are also examined here, including the conditional probability mass function (PMF) and moments of the new class. The techniques of maximum likelihood and pseudolikelihood are used to estimate the model parameters. Additionally, the effectiveness of the bivariate Poisson exponential conditional (BPEC) distribution is compared to that of the bivariate Poisson conditional (BPC), the bivariate Poisson (BP), the bivariate Poisson–Lindley (BPL), and the bivariate negative binomial (BNB) distributions using a real-world dataset. The findings of Akaike information criterion (AIC) and Bayesian information criterion (BIC) reveal that the BPEC distribution performs better than the other distributions considered in this study. As a result, the authors claim that this distribution may be used to fit dependent and overspread count data.

1. Introduction

In many areas of application, it is appropriate to study discrete bivariate variables. For example, problems arise in many social, economic, and physical phenomena [1], and in insurance risk applications, those number of cases in distinctive classifications will be regularly randomized (the readers are referred to Wu and Yuen [2], Yuen et al. [3], and Morata [4] for more details). Several authors have discussed these problems from different points of view, which include traffic accidents by Cacoullos and Papageorgiou [5] and Papageorgiou [6, 7] and the problem associated with crime utilizing the method of Miethe et al. [8]. Also, Lee [9] and Karlis and Ntzoufras [10] modeled scores "points and goals" of two competing teams in sports and pointed out that they are highly correlated. Modeling dependence on goals scored by teams competing in international football matches was studied by McHale and Scarf [11], and evaluated risks and spot errors using scarce data were discussed by Ahooyi et al. [12]. Several discrete bivariate models have been proposed in the literature (see, for example, Marshall and Olkin [13], Mishra [14], Özel [15–17], Reilly and Sapkota [18], Lee and Cha [19], and Jiang et al. [20]). The specific conditional distributions are one of the most important ways to get flexible bivariate distributions. Moreover, the important role of functional equations has been emphasized in establishing results in this regard which is highlighted by Castillo and Galambos [21–23], Arnold [24], Arnold et al. [25–27], Kottas et al. [28], and Gharib and Mohammed [29]. The use of this type of distribution in risk analysis and economics is relatively new; however, some applications were done by Sarabia et al. [30, 31].

In this paper, another class of bivariate model for Poisson exponential conditionals will be considered. A discrete random variable X is said to have a one-parameter Poisson exponential distribution (PED) for modeling countable data if its probability mass function (PMF) is

$$P(Z=z)=\frac{\alpha}{(1+\alpha)^{z}},\quad \alpha>0, z=0,1,\ldots. \tag{1}$$

If the parameter of the Poisson model follows a continuous exponential distribution, then equation (1) is a mixture of Poisson and exponential distributions denoted by $Z\sim \mathrm{PE}(\alpha)$. This distribution is applicable to biological datasets, traffic datasets, thunderstorm datasets, and other discrete datasets. The scientific properties and estimation for parameter have been examined by Fazal and Bashir [32]; also, its requisition turns out that it will be a great substitution cost from claiming Poisson and Lindley distributions. In this paper, a new class of bivariate distribution has been proposed which is fully characterized by specifying its conditionals as Poisson exponential distribution. Finally, the performance of this distribution is compared with other distributions considering a real-life dataset.

2. Bivariate Poisson Exponential Conditionals

Consider a general bivariate model (J,K) whose conditional distributions must satisfy the following two conditions:

$$J\mid K=k\sim \mathrm{PE}(\eta_1(k)), \tag{2}$$

$$K\mid J=j\sim \mathrm{PE}(\eta_2(j)), \tag{3}$$

where $\eta_1(k)$ and $\eta_2(j)$ are some positive functions and PE denotes a Poisson exponential distribution. These equations lead us to discuss the next theorem.

Theorem 1. *The discrete bivariate model with $J\mid K=k\sim PE(\eta_1(k))$ and $K\mid J=j\sim PE(\eta_2(j))$ can be described by the following distribution:*

$$P_{J,K}(j,k)=[N(\theta_1,\theta_2,\theta_3)]^{-1}\exp[\theta_1 k-\theta_3 jk+\theta_2 j],\quad j,k=0,1,\ldots,\theta_1,\theta_2<0,\theta_3\in\mathscr{R}, \tag{4}$$

where $[N(\theta_1,\theta_2,\theta_3)]^{-1}$ is the normalizing constant such that $P_{J,K}(j,k)$ summates to 1.

Proof. According to (2) and (3), we can write the joint density $P(j,k)$ as a product of a marginal and a conditional density in both ways to get

$$\left(\frac{\eta_1(k)}{(1+\eta_1(k))^{j}}\right)h_K(k)=\left(\frac{\eta_2(j)}{(1+\eta_2(j))^{k}}\right)h_J(j), \tag{5}$$

where $h_J(j)$ and $h_K(k)$ are the marginal PMFs of J and K, respectively.

Denoting

$$g(k)=\log[\eta_1(k)h_K(k)], \tag{6}$$

$$f(j)=\log[\eta_2(j)h_J(j)], \tag{7}$$

equation (5) readily reduces to

$$g(k)-x\log(1+\eta_1(k))-f(j)-k\log(1+\eta_2(j))=0, \tag{8}$$

which is a special case of the functional equation $\sum_{m=1}^{n} f_m(x)g_m(y)=0$, whose most general solution is given by Aczel [33] as follows:

$$\alpha_2(j)=\exp(-\theta_1+\theta_3 j)-1,\ldots,\eta_1(k)=\exp(-\theta_2+\theta_3 k)-1. \tag{9}$$

Substituting these expressions in (6) and (7), we can get the marginal PMF as

$$h_J(j)=\frac{[N(\theta_1,\theta_2,\theta_3)]^{-1}}{\exp(-\theta_1+\theta_3 j)-1}\exp(-\theta_2 j),\quad j=0,1,\ldots, \tag{10}$$

$$h_K(k)=\frac{[N(\theta_1,\theta_2,\theta_3)]^{-1}}{\exp(-\theta_2+\theta_3 k)-1}\exp(-\theta_1 k),\quad k=0,1,\ldots. \tag{11}$$

Finally, in accordance with (10) and (11), the class of discrete bivariate distribution with Poisson exponential conditionals is that given by (4), which describes the complete class of the BPEC distribution that has the three parameters θ_1,θ_2 (intensity parameters for K and J, respectively), and $\theta_3\in\mathscr{R}$ (interaction or dependence parameter), where $\theta_3=0$ corresponds to independence between J and K.

Figure 1 shows the three-dimensional curve of the BPLC given by (4) for the special cases for θ_1,θ_2, and θ_3. □

3. Properties of the Bivariate Poisson Exponential Class

In this part, the fundamental properties of the new bivariate distribution are contemplated.

We first know that the class (4) has the three parameters θ_1,θ_2, and θ_3, while $[N(\theta_1,\theta_2,\theta_3)]^{-1}$ is the normalizing constant and is given by

$$N(\theta_1,\theta_2,\theta_3)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\mathrm{Exp}[\theta_1 k-\theta_3 jk+\theta_2 j]=\sum_{j=0}^{\infty}\frac{e^{-\theta_1+(\theta_3-\theta_2)j}}{e^{\theta_1+\theta_3 j}-1}. \tag{12}$$

3.1. Conditional PMF and Moments. The particular manifestations of the conditional distributions to the new model would provide

$$P_{J|K}(j\mid k)=\frac{\exp(-\theta_2+\theta_3 k)-1}{(\exp(-\theta_2+\theta_3 k))^{j}},\quad j,k=0,1,\ldots, \tag{13}$$

$$P_{K|J}(k\mid j)=\frac{\exp(-\theta_1+\theta_3 j)-1}{(\exp(-\theta_1+\theta_3 j))^{k}},\quad j,k=0,1,\ldots, \tag{14}$$

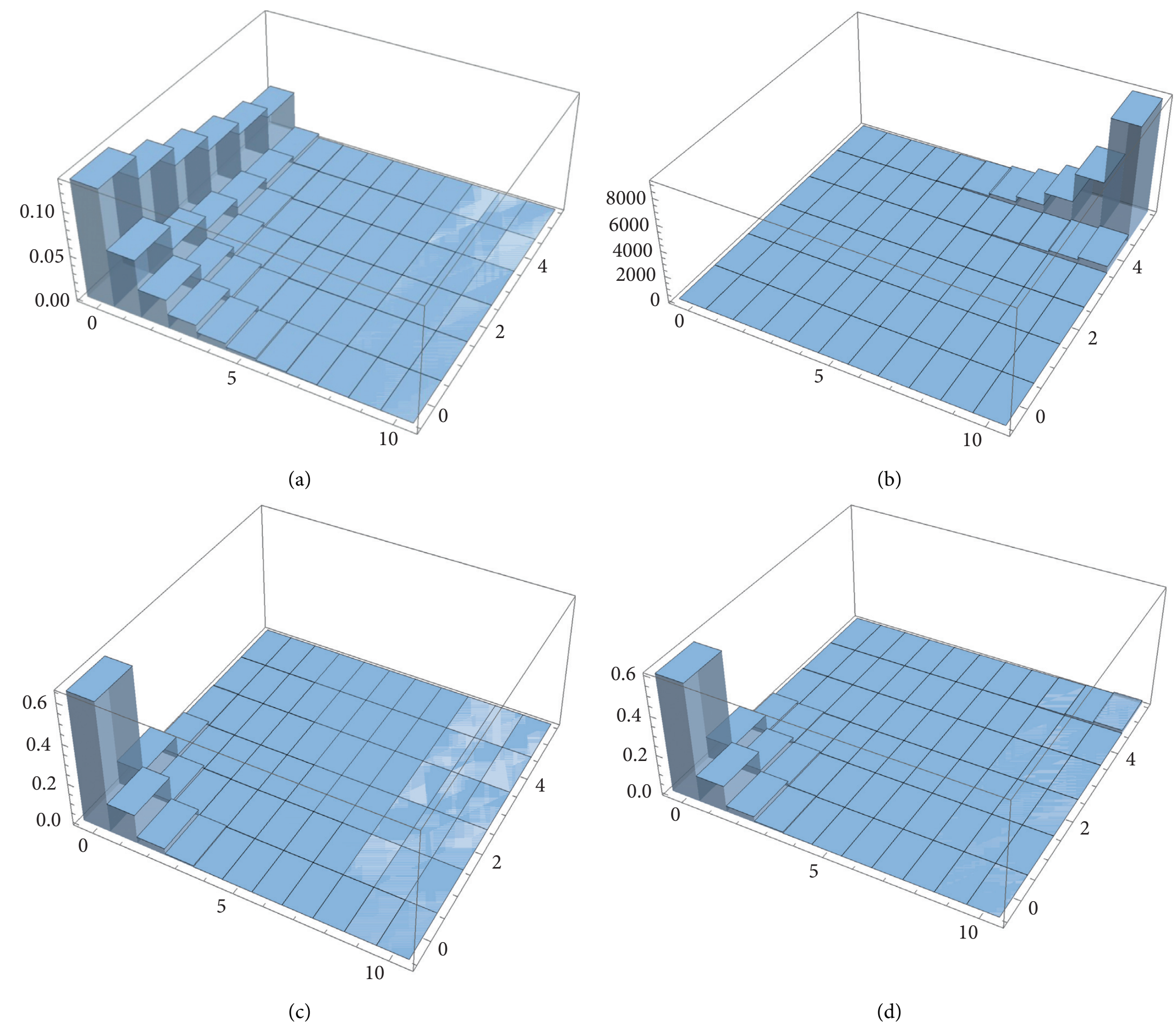

FIGURE 1: The three-dimensional curve of the BPLC under different scenarios. (a) BPEC with $\theta_1 = -0.2, \theta_2 = -0.7$, and $\theta_3 = 0.3$. (b) BPEC with $\theta_1 = -0.2, \theta_2 = -0.7$, and $\theta_3 = -0.3$. (c) BPEC with $\theta_1 = -1.7, \theta_2 = -1.5$, and $\theta_3 = 0.4$. (d) BPEC with $\theta_1 = -1.7, \theta_2 = -1.5$, and $\theta_3 = -0.4$.

i.e.,

$$\begin{aligned} J \mid K = k &\sim PE\left(\exp\left(-\theta_2 + \theta_3 k\right) - 1\right), \\ K \mid J = j &\sim \mathrm{PE}\left(\exp\left(-\theta_1 + \theta_3 j\right) - 1\right). \end{aligned} \tag{15}$$

The conditional distributions are given by (13) and (14), satisfying the compatibility conditions, and are studied by Arnold et al. [26], which guarantees the existence of the discrete bivariate model (4).

The regression functions for these conditional distributions are

$$\begin{aligned} E(J \mid K = k) &= \frac{1}{\exp\left(-\theta_2 + \theta_3 k\right) - 1}, \\ E(K \mid J = j) &= \frac{1}{\exp\left(-\theta_1 + \theta_3 j\right) - 1}. \end{aligned} \tag{16}$$

These regression functions are nonlinear and decreasing (increasing) if $\theta_3 > 0$ ($\theta_3 < 0$) (see Figure 2).

The first moment of the pair (J, K) is obtained by direct calculations using (4), and we find that

$$E(JK) = \sum_{j=0}^{\infty} \frac{\left[N\left(\theta_1, \theta_2, \theta_3\right)\right]^{-1} j e^{-\left(\theta_1 + \left(\theta_2 + \theta_3\right) j\right)}}{\left(e^{-\theta_1} + e^{\theta_3 j}\right)^2}. \tag{17}$$

Special Classes. Class (4) can be classified by suitable selections for the parameters θ_1, θ_2, and θ_3 into the following two subclasses.

(a) Subclass I (subclass with two parameters):

$$P_{J,K}(j,k) = \left[N\left(\theta_1, \theta_2, \theta_3\right)\right]^{-1} \exp\left[\theta_1 k - \theta_3 jk + \theta_2 j\right], \quad j, k = 0, 1, \ldots, \theta_1, \theta_2 < 0, \theta_3 \in \mathscr{R}. \tag{18}$$

(1) $\theta_3 = 0$, J, K are independent, and (4) reduces to

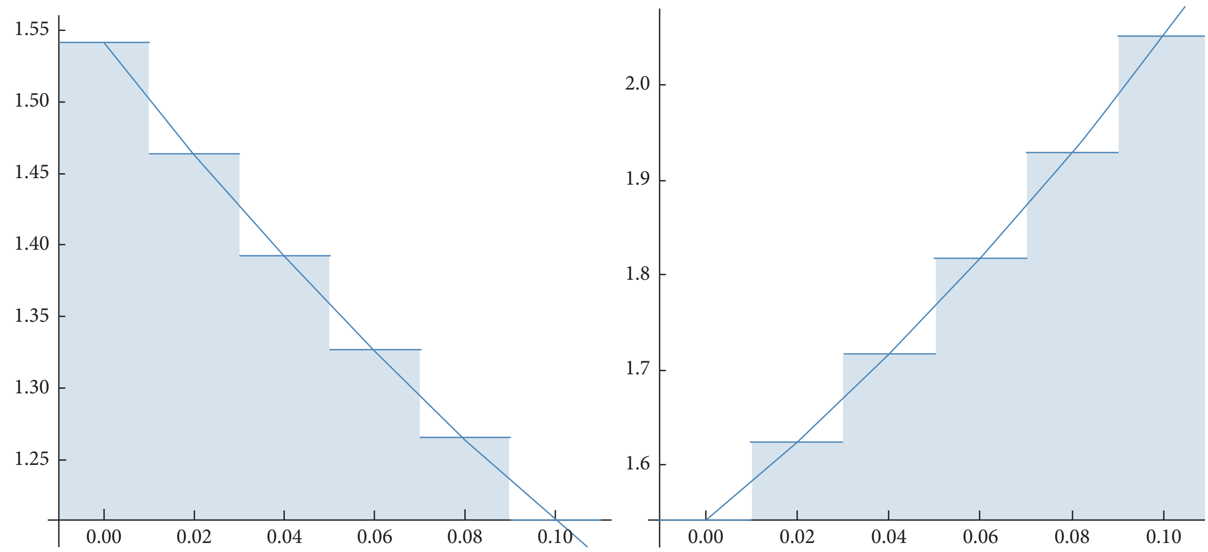

Figure 2: The regression curve of j on k (k on j) of BPEC distribution for $\theta_3 = 0.5 > 0$ ($\theta_3 = -0.5 < 0$).

$$P_{J,K}(j,k) = [N(\theta_1,\theta_2,0)]^{-1}\exp(\theta_2 j)\exp(\theta_1 k); \quad j,k = 0,1,\ldots,\theta_1 < 0, \theta_2 \in \mathscr{R}, \tag{19}$$

where $[N(\theta_1,\theta_2,0)]^{-1} = 1/(e^{\theta_1}-1)(e^{\theta_2}-1)$.

It is clear from (19) that the two random variables (RVs) J and K are independent, with the following marginal densities:

$$P_{J|K}(j \mid k) = \frac{\exp(-\theta_2)-1}{(\exp(-\theta_2))^{j}}, \quad j,k = 0,1,\ldots, \tag{20}$$

$$P_{K|J}(k \mid j) = \frac{\exp(-\theta_1)-1}{(\exp(-\theta_1))^{k}}, \quad j,k = 0,1,\ldots, \tag{21}$$

i.e.,

$$J \mid K = k \sim \mathrm{PE}(\exp(-\theta_2)-1), \quad K \mid J = j \sim \mathrm{PE}(\exp(-\theta_1)-1). \tag{22}$$

(2) $\theta_1 = \theta_2$, and (4) reduces to

$$P_{J,K}(j,k) = [N(\theta_2,\theta_2,\theta_3)]^{-1}\exp[(j+k)\theta_2 - \theta_3 jk], \quad j,k = 0,1,\ldots,\theta_2 < 0, \theta_3 \in \mathscr{R}, \tag{23}$$

where

$$[N(\theta_2,\theta_2,\theta_3)]^{-1} = \sum_{x=0}^{\infty}\frac{e^{(\theta_2+\theta_3)j}}{-e^{\theta_2}+e^{\theta_3 j}},$$

$$P_{J|K}(j \mid k) = \frac{\exp(-\theta_2+\theta_3 k)-1}{(\exp(-\theta_2+\theta_3 k))^{j}}, \quad j,k = 0,1,\ldots,$$

$$P_{K|J}(k \mid j) = \frac{\exp(-\theta_1+\theta_3 j)-1}{(\exp(-\theta_1+\theta_3 j))^{k}}, \quad j,k = 0,1,\ldots, \tag{24}$$

i.e.,

$$J \mid K = k \sim \mathrm{PE}(\exp(-\theta_2+\theta_3 k)-1), \quad K \mid J = j \sim \mathrm{PE}(\exp(-\theta_2+\theta_3 j)-1). \tag{25}$$

(3) $\theta_2 = 0$, and (4) becomes

$$P_{J,K}(j,k) = [N(\theta_1,0,\theta_3)]^{-1}\exp[\theta_1 k - \theta_3 jk], \quad j,k = 0,1,\ldots,\theta_1 < 0, \theta_3 \in \mathscr{R}, \tag{26}$$

where

$$[N(\theta_1,0,\theta_3)]^{-1} = \sum_{x=0}^{\infty}\frac{e^{\theta_3 j}}{-e^{\theta_1}+e^{\theta_3 j}},$$

$$P_{J|K}(j \mid k) = \frac{\exp(\theta_3 k)-1}{(\exp(\theta_3 k))^{j}}, \quad j,k = 0,1,\ldots,$$

$$P_{K|J}(k \mid j) = \frac{\exp(-\theta_1+\theta_3 j)-1}{(\exp(-\theta_1+\theta_3 j))^{k}}, \quad j,k = 0,1,\ldots, \tag{27}$$

i.e.,

$$J \mid K = k \sim \mathrm{PE}(\exp(\theta_3 k)-1), \quad K \mid J = j \sim \mathrm{PE}(\exp(-\theta_1+\theta_3 j)-1). \tag{28}$$

(4) $\theta_1 = 0$, and (4) reduces to

$$P_{J,K}(j,k) = [N(0,\theta_2,\theta_3)]^{-1}\exp[-\theta_3 jk + \theta_2 j], \quad j,k = 0,1,\ldots,\theta_2 < 0, \theta_3 \in \mathscr{R}, \tag{29}$$

where

$$[N(0,\theta_2,\theta_3)]^{-1} = \sum_{k=0}^{\infty} \frac{e^{\theta_3 k}}{-e^{\theta_2} + e^{\theta_3 k}},$$

$$P_{J|K}(j \mid k) = \frac{\exp(-\theta_2 + \theta_3 k) - 1}{(\exp(-\theta_2 + \theta_3 k))^j}, \quad j,k = 0,1,\ldots,$$

$$P_{K|J}(k \mid j) = \frac{\exp(\theta_3 j) - 1}{(\exp(\theta_3 j))^k}, \quad j,k = 0,1,\ldots, \tag{30}$$

i.e.,

$$J \mid K = k \sim \mathrm{PE}(\exp(-\theta_2 + \theta_3 k) - 1),$$
$$K \mid J = j \sim \mathrm{PE}(\exp(\theta_3 j) - 1). \tag{31}$$

(b) Subclass II (subclass with one parameter):
(1) $\theta_1 = \theta_2 = \theta_3, \theta_3 > 0$, and (4) reduces to

$$P_{J,K}(j,k) = [N(\theta_3)]^{-1} \exp[(j - jk + k)\theta_3],$$
$$j,k = 0,1,\ldots,\theta_3 < 0, \tag{32}$$

where

$$[N(\theta_3)]^{-1} = \sum_{y=0}^{\infty} \frac{e^{2\theta_3 k}}{-e^{\theta_3} + e^{\theta_3 k}},$$

$$P_{J|K}(j \mid k) = \frac{\exp[(k-1)\theta_3] - 1}{(\exp[(k-1)\theta_3])^j}, \quad j,k = 0,1,\ldots, \tag{33}$$

$$P_{K|J}(k \mid j) = \frac{\exp[(j-1)\theta_3] - 1}{(\exp[(j-1)\theta_3])^k}, \quad j,k = 0,1,\ldots,$$

i.e.,

$$J \mid K = k \sim \mathrm{PE}(\exp[(k-1)\theta_3] - 1),$$
$$K \mid J = j \sim \mathrm{PE}(\exp((j-1)\theta_3) - 1). \tag{34}$$

4. Estimation of the Parameters of BPLC

Suppose that $(j_1,k_1), (j_2,k_2), \ldots, (j_n,k_n)$ are random samples from $\mathrm{BPEC}(\theta_1,\theta_2,\theta_3)$ class with density function given in (4).

4.1. Maximum Likelihood Estimation (MLE) for the Parameters. The log-likelihood function $l(\theta)$ of BPEC $(\theta_1,\theta_2,\theta_3)$ is given by

$$l(\theta) = -n \log(N(\theta_1,\theta_2,\theta_3)) + \theta_1 \sum_{i=1}^{n} k_i - \theta_3 \sum_{i=1}^{n} j_i k_i + \theta_2 \sum_{i=1}^{n} j_i. \tag{35}$$

The maximum likelihood estimates of θ_1, θ_2, and θ_3 can be obtained by solving

$$\frac{\partial N(\theta)/\partial\theta_1}{N(\theta)} = \frac{1}{n} \sum_{i=1}^{n} j_i, \tag{36}$$

$$\frac{\partial N(\theta)/\partial\theta_2}{N(\theta)} = \frac{1}{n} \sum_{i=1}^{n} j_i, \tag{37}$$

$$\frac{\partial N(\theta)/\partial\theta_3}{N(\theta)} = -\frac{1}{n} \sum_{i=1}^{n} j_i k_i, \tag{38}$$

where $N(\theta)$ is given by (12).

The implicit nature of systems (36)–(38) suggests the numerical derivation of the MLE of parameters θ_1, θ_2, and θ_3.

4.2. Pseudolikelihood Estimation for the Parameters. The pseudolikelihood method is an alternative estimation technique that does not include the normalizing constant (see Besag [34, 35] and Arnold and Strauss [36, 37]). The pseudolikelihood function can be written as

$$\mathrm{PL}(\theta) = \prod_{i=1}^{n} P_{j|k}(j_i \mid k_i) P_{k|j}(k_i \mid j_i),$$

$$\mathrm{PL}(\theta) = \prod_{i=1}^{n} \frac{\exp(-\theta_2 + \theta_3 k) - 1}{(\exp(-\theta_2 + \theta_3 k))^j} \frac{\exp(-\theta_1 + \theta_3 j) - 1}{(\exp(-\theta_1 + \theta_3 j))^k}. \tag{39}$$

Therefore, we have the following logarithmic form of the pseudolikelihood function:

$$\begin{aligned} \log\mathrm{PL}(\theta) = {} & \sum_{i=1}^{n} \log[\exp(-\theta_2 + \theta_3 k_i) - 1] \\ & - \sum_{i=1}^{n} j_i(-\theta_2 + \theta_3 k_i) \\ & + \sum_{i=1}^{n} \log[\exp(-\theta_1 + \theta_3 j_i) - 1] \\ & - \sum_{i=1}^{n} k_i(-\theta_1 + \theta_3 j_i). \end{aligned} \tag{40}$$

The maximum pseudolikelihood estimates of θ_1, θ_2, and θ_3 can be obtained by solving the following:

$$\frac{\partial \log \mathrm{PL}(\theta_1,\theta_2,\theta_3)}{\partial\theta_1} = -\sum_{i=1}^{n}\frac{\exp(-\theta_1+\theta_3 j_i)}{\exp(-\theta_1+\theta_3 j_i)-1}+\sum_{i=1}^{n}k_i,$$
$$\frac{\partial \log \mathrm{PL}(\theta_1,\theta_2,\theta_3)}{\partial\theta_2} = -\sum_{i=1}^{n}\frac{\exp(-\theta_2+\theta_3 k_i)}{\exp(-\theta_2+\theta_3 k_i)-1}+\sum_{i=1}^{n}x_i, \tag{41}$$
$$\frac{\partial \log \mathrm{PL}(\theta_1,\theta_2,\theta_3)}{\partial\theta_3} = \sum_{i=1}^{n}\frac{j_i\exp(-\theta_1+\theta_3 j_i)}{\exp(-\theta_1+\theta_3 j_i)-1}+\sum_{i=1}^{n}\frac{k_i\exp(-\theta_2+\theta_3 k_i)}{\exp(-\theta_2+\theta_3 k_i)-1}+2\sum_{i=1}^{n}j_i k_i.$$

5. Application

We consider a dataset in this paper which was obtained from Mitchell and Paulson [38] and is presented in Table 1. Utilizing these data, we should gauge and estimate the parameters θ_1, θ_2, and θ_3 of class (4). The information includes flight aborts count data from 109 aircrafts, and the variables J and K represent the flight aborts in the first and second sequential six months of a one-year period.

The frequencies of the observed data provide several (j, 0) and (0, k) data, indicating a negative correlation between j and k. Therefore, we fit BPC, BP, BPL, and BNB distributions to the data since these distributions can be fitted to bivariate data with positive, zero, or negative correlation.

The statistic measures for the given data are $\overline{j} = 0.62$, $s_1^2 = 1.03$; $\overline{k} = 0.72$, $s_2^2 = 1.07$, $\mathrm{Cov}(j,k) = -0.169$, and $\rho = -0.16$. Table 2 presents the estimated parameters of the BPEC model and its mean square error (MSE).

The joint PMF of bivariate Poisson conditional distribution can be defined as $\mathrm{BPC}(\lambda_1,\lambda_2,\lambda_3)$ and defined as follows (Arnold and Strauss [36]):

$$P(J=j,K=k) = C(\lambda_1,\lambda_2,\lambda_3)\frac{\lambda_1^j\lambda_2^k\lambda_3^{jk}}{j!k!},\quad j,k=0,1,2,\ldots,\lambda_1,\lambda_2>0, 0<\lambda_3\le 1, \tag{42}$$

where $C(\lambda_1,\lambda_2,\lambda_3)$ is constant. The conditionals $K\,|\,J$ and $J\,|\,K$ are $\lambda_2\lambda_3^j$ and $\lambda_1\lambda_3^k$, respectively.

The joint PMF of BP $(\lambda_1,\lambda_2,\alpha)$ distribution is (Lakshminarayana et al. [39])

$$P(J=j,K=k) = e^{-\lambda_1-\lambda_2}\frac{\lambda_1^j\lambda_2^k}{jk}\left[1+\alpha\left(e^{-j}-e^{-c\lambda_1}\right)\left(e^{-k}-e^{-c\lambda_2}\right)\right],\quad j,k=0,1,2,\ldots,\lambda_1,\lambda_2>0, \tag{43}$$

where $c = 1-e^{-1}$, $E(J) = \mathrm{Var}(J) = \lambda_1, E(K) = \mathrm{Var}(K) = \lambda_2$, $\mathrm{Cov}\ (J,K) = \alpha\lambda_1\lambda_2 c^2 e^{-c(\lambda_1+\lambda_2)}$, and α can be chosen such that $P(Y_1,Y_2)$ will be the PMF.

The joint PMF of $\mathrm{BPL}(\theta_1,\theta_2,m_1,m_2,\gamma)$ distribution is (Zamani et al. [40])

$$P(J=j,K=k) = \frac{\theta_1^2(j+\theta_1+2)\theta_1^2(j+\theta_1+2)}{(\theta_1+1)^{j+3}(\theta_2+1)^{k+3}}\cdot\left[1+\alpha\left(e^{-j}-c_1\right)\left(e^{-k}-c_2\right)\right],\quad j,k=0,1,2,\ldots,\theta_1,\theta_2>0, \tag{44}$$

where

$$c_1 = E\left(e^{-J}\right) = \frac{\theta_1^2}{1+\theta_1}\frac{\left(\theta_1+2-e^{-1}\right)}{\left(\theta_1-e^{-1}+1\right)^2},\quad c_2 = E\left(e^{-K}\right) = \frac{\theta_2^2}{1+\theta_2}\frac{\left(\theta_2+2-e^{-1}\right)}{\left(\theta_2-e^{-1}+1\right)^2}. \tag{45}$$

The joint PMF of $\mathrm{BNB}(\theta_1,\theta_2,m_1,m_2,\gamma)$ distribution is (Famoye [41])

$$P(J=j,K=k) = \binom{m_1^{-1}+j-1}{k}\theta_1^j(1-\theta_1)^{m_1^{-1}}\binom{m_2^{-1}+j-1}{k}\theta_2^k(1-\theta_2)^{m_2^{-1}}\left[1+\gamma\left(e^{-j}-c_1\right)\left(e^{-k}-c_2\right)\right],\quad j,k=0,1,2,\ldots,\theta_1,\theta_2>0, 0<m_1,m_2<1, \tag{46}$$

Table 1: The flight abort check information starting with 109 aircraft.

j	K 0	1	2	3	4	$\sum$
0	34	20	4	6	4	68
1	17	7	0	0	0	24
2	6	4	1	0	0	11
3	0	4	0	0	0	4
4	0	0	0	0	0	0
5	2	0	0	0	0	2
$\sum$	59	35	5	6	4	109

Table 2: Estimation of parameters of the BPEC.

Parameters	MLE	MSE	MPLE	MSE
θ_1	−0.6712	0.271675	−0.932498	0.000506142
θ_2	−0.7664	0.0709704	−0.824366	0.000593723
θ_3	−1.6437	0.0206583	0.57096	0.00167806

where

$$c_1 = E\left(e^{-J}\right) = \left(\frac{1-\theta_1}{1-\theta_1 e^{-1}}\right)^{m_1^{-1}},$$
$$c_2 = E\left(e^{-K}\right) = \left(\frac{1-\theta_2}{1-\theta_2 e^{-1}}\right)^{m_2^{-1}}, \tag{47}$$

and the mean, variance, and covariance are

$$E(J) = m_1^{-1}\frac{\theta_1}{1-\theta_1},$$
$$E(K) = m_2^{-1}\frac{\theta_2}{1-\theta_2},$$
$$\operatorname{Var}(J) = m_1^{-1}\frac{\theta_1}{(1-\theta_1)^2}, \tag{48}$$
$$\operatorname{Var}(K) = m_2^{-1}\frac{\theta_2}{(1-\theta_2)^2},$$
$$\operatorname{Cov}(J,K) = \gamma c_1 c_2 \prod_i\left(\frac{m\alpha_i^{-1}\theta_i e^{-1}}{1-\theta_i e^{-1}} - \frac{m_i^{-1}\theta_i}{1-\theta_i}\right).$$

We used the Mathematica package to estimate the parameters of BPEC distribution.

The new distribution BPEC is more appropriate as we can see in Table 3 as compared to the BPC, BP, BPL, and BNB distributions, where the BPEC distribution gives the largest value for the AIC and BIC statistics compared to other models.

Table 3: Parameter estimators and AIC and BIC of BPEC, BPC, BP, BNB, and BPL distributions.

Model	Parameter	MLE	Log-likelihood	AIC	BIC
	θ_1	−0.6712			
BPEC	θ_2	−0.7664	−114.821	−120.821	−121.858
	θ_3	−1.6437			
	λ_1	0.14388			
BPC	λ_2	0.208112	−252.078	−258.078	−259.115
	λ_3	0.426338			
	λ_1	0.6129			
BP	λ_2	0.7131	−254.99	−260.99	−262.027
	α	−0.9290			
	θ_1	2.1165			
BPL	θ_2	1.8607	−244.62	−250.62	−251.657
	α	−1.0363			
	θ_1	0.4045			
	θ_2	0.3138			
BNB	m_1	1.0977	−244.27	−254.27	−255.998
	m_2	0.6305			
	γ	−1.1109			

6. Conclusion

In this work, a BPEC model is presented by determining conditional discrete Poisson exponential distributions. Therefore, we obtained the statistical properties and special classes for BPEC distribution. The estimation of BPEC parameters through the techniques for MLE and MPLE is presented. In view of the findings presented in Table 1, the MPLE is better than MLE because the MPLE technique uses conditional distributions which in our case do not suffer from the problem caused by the normalizing constant. Moreover, the AIC and BIC depict that BPEC distribution adequately fits the considered dataset compared to the BPC, BP, BPL, and BNB distributions.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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