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A new lifetime model with variable shapes for the hazard rate

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Abstract. We define and study a new generalization of the complementary Weibull geometric distribution introduced by Tojeiro et al. (*J. Stat. Comput. Simul.* **84** (2014) 1345–1362). The new lifetime model is referred to as the Kumaraswamy complementary Weibull geometric distribution and includes twenty three special models. Its hazard rate function can be constant, increasing, decreasing, bathtub and unimodal shaped. Some of its mathematical properties, including explicit expressions for the ordinary and incomplete moments, generating and quantile functions, Rényi entropy, mean residual life and mean inactivity time are derived. The method of maximum likelihood is used for estimating the model parameters. We provide some simulation results to assess the performance of the proposed model. Two applications to real data sets show the flexibility of the new model compared with some nested and non-nested models.

1 Introduction

The use of new generators of continuous distributions from classic ones has become very common in recent years. The procedure of expanding a class of distributions by adding new shape parameters is well-known in the statistical literature. In many applied sciences such as medicine, engineering and finance, among others, modeling and analyzing lifetime data are crucial. Several lifetime distributions have been adopted to model different types of survival data. The quality of the procedures used in a statistical analysis depends heavily on the generated family of distributions and considerable effort has been directed to define new statistical models. However, there still remain many important problems involving real data, which do not follow any of the popular statistical models. The chief motivation of the generalized distributions for modeling lifetime data lies in the flexibility to model both monotonic and non-monotonic failure rates even though the baseline failure rate may be monotonic. The role of the extra shape parameters is to

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introduce skewness and to vary tail weights. Further, various models have been constructed by extending some useful lifetime distributions and investigated them with respect to different characteristics.

Several distributions have been proposed to model real lifetime data. The Weibull distribution is one of the most commonly used distributions for this purpose. In practice, it has been shown to be very flexible in modeling various types of lifetime data with monotone failure rates but it is not useful for modeling the bathtub shaped and the unimodal failure rates, which are common in reliability and biological studies. It is of utmost interest because of its great number of special features and its ability to fit data from various fields, ranging from life data to observations made in economics and business administration, meteorology, hydrology, quality control, acceptance sampling, statistical process control, inventory control, physics, chemistry, geology, geography, astronomy, medicine, psychology, material science, engineering, biology, see, for example, Rinne (2009).

Louzada et al. (2011) proposed the complementary exponential geometric distribution as the complementary of the exponential geometric model (Adamidis and Loukas, 1998). Their model is based on a complementary risk problem in presence of latent risks in the sense that there is no information about which factor is responsible for the component failure but only the maximum lifetime value among all risks is observed. Further, Louzada et al. (2013) studied the complementary exponentiated exponential geometric model by extending the complementary exponential geometric distribution. Tojeiro et al. (2014) studied the complementary Weibull geometric (CWG) model as the complementary distribution to the Weibull geometric (WG) model (Barreto-Souza et al., 2011). Afify et al. (2014) defined the transmuted complementary Weibull geometric (TCWG) distribution and studied its various structural properties.

The cumulative distribution function (cdf) of the CWG distribution (for $x > 0$) is given by

$$G(x) = \frac{\alpha\{1 - \exp[-(\gamma x)^{\beta}]\}}{\alpha + (1-\alpha)\exp[-(\gamma x)^{\beta}]}, \tag{1}$$

where $\gamma > 0$ is a scale parameter and $0 < \alpha < 1$ and $\beta > 0$ are shape parameters. The corresponding probability density function (pdf) is given by

$$g(x) = \frac{\alpha\beta\gamma(\gamma x)^{\beta-1}\exp[-(\gamma x)^{\beta}]}{\{\alpha + (1-\alpha)\exp[-(\gamma x)^{\beta}]\}^{2}}. \tag{2}$$

The aim of this paper is to study a new lifetime model called the *Kumaraswamy complementary Weibull geometric* (Kw–CWG) distribution. The main feature of this model is that two additional shape parameters inserted in (2) can give greater flexibility in the form of the generated density. By using the Kumaraswamy-generalized (Kw–G) family proposed by Cordeiro and de Castro (2011), we construct the five-parameter Kw–CWG model. We provide a comprehensive description of some of its mathematical properties with the hope that it will attract wider applications in reliability, engineering and other areas of research.

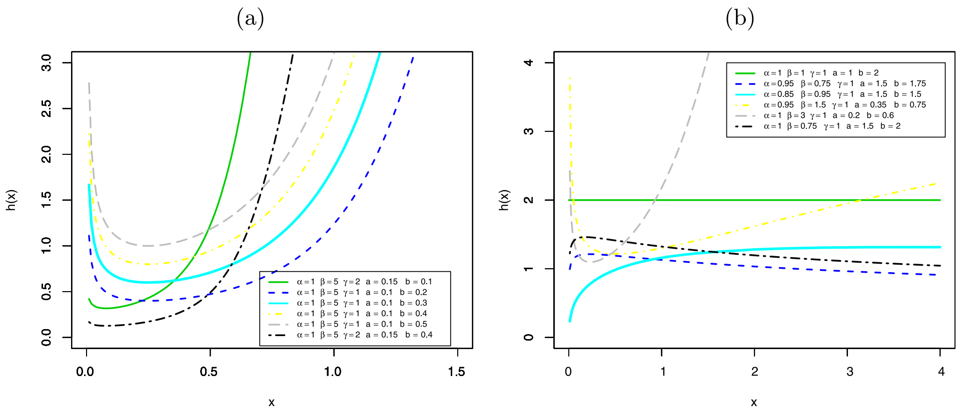

Figure 1 *Plots of the hrf for some parameter values.*

For an arbitrary parent cdf $G(x)$, Cordeiro and de Castro (2011) defined the Kw–G class of distributions by the cdf $F(x)$ and pdf $f(x)$

$$F(x) = 1 - [1 - G(x)^a]^b \tag{3}$$

and

$$f(x) = abg(x)G(x)^{a-1}[1 - G(x)^a]^{b-1}, \tag{4}$$

respectively, where $g(x) = dG(x)/dx$ and a and b are two extra positive shape parameters. Clearly, for $a = b = 1$, we obtain the baseline distribution. If X is a random variable with pdf (4), we write $X \sim$ Kw–G(a, b), where a and b govern the skewness and tail weights. An attractive feature of this model is that these parameters can afford greater control over the weights in both tails and in its center. Equation (4) does not involve any special function, such as the incomplete beta function, as is the case of the beta-G class of distributions (Eugene et al., 2002). The generalization (4) contains distributions with unimodal and bathtub shaped hazard functions. It also contemplates several models with monotonic and non-monotonic hazard rate functions (hrfs) as shown in the plots of Figures 1 and 2.

Next, we define the new model by inserting (1) in equation (3). Then, the cdf (for $x > 0$) of the Kw–CWG model, say $F(x) = F(x; \alpha, \beta, \gamma, a, b)$, reduces to

$$F(x) = 1 - \{1 - \alpha^a(1 - \exp[-(\gamma x)^\beta])^a\{\alpha + (1-\alpha)\exp[-(\gamma x)^\beta]\}^{-a}\}^b. \tag{5}$$

The Kw–CWG pdf follows by inserting (1) and (2) in equation (4)

$$\begin{aligned} f(x) = \alpha^a \beta\gamma ab(\gamma x)^{\beta-1} \exp[-(\gamma x)^\beta] \frac{\{1 - \exp[-(\gamma x)^\beta]\}^{a-1}}{\{\alpha + (1-\alpha)\exp[-(\gamma x)^\beta]\}^{a+1}} \\ \times \left\{1 - \frac{\alpha^a[1 - \exp[-(\gamma x)^\beta]]^a}{\{\alpha + (1-\alpha)\exp[-(\gamma x)^\beta]\}^a}\right\}^{b-1}. \end{aligned} \tag{6}$$

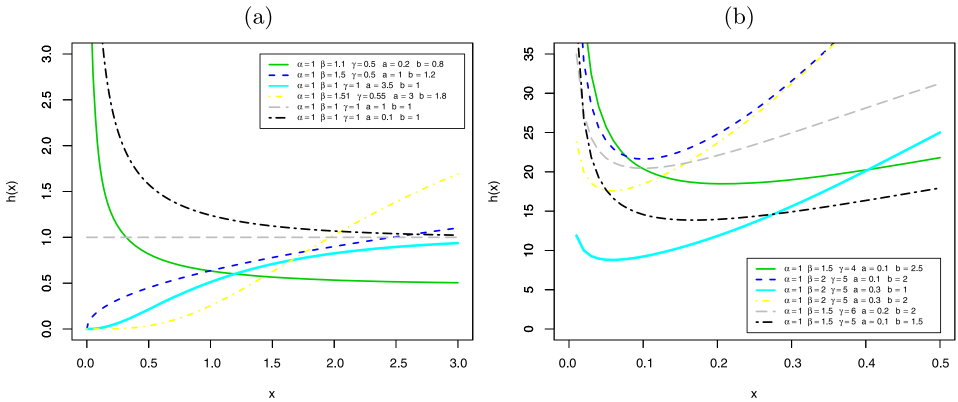

Figure 2 *Plots of the hrf for some parameter values.*

Henceforth, let X be a random variable having the pdf (6), that is, $X \sim$Kw–CWG$(\alpha, \beta, \gamma, a, b)$. We derive linear representations for the pdf and cdf of the Kw–CWG model in the Appendix.

The survival function (sf), hrf, cumulative hazard rate function (chrf) and reversed hazard rate function (rhrf) of X are given by

$$S(x) = \{1 - \alpha^a[1 - \exp[-(\gamma x)^\beta]]^a\{\alpha + (1-\alpha)\exp[-(\gamma x)^\beta]\}^{-a}\}^b, \tag{7}$$

$$h(x) = \alpha^a \beta\gamma a b(\gamma x)^{\beta-1}\{\alpha + (1-\alpha)\exp[-(\gamma x)^\beta]\}^{-a-1}\exp[-(\gamma x)^\beta] \times \{1 - \exp[-(\gamma x)^\beta]\}^{a-1}\left(1 - \frac{\alpha^a\{1-\exp[-(\gamma x)^\beta]\}^a}{\{\alpha + (1-\alpha)\exp[-(\gamma x)^\beta]\}^a}\right)^{-1},$$

$$H(x) = -b\log\{1 - \alpha^a[1 - \exp[-(\gamma x)^\beta]]^a\{\alpha + (1-\alpha)\exp[-(\gamma x)^\beta]\}^{-a}\}$$

and

$$r(x) = \frac{\alpha^a\beta\gamma a b(\gamma x)^{\beta-1}\exp[-(\gamma x)^\beta]\{1 - \exp[-(\gamma x)^\beta]\}^{a-1}}{\{\alpha + (1-\alpha)\exp[-(\gamma x)^\beta]\}^{a+1}} \times \left(1 - \frac{\alpha^a\{1-\exp[-(\gamma x)^\beta]\}^a}{\{\alpha + (1-\alpha)\exp[-(\gamma x)^\beta]\}^a}\right)^{b-1} \times \left[1 - \left(1 - \frac{\alpha^a\{1-\exp[-(\gamma x)^\beta]\}^a}{\{\alpha + (1-\alpha)\exp[-(\gamma x)^\beta]\}^a}\right)^b\right]^{-1},$$

respectively. The Kw–CWG distribution is a very flexible model having several special cases. It contains 23 sub-models listed in Table 1. It also includes eight important special models, namely: the Kumaraswamy complementary exponential geometric (Kw–CEG), Kumaraswamy complementary Rayleigh geometric (Kw–CRG), generalized complementary Weibull geometric (GCWG), generalized complementary exponential geometric (GCEG), generalized complementary Rayleigh

Table 1 *Special models of the Kw–CWG distribution*

No.	Reduced model	Parameters α	β	γ	a	b	Author
1	Kw–CEG	α	1	γ	a	b	New
2	Kw–CRG	α	2	γ	a	b	New
3	Kw–W	1	β	γ	a	b	Cordeiro et al. (2010)
4	Kw–E	1	1	γ	a	b	–
5	Kw–R	1	2	γ	a	b	Gomes et al. (2014)
6	GCWG	α	β	γ	1	b	New
7	GCEG	α	1	γ	1	b	New
8	GCRG	α	2	γ	1	b	New
9	GW	1	β	γ	1	b	Mudholkar et al. (1996)
10	GE	1	1	γ	1	b	Gupta and Kundu (1999)
11	GR	1	2	γ	1	b	Kundu and Raqab (2005)
12	ECWG	α	β	γ	a	1	New
13	ECEG	α	1	γ	a	1	New
14	ECRG	α	2	γ	a	1	New
15	EW	1	β	γ	a	1	Nassar and Eissa (2003)
16	EE	1	1	γ	a	1	Gupta and Kundu (2001)
17	ER	1	2	γ	a	1	Kundu and Raqab (2005)
18	CWG	α	β	γ	1	1	Tojeiro et al. (2014)
19	CRG	α	2	γ	1	1	–
20	CEG	α	1	γ	1	1	Louzada et al. (2011)
21	W	1	β	γ	1	1	Weibull (1951)
22	E	1	1	γ	1	1	–
23	R	1	2	γ	1	1	Rayleigh (1880)

geometric (GCRG), exponentiated complementary Weibull geometric (ECWG), exponentiated complementary exponential geometric (ECEG) and exponentiated complementary Rayleigh geometric (ECRG) distributions.

Figures 3(a), (b), 4(a), (b) and 5(a), (b) display some plots of the Kw–CWG density for some values of the parameters α, β, γ, a and b. Further, the hrf of the new distribution is very flexible in accommodating all different forms (see Figures 1 and 2) and thus it becomes an important model to fit real lifetime data.

The paper is organized as follows. In Section 2, we obtain the quantile function (qf), ordinary and incomplete moments, moment generating function (mgf), Rényi and q-entropies, mean residual life (MRL) and mean inactivity time (MIT) of X. In Section 3, the moments of the order statistics are determined. In Section 4, we obtain the maximum likelihood estimates (MLEs) of the model parameters. In Section 5, some simulation results investigate the performance of these estimates. In Section 6, we illustrate the potentiality of the new distribution by means of two real data analyzes. Finally, in Section 7, we offer some concluding remarks.

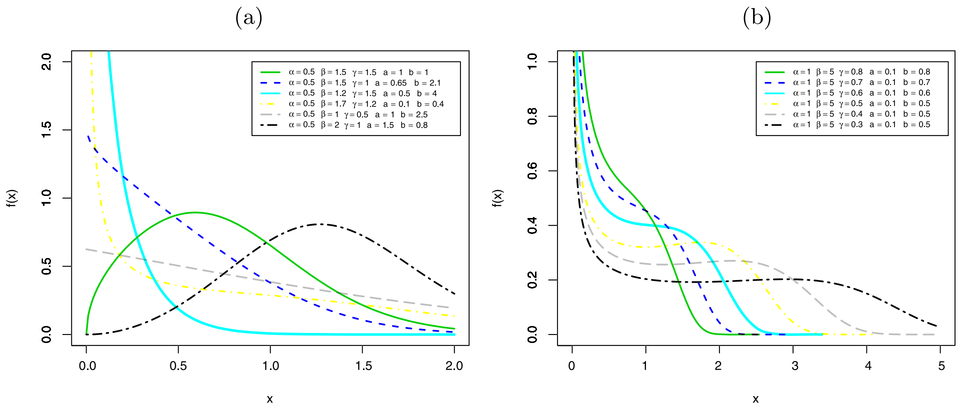

Figure 3 *Plots of the Kw–CWG density function for some parameter values.*

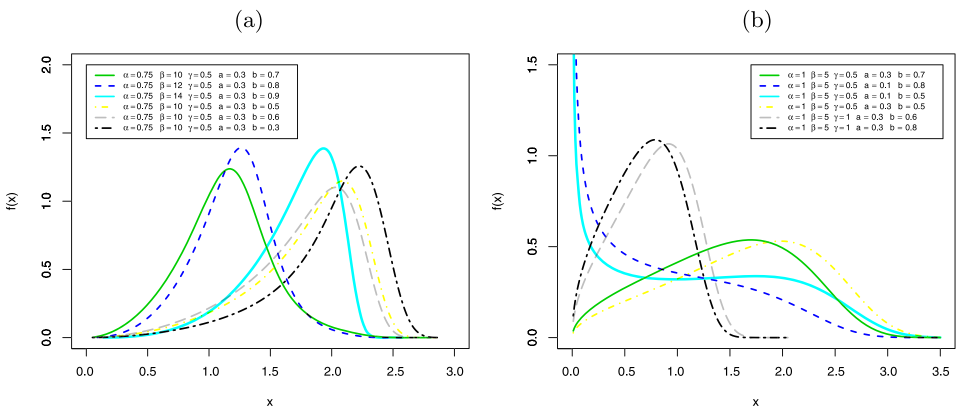

Figure 4 *Plots of the Kw–CWG density function for some parameter values.*

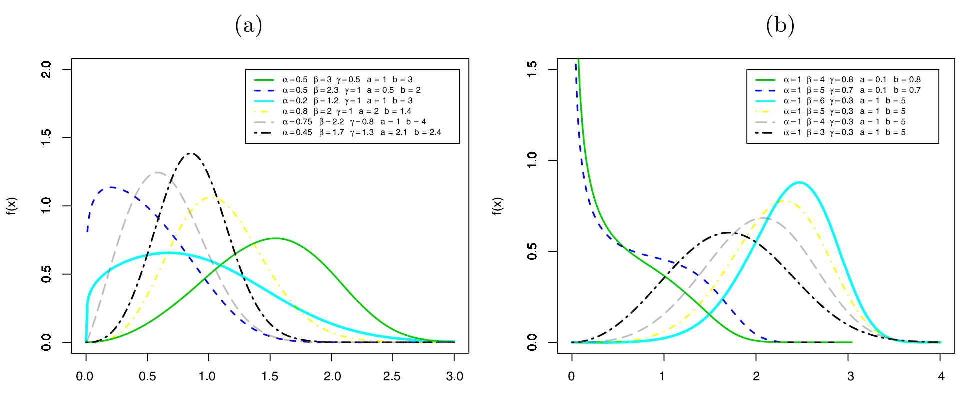

Figure 5 *Plots of the Kw–CWG density function for some parameter values.*

2 Mathematical properties

Established algebraic expansions to determine some structural quantities of the Kw–CWG distribution can be more efficient than computing those directly by numerical integration of its density function.

2.1 Quantile and random number generation

The qf of X, say $Q(u)$, is obtained by inverting (5)

$$Q(u) = \gamma^{-1}\left\{\log\left[\frac{\alpha + (1-\alpha)\sqrt[a]{1-\sqrt[b]{1-u}}}{\alpha(1-\sqrt[a]{1-\sqrt[b]{1-u}})}\right]\right\}^{1/\beta}, \qquad 0 < u < 1.$$

Simulating the Kw–CWG random variable is straightforward. If U is a uniform variate on the unit interval (0, 1), then the random variable $X = Q(U)$ has density (6).

2.2 Moments

Henceforth, let Z be a random variable having the Weibull distribution with scale $\gamma > 0$ and shape $\beta > 0$. Then, the pdf of Z is given by

$$f(z) = \beta\gamma^{\beta} z^{\beta-1}\{1 - \exp[-(\gamma z)^{\beta}]\}.$$

The rth ordinary and incomplete moments of Z are given by

$$\mu'_{r,Z} = \gamma^{-r}\Gamma\left(1+\frac{r}{\beta}\right) \quad \text{and} \quad \varphi_{r,Z}(t) = \gamma^{-r}\gamma\left(1+\frac{r}{\beta}, (\gamma t)^{\beta}\right),$$

respectively, where $\gamma(s,t) = \int_0^t x^{s-1}\mathrm{e}^{-x}\,dx$ is the lower incomplete gamma function.

Then, the rth ordinary moment of X, say μ'_r, can be expressed from equation (19) (see Appendix) as

$$\mu'_r = \gamma^{-r}\sum_{k,i=0}^{\infty} s_{k,i}(k+i+1)^{-r/\beta}\Gamma(1+r/\beta). \tag{8}$$

Using the relation between the central and non-central moments, we obtain the nth central moment of X, say μ_n, as follows

$$\mu_n = \gamma^{-r}\sum_{r=0}^{n}\sum_{k,i=0}^{\infty}\binom{n}{r}\frac{s_{k,i}(-\mu'_1)^{n-r}}{(k+i+1)^{r/\beta}}\Gamma(1+r/\beta).$$

The skewness and kurtosis measures of X can be determined from the central moments using well-known relationships.

2.3 Generating function

First, we provide the generating function of the Weibull model as discussed by Nadarajah et al. (2013). We can write the mgf of Z as

$$M(t;\gamma,\beta)=\beta\gamma^{\beta}\int_0^{\infty}\exp(tx)x^{\beta-1}\exp[-(\gamma x)^{\beta}]\,dx.$$

By expanding the first exponential and determining the integral, we obtain

$$M(t;\gamma,\beta)=\sum_{m=0}^{\infty}\frac{(t/\gamma)^m}{m!}\Gamma(\beta+m/\beta).$$

Consider the Wright generalized hypergeometric function defined by

$${}_p\Psi_q\left[\begin{matrix}(\gamma_1,A_1),\ldots,(\gamma_p,A_p)\\(\beta_1,B_1),\ldots,(\beta_q,B_q)\end{matrix};x\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(\gamma_j+A_jn)}{\prod_{j=1}^{q}\Gamma(\beta_j+B_jn)}\frac{x^n}{n!}.$$

Then, we can write $M(t;\gamma,\beta)$ as

$$M(t;\gamma,\beta)={}_1\Psi_0\left[\begin{matrix}(1,-\beta^{-1})\\-\end{matrix};\left(\frac{t}{\gamma}\right)\right].$$

Combining the last expression and equation (19), the mgf of X reduces to

$$M_X(t)=\sum_{k,i=0}^{\infty}s_{k,i}\,{}_1\Psi_0\left[\begin{matrix}(1,-\beta^{-1})\\-\end{matrix};(k+i+1)^{-1/\beta}t/\gamma\right].$$

2.4 Rényi and q-entropies

The Rényi entropy of a random variable represents a measure of variation of the uncertainty. The Rényi entropy X is defined by

$$I_\delta=\frac{1}{1-\delta}\log\left(\int_{-\infty}^{\infty}f^{\delta}(x)\,dx\right),\qquad \delta>0\text{ and }\delta\neq 1.$$

We can write from equation (6)

$$f^{\delta}(x)=K\sum_{k,i=0}^{\infty}d_{k,i}\alpha^{2aj-\delta}\left(1-\frac{1}{\alpha}\right)^{k}x^{\delta(\beta-1)}\exp[-(k+i+\delta)(\gamma x)^{\beta}],$$

where $K=(ab\beta\gamma^{\beta})^{\delta}$ and

$$d_{k,i}=\sum_{j=0}^{\infty}\frac{(-1)^{j+i}\Gamma(aj+a\delta+\delta+k)\Gamma(aj+a\delta-\delta+1)}{j!k!i!\Gamma(\delta b-\delta-j+1)\Gamma(aj+a\delta+\delta)}$$
$$\times\frac{\Gamma(\delta b-\delta+1)(ab)^{\delta}(\beta\gamma)^{\delta-1}}{\alpha^{\delta}\Gamma(aj+a\delta-\delta-i+1)}\left(1-\frac{1}{\alpha}\right)^{k}.$$

Then, I_δ reduces to

$$I_\delta = \frac{1}{1-\delta} \log\left\{ K \sum_{k,i=0}^{\infty} d_{k,i} \alpha^{2aj-\delta} \left(1 - \frac{1}{\alpha}\right)^k \right.$$
$$\left. \times \int_0^\infty x^{\delta(\beta-1)} \exp[-(k+i+\delta)(\gamma x)^\beta]\, dx \right\}.$$

Further,

$$\int_0^\infty x^{\delta(\beta-1)} \exp[-(k+i+\delta)(\gamma x)^\beta]\, dx$$
$$= \frac{\gamma^{\delta(1-\beta)-1}}{\beta} (k+i+\delta)^{(\delta(1-\beta)-1)/\beta} \Gamma\left(\frac{\delta(\beta-1)+1}{\beta}\right)$$

and then

$$I_\delta = (1-\delta)^{-1} \log\left\{ K \sum_{k,i=0}^{\infty} d_{k,i} (k+i+\delta)^{-n} \Gamma\left(\frac{\delta(\beta-1)+1}{\beta}\right)\right\}. \tag{9}$$

The q-entropy, say H_q, is defined by

$$H_q = \frac{1}{q-1} \log(1 - J_q),$$

where $J_q = \int_{\Re} f^q(x)\, dx$ ($q > 0$ and $q \neq 1$), follows from (9) as $J_q = (1-q) I_q$.

2.5 Mean residual life and mean inactivity time

The MRL has many applications in biomedical sciences, life insurance, maintenance and product quality control, economics and social studies, demography and product technology (see Lai and Xie, 2006). Guess and Proschan (1988) gave an extensive coverage of possible applications this quantity. The MRL (or the life expectancy at age t) is given by $m_X(t) = E(X - t \mid X > t)$, for $t > 0$, and it represents the expected additional life length for a unit, which is alive at age t.

The MRL of X can be obtained as

$$m_X(t) = [1 - \varphi_1(t)]/R(t) - t, \tag{10}$$

where $\varphi_1(t) = \int_0^t x f(x)\, dx$ is the first incomplete moment of X.

By using equation (19), we obtain

$$\varphi_1(t) = \sum_{k,i=0}^{\infty} \frac{(k+i+1)^{-1/\beta} s_{k,i}}{\gamma} \gamma\left(1 + \frac{1}{\beta}, (k+i+1)(\gamma t)^\beta\right).$$

By substituting this equation in (10), we can write

$$m_X(t) = \frac{1}{\gamma R(t)} \sum_{k,i=0}^{\infty} \frac{s_{k,i}}{(k+i+1)^{1/\beta}} \gamma\left(1 + \frac{1}{\beta}, (k+i+1)(\gamma t)^\beta\right) - t.$$

The MIT defined by $M_X(t) = E(t - X \mid X \le t)$ (for $t > 0$) represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0, t)$.

The MIT of X is given by

$$M_X(t) = t - \left[\varphi_1(t)/F(t)\right]. \tag{11}$$

By inserting the first incomplete moment in equation (11), the MIT of X is given by

$$M_X(t) = t - \frac{1}{\gamma F(t)} \sum_{k,i=0}^{\infty} \frac{s_{k,i}}{(k+i+1)^{1/\beta}} \gamma\left(1 + \frac{1}{\beta}, (k+i+1)(\gamma t)^{\beta}\right).$$

3 Order statistics

The order statistics and their moments have great importance in many statistical problems and applications in reliability analysis and life testing. Let $X_1, \ldots, X_n$ be a random sample of size n from the Kw–CWG$(\alpha, \beta, \gamma, a, b)$ with cdf (5) and pdf (6), respectively. Let $X_{1:n}, \ldots, X_{n:n}$ be the corresponding order statistics. Then, the pdf of rth order statistic, say $X_{r:n}$, $1 \le r \le n$, denoted by $f_{r:n}(x)$, can be expressed as

$$\begin{aligned} f_{r:n}(x) &= C_{r:n}\alpha^a \beta\gamma ab(\gamma x)^{\beta-1} \exp[-(\gamma x)^{\beta}] d_x^{a-1} l_x^{-(a+1)} \\ &\quad \times \left[1 - \left(1 - \alpha^a d_x^a l_x^{-a}\right)^b\right]^{r-1} \left(1 - \alpha^a d_x^a l_x^{-a}\right)^{b(n-r+1)-1}, \end{aligned}$$

where $C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$, $d_x = \{1 - \exp[-(\gamma x)^{\beta}]\}$ and $l_x = \{\alpha + (1-\alpha) \times \exp[-(\gamma x)^{\beta}]\}$.

The pdf of $X_{r:n}$ can also be expressed as

$$f_{r:n}(x) = \frac{f(x)}{\mathrm{B}(r, n-r+1)} \sum_{s=0}^{n-1} (-1)^s \binom{n-1}{s} F^{s+r-1}(x). \tag{12}$$

Further, we can write

$$F^{s+r-1}(x) = \sum_{m=0}^{\infty} (-1)^m \binom{s+r-1}{m} \left(1 - \frac{\alpha^a \{1 - \exp[-(\gamma x)^{\beta}]\}^a}{\{\alpha + (1-\alpha)\exp[-(\gamma x)^{\beta}]\}^a}\right)^b. \tag{13}$$

By inserting (6) and (13) in equation (12) and, after some simplification, we obtain

$$f_{r:n}(x) = \sum_{k,i=0}^{\infty} b_{k,i} h_{k+i+1}(x), \tag{14}$$

where $h_{k+i+1}(x)$ denotes to the Weibull pdf with shape parameter β and scale parameter $\gamma(k+i+1)^{1/\beta}$ and

$$b_{k,i}=\sum_{s=0}^{r-1}\sum_{m,j=0}^{\infty}\frac{(-1)^{j+i+m+s}b\Gamma(bm+b)\Gamma(aj+a+k+1)}{(j+1)!k!i!\alpha\Gamma(bm+m-j)\Gamma(aj+a-i)}$$
$$\times B(r,n-r+1)\left(1-\frac{1}{\alpha}\right)^k\binom{s+r-1}{m}\binom{n-1}{s}.$$

Equation (14) reveals that the pdf of the Kw–CWG order statistics is a mixture of Weibull densities. So, some of their mathematical properties can also be obtained from those of the Weibull distribution. For example, the pth moment of $X_{r:n}$ can be expressed as

$$E\left(X_{r:n}^p\right)=\gamma^{-p}\Gamma(1+p/\beta)\sum_{s=0}^{n-1}\frac{(-1)^s\binom{n-1}{s}}{\mathrm{B}(r,n-r+1)}\sum_{k,i=0}^{\infty}\frac{b_{k,i}}{(k+i+1)^{p/\beta}}.$$

The joint pdf of $X_{(r:n)}$ and $Y_{(j:n)}$, $1\le r\le j\le n$, is given by

$$\begin{aligned}f_{r:j:n}(x,y)=&C_{r:j:n}\left(\alpha^a\beta ab\right)^2\gamma^{2\beta}(xy)^{\beta-1}\\&\times\exp\left[-(\gamma x)^\beta\right]e^{-(\gamma y)^\beta}(d_xd_y)^{a-1}(l_xl_y)^{-a-1}\\&\times\left[\left(1-\alpha^ad_x^al_x^{-a}\right)\left(1-\alpha^ad_y^al_y^{-a}\right)\right]^{b-1}\left[1-\left(1-\alpha^ad_x^al_x^{-a}\right)^b\right]^{r-1}\\&\times\left[\left(1-\alpha^ad_x^al_x^{-a}\right)^b\right]^{n-j}\left[\left(1-\alpha^ad_y^al_y^{-a}\right)^b-\left(1-\alpha^ad_x^al_x^{-a}\right)^b\right]^{j-r-1},\end{aligned}$$

where $C_{r:j:n}=n!/(r-1)!(j-r-1)!(n-j)!$.

4 Maximum likelihood estimation

Several approaches for parameter estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. The MLEs enjoy desirable properties and can be used for constructing confidence intervals for the model parameters. The normal approximation for these estimators in large sample distribution theory is easily handled either analytically or numerically.

Let X_i be a random variable following (6) with vector of parameters $\boldsymbol{\theta}=(\alpha,\beta,\gamma,a,b)^T$. The data encountered in survival analysis and reliability studies are often censored. A very simple random censoring mechanism that is often realistic is one in which each individual i is assumed to have a lifetime X_i and a censoring time C_i, where X_i and C_i are independent random variables. Suppose that the data consist of n independent observations $x_i=\min(X_i,C_i)$ for $i=1,\dots,n$. The distribution of C_i does not depend on any of the unknown parameters of X_i.

Let F and C be the sets of individuals for which x_i is the lifetime or censoring, respectively. Parametric inference for such data is usually based on likelihood methods and their asymptotic theory.

The log-likelihood function for the vector of parameters $\boldsymbol{\theta}$ from model (6) has the form $l(\boldsymbol{\theta}) = \sum_{i\in F} l_i(\boldsymbol{\theta}) + \sum_{i\in C} l_i^{(c)}(\boldsymbol{\theta})$, where $l_i(\boldsymbol{\theta}) = \log[f(x_i)]$, $l_i^{(c)}(\boldsymbol{\theta}) = \log[S(x_i)]$, $f(x_i)$ is the density (6) and $S(x_i)$ is survival function (7) of X_i. The total log-likelihood function for $\boldsymbol{\theta}$ is given by

$$
\begin{aligned}
l(\boldsymbol{\theta}) = r\log\big(\alpha^a\beta\gamma ab\big) &+ (\beta-1)\sum_{i\in F}\log(\gamma x_i) + \sum_{i\in F}\log(1-u_i)\\
&+ (a-1)\sum_{i\in F}\log(u_i) - (a+1)\sum_{i\in F}\log(z_i) \qquad (15)\\
&+ (b-1)\sum_{i\in F}\log\left[1-\alpha^a\left(\frac{u_i}{z_i}\right)^a\right] + b\sum_{i\in C}\log\left[1-\alpha^a\left(\frac{u_i}{z_i}\right)\right],
\end{aligned}
$$

where r is the number of uncensored observations (failures),

$$
z_i = \alpha + (1-\alpha)\exp\big[-(\gamma x_i)^\beta\big] \quad \text{and} \quad u_i = 1-\exp\big[-(\gamma x_i)^\beta\big].
$$

The MLE $\widehat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ can be determined by maximizing the log-likelihood (15). We can use the MATHCAD program, R (optim function), SAS (NLMixed procedure), Ox program (sub-routine MaxBFGS) or, alternatively, by solving the nonlinear equations obtained by differentiating the log-likelihood.

The score vector $\mathbf{U}(\boldsymbol{\theta}) = (\mathbf{U}(\alpha), \mathbf{U}(\beta), \mathbf{U}(\gamma), \mathbf{U}(a), \mathbf{U}(b))^T$ has components given by

$$
\begin{aligned}
\mathbf{U}(\alpha) = \frac{ar}{\alpha} &- (a+1)\sum_{i\in F}\frac{u_i}{z_i} + b\alpha^{a-1}\sum_{i\in C}\frac{\alpha u_i^2 z_i^{-2} - a u_i z_i^{-1}}{1-\alpha^a u_i z_i^{-1}}\\
&+ a\alpha^{a-1}(b-1)\sum_{i\in F}^{n}\frac{\alpha u_i^{a+1} z_i^{-a-1} - u_i^a z_i^{-a}}{1-\alpha^a u_i^a z_i^{-a}},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{U}(\beta) = \sum_{i\in F}\log(\gamma x_i) &- a\alpha^a(b-1)\sum_{i\in F}\frac{p_i u_i^{a-1} z_i^{-a} + (1-\alpha)p_i u_i^a z_i^{-a-1}}{1-\alpha^a u_i^a z_i^{-a}}\\
&- \sum_{i\in F}\frac{p_i}{1-u_i} + \frac{r}{\beta} - b\alpha^a\sum_{i\in C}\frac{p_i z_i^{-1} + (1-\alpha)p_i u_i z_i^{-2}}{1-\alpha^a u_i z_i^{-1}}\\
&+ (a-1)\sum_{i\in F}\frac{p_i}{u_i} + (a+1)(1-\alpha)\sum_{i\in F}\frac{p_i}{z_i},
\end{aligned}
$$

$$
\mathbf{U}(\gamma) = \sum_{i\in F}\frac{(\beta-1)}{\gamma} + (a-1)\sum_{i\in F}\frac{q_i}{u_i}
$$

$$- a\alpha^{a}(b-1)\sum_{i\in F}\frac{q_i[(1-\alpha)u_i^{a}z_i^{-a-1}+u_i^{a-1}z_i^{-a}]}{1-\alpha^{a}u_i^{a}z_i^{-a}}+\frac{r}{\gamma}-\sum_{i\in F}\frac{q_i}{1-u_i}$$

$$+(a+1)\sum_{i\in F}\frac{(1-\alpha)q_i}{z_i}-b\alpha^{a}\sum_{i\in C}\frac{q_i[z_i^{-1}+(1-\alpha)u_i z_i^{-2}]}{1-\alpha^{a}u_i z_i^{-1}},$$

$$\mathbf{U}(a)=\frac{r}{a}\big[1+a\log(\alpha)\big]+\sum_{i\in F}\log(u_i)-\sum_{i\in F}\log(z_i)-\sum_{i\in C}\frac{b\alpha^{a}u_i}{z_i-\alpha^{a}u_i}$$

$$+(b-1)\alpha^{a}\sum_{i\in F}\frac{u_i^{a}[\log(z_i)-\log(u_i)-\log(\alpha)]}{z_i^{a}-\alpha^{a}u_i^{a}}$$

and

$$\mathbf{U}(b)=\frac{r}{b}+\sum_{i\in F}\log\big(1-\alpha^{a}u_i^{a}z_i^{-a}\big)+\sum_{i\in C}\log\big(1-\alpha^{a}u_i z_i^{-1}\big),$$

where

$$p_i=(\gamma x_i)^{\beta}\exp\big[-(\gamma x_i)^{\beta}\big]\log(\gamma x_i)\quad\text{and}\quad q_i=\frac{\beta}{\gamma}(\gamma x_i)^{\beta}\exp\big[-(\gamma x_i)^{\beta}\big].$$

Initial values for β and γ are usually taken from the fit of the Weibull model with $a=1$, $b=1$ and $\alpha=1$. Then, the estimated survival function for X_i is given by

$$\hat{S}(x_i;\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{a},\hat{b})=\big\{1-\hat{\alpha}^{\hat{a}}\big[1-\exp\big[-(\hat{\gamma}x)^{\beta}\big]\big]^{\hat{a}}$$
$$\times\big\{\hat{\alpha}+(1-\hat{\alpha})\exp\big[-(\gamma x)^{\hat{\beta}}\big]\big\}^{-\hat{a}}\big\}^{\hat{b}}.$$

Under general regularity conditions, the asymptotic distribution of $(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta})$ is multivariate normal $N_{p+3}(0,K(\boldsymbol{\theta})^{-1})$, where $K(\boldsymbol{\theta})$ is the expected information matrix. The asymptotic covariance matrix $K(\boldsymbol{\theta})^{-1}$ of $\widehat{\boldsymbol{\theta}}$ can be approximated by the inverse of the 5×5 observed information matrix $-\ddot{\mathbf{L}}(\boldsymbol{\theta})$. The elements of the observed information matrix $-\ddot{\mathbf{L}}(\boldsymbol{\theta})$, namely $-\mathbf{L}_{\alpha\alpha}$,$-\mathbf{L}_{\alpha\beta}$, $-\mathbf{L}_{\alpha\gamma}$,$-\mathbf{L}_{\alpha a}$, $-\mathbf{L}_{\alpha b}$,$-\mathbf{L}_{\beta\beta}$, $-\mathbf{L}_{\beta\gamma}$,$-\mathbf{L}_{\beta a}$, $-\mathbf{L}_{\beta b}$, $-\mathbf{L}_{\gamma\gamma}$, $-\mathbf{L}_{\gamma a}$, $-\mathbf{L}_{\gamma b}$, $-\mathbf{L}_{aa}$, $-\mathbf{L}_{ab}$ and $-\mathbf{L}_{bb}$ can be evaluated numerically.

The approximate multivariate normal distribution $N_5(0,-\ddot{\mathbf{L}}(\boldsymbol{\theta})^{-1})$ for $\widehat{\boldsymbol{\theta}}$ can be used in the classical way to construct approximate confidence intervals for the parameters in $\boldsymbol{\theta}$. Further, we can use the likelihood ratio (LR) statistic for comparing some special models with the Kw–CWG model. We consider the partition $\boldsymbol{\theta}=(\boldsymbol{\theta}_1^T,\boldsymbol{\theta}_2^T)^T$, where $\boldsymbol{\theta}_1$ is a subset of parameters of interest and $\boldsymbol{\theta}_2$ is a subset of remaining parameters. The LR statistic for testing the null hypothesis $H_0:\boldsymbol{\theta}_1=\boldsymbol{\theta}_1^{(0)}$ versus the alternative hypothesis $H_1:\boldsymbol{\theta}_1\neq\boldsymbol{\theta}_1^{(0)}$ is given by $w=2\{\ell(\widehat{\boldsymbol{\theta}})-\ell(\widetilde{\boldsymbol{\theta}})\}$, where $\widetilde{\boldsymbol{\theta}}$ and $\widehat{\boldsymbol{\theta}}$ are the estimates under the null and alternative hypotheses, respectively. The statistic w is asymptotically (as $n\to\infty$) distributed as χ_k^2, where k is the dimension of the subset of parameters $\boldsymbol{\theta}_1$ of interest. For example, we can compute the maximum values of the unrestricted and restricted log-likelihoods to construct LR statistics for testing some sub-models such as those listed in Table 1 of the Kw–CWG distribution.

5 Simulation study

We obtain some frequentist properties of the MLEs by means of a simulation study for $n = 50$, 100, 200 and 300, each one with 1,000 generated data sets. We simulate the Kw–CWG model under two setups for the model parameters: $\alpha = 0.25$, $\beta = 0.5$, $\gamma = 0.25$, $a = 0.5$ and $b = 0.75$ (setup 1) and $\alpha = 0.5$, $\beta = 2.0$, $\gamma = 0.75$, $a = 1.25$ and $b = 1.5$ (setup 2).

The censoring times C_i are sampled from the uniform distribution in the interval $(0, \tau)$, where τ denotes the proportion of censored observations. In this study, the proportions of censored observations are approximately equal to 20% in both setups.

Table 2 lists the averages of the MLEs (Mean), the biases and the mean square errors (MSEs) and 95% coverage probabilities (PCs). The figures in this table indicate that the MSEs increase when the censoring percentage increases. Further, the MSEs of the MLEs of α, β, γ, a and b decay toward zero as the sample size increases, as expected under standard asymptotic theory. In fact, the biases of the estimates tend to be closer to the true parameter values if n increases. This fact supports that the asymptotic normal distribution provides an adequate approximation to the finite sample distribution of the MLEs. The normal approximation can be oftentimes improved by using bias adjustments to these estimators. Approximations to their biases in simple models may be obtained analytically. Bias correction typically does a very good job for correcting the MLEs. We do not encounter any non-identifiability problems in the simulation study.

6 Applications

In this section, we provide two applications to real data to prove empirically the flexibility of the Kw–CWG distribution.

6.1 Application 1: The gauge lengths data

The first data set refers to the gauge lengths of 20 mm (Kundu and Raqab, 2009) and consists of $n = 74$ observations. These data were previously analyzed by Nofal et al. (2017). For these data, we compare the fits of the new model with some sub-models and other non-nested competitive models, namely: Kw–CRG, Kw–CEG, CWG, beta Weibull (BW), Kumaraswamy transmuted log-logistic (Kw–TLL), McDonald Weibull (McW), modified beta Weibull (MBW) and exponentiated transmuted generalized Rayleigh (ETGR) distributions.

Table 2 *Summaries of the estimates for the Kw–CWG model*

Setup	Mean of censored observation (in %)	Sample size (n)	Parameter	Summaries of parameters		
				Mean	Bias	MSE
Setup 1	0%	50	α	0.2748	0.0248	0.0249
			β	0.5623	0.0623	0.0201
			γ	0.3748	0.1248	0.2614
			a	0.4951	−0.0049	0.0379
			b	0.8401	0.0901	0.1449
		100	α	0.2575	0.0075	0.0176
			β	0.5336	0.0336	0.0112
			γ	0.3462	0.0962	0.1468
			a	0.4945	−0.0055	0.0173
			b	0.8501	0.1001	0.1159
		200	α	0.2605	0.0105	0.0146
			β	0.5162	0.0162	0.0063
			γ	0.3012	0.0512	0.0612
			a	0.5020	0.0020	0.0103
			b	0.8515	0.1015	0.0980
		300	α	0.2608	0.0108	0.0117
			β	0.5107	0.0107	0.0050
			γ	0.2793	0.0293	0.0394
			a	0.5102	0.0102	0.0074
			b	0.8593	0.1093	0.0802
	20%	50	α	0.2453	−0.0047	0.0194
			β	0.5497	0.0497	0.0262
			γ	0.5419	0.2919	0.5019
			a	0.4975	−0.0025	0.0505
			b	0.7445	−0.0055	0.1303
		100	α	0.2595	0.0095	0.0141
			β	0.5278	0.0278	0.0139
			γ	0.4755	0.2255	0.3739
			a	0.5007	0.0007	0.0252
			b	0.7556	0.0056	0.1008
		200	α	0.2609	0.0109	0.0085
			β	0.5183	0.0183	0.0082
			γ	0.3772	0.1272	0.1244
			a	0.4976	−0.0024	0.0122
			b	0.7535	0.0035	0.0697
		300	α	0.2586	0.0086	0.0056
			β	0.5136	0.0136	0.0064
			γ	0.3644	0.1144	0.1146
			a	0.4969	−0.0031	0.0081
			b	0.7503	0.0003	0.0642

Table 2 (*Continued*)

Setup	Mean of censored observation (in %)	Sample size (n)	Parameter	Summaries of parameters Mean	Bias	MSE
Setup 2	0%	50	α	0.3797	−0.1203	0.0549
			β	2.3339	0.3339	0.4965
			γ	0.9356	0.1856	0.1145
			a	1.0077	−0.2423	0.2282
			b	1.0124	−0.4876	0.4806
		100	α	0.4114	−0.0886	0.0466
			β	2.1955	0.1955	0.3007
			γ	0.8913	0.1413	0.0799
			a	1.0853	−0.1647	0.1517
			b	1.1878	−0.3122	0.3975
		200	α	0.4637	−0.0363	0.0354
			β	2.1244	0.1244	0.1714
			γ	0.843	0.0930	0.0514
			a	1.1536	−0.0964	0.0932
			b	1.3105	−0.1895	0.3486
		300	α	0.4763	−0.0237	0.0322
			β	2.1157	0.1157	0.1561
			γ	0.8245	0.0745	0.0441
			a	1.1713	−0.0787	0.0828
			b	1.3631	−0.1369	0.3158
	20%	50	α	0.3683	−0.1317	0.0567
			β	2.4279	0.4279	0.6717
			γ	0.9443	0.1943	0.1237
			a	0.9656	−0.2844	0.2483
			b	0.9688	−0.5312	0.6026
		100	α	0.4247	−0.0753	0.0472
			β	2.2651	0.2651	0.3602
			γ	0.8901	0.1401	0.0849
			a	1.0714	−0.1786	0.1637
			b	1.1563	−0.3437	0.4604
		200	α	0.4480	−0.0520	0.0354
			β	2.1430	0.1430	0.2130
			γ	0.8716	0.1216	0.0665
			a	1.1450	−0.1050	0.1044
			b	1.2367	−0.2633	0.3738
		300	α	0.4673	−0.0327	0.0313
			β	2.1276	0.1276	0.1812
			γ	0.8449	0.0949	0.0528
			a	1.1586	−0.0914	0.0837
			b	1.2937	−0.2063	0.3233

The density functions (for $x > 0$) corresponding to these alternative models are presented below.

- The BW density function (Lee et al., 2007 and Cordeiro et al., 2013) is given by

$$f(x) = \frac{\beta\alpha^{\beta}}{B(a,b)} x^{\beta-1} \exp[-b(\alpha x)^{\beta}]\{1-\exp[-(\alpha x)^{\beta}]\}^{a-1}.$$

- The Kw–TLL density function (Afify et al., 2016) is given by

$$\begin{aligned} f(x) = {} & \frac{ab\beta x^{\beta-1}}{\alpha^{\beta}[1+(\frac{x}{\alpha})^{\beta}]^2}\left\{1-\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-1}\right\}^{a-1}\left(1-\lambda\left\{1-\frac{2}{[1+(\frac{x}{\alpha})^{\beta}]}\right\}\right) \\ & \times\left\{1+\frac{\lambda}{[1+(\frac{x}{\alpha})^{\beta}]}\right\}^{a-1}\left(1-\left\{1+\frac{\lambda}{[1+(\frac{x}{\alpha})^{\beta}]}\right\}^{a} \\ & \times\left\{1-\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-1}\right\}^{a}\right)^{b-1}. \end{aligned}$$

- The McW density function (Cordeiro et al., 2014) is given by

$$\begin{aligned} f(x) = {} & \frac{\beta c\alpha^{\beta}}{B(a/c,b)} x^{\beta-1} e^{-(\alpha x)^{\beta}} 1\{-\exp[-(\alpha x)^{\beta}]\}^{a-1} \\ & \times\{1-[1-\exp[-(\alpha x)^{\beta}]]^{c}\}^{b-1}. \end{aligned}$$

- The MBW density function (Khan, 2015) is given by

$$\begin{aligned} f(x) = {} & \frac{\beta c^{a}}{\alpha^{\beta}B(a,b)} x^{\beta-1}\exp\left[-b\left(\frac{x}{\alpha}\right)^{\beta}\right]\left\{1-\exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]\right\}^{a-1} \\ & \times\left(1-(1-c)\left\{1-\exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]\right\}\right)^{-a-b}. \end{aligned}$$

- The ETGR density function (Afify et al., 2015) is given by

$$\begin{aligned} f(x) = {} & 2\alpha\delta\beta^{2}x\exp[-(\beta x)^{2}]\{1+\lambda-2\lambda[1-\exp[-(\beta x)^{2}]]^{\alpha}\} \\ & \times\{1-\exp[-(\beta x)^{2}]\}^{\alpha\delta-1}\{1+\lambda-\lambda[1-\exp\{-(\beta x)^{2}\}]^{\alpha}\}^{\delta-1}. \end{aligned}$$

The parameters of the densities above are all positive real numbers except for the Kw–TLL and ETGR distributions, where $|\lambda| \leq 1$.

In order to compare these distributions, we consider the goodness-of-fit measures including $-2\widehat{\ell}$, where $\widehat{\ell}$ is the maximized log-likelihood, the Cramér–von Mises (W^*) and Anderson–Darling (A^*) statistics. The statistics W^* and A^* are described in details in Chen and Balakrishnan (1995). In general, the smaller their values, the better the fit to the data.

Table 3 *MLEs, their SEs (in parentheses) and $-2\widehat{\ell}$, W^* and A^* statistics for the gauge length data*

Model	Estimates					-2ℓ	W^*	A^*
Kw–CWG$(\alpha, \beta, \gamma, a, b)$	0.5297	2.0499	0.4097	3.3854	7.2202	102.26	0.026	0.204
	(1.54)	(3.355)	(0.544)	(4.722)	(31.678)			
Kw–CRG(α, γ, a, b)	0.1963	0.5774	2.6483	2.6155		102.32	0.026	0.191
	(0.412)	(0.278)	(1.963)	(3.957)				
Kw–CEG(α, γ, a, b)	0.0244	1.4785	2.9423	5.4372		102.43	0.026	0.193
	(0.053)	(0.708)	(1.737)	(2.995)				
BW(α, β, a, b)	0.417	4.575	1.563	0.835		102.30	0.027	0.214
	(0.367)	(2.461)	(1.195)	(4.334)				
CWG(α, β, γ)	0.0965	3.2134	0.5239			104.69	0.042	0.327
	(0.166)	(1.374)	(0.17)					
Kw–TLL$(\alpha, \beta, a, b, \lambda)$	0.4909	1.0608	73.3394	100.0885	0.6761	104.53	0.066	0.44
	(0.448)	(0.223)	(87.092)	(91.115)	(0.439)			
Mc–W(α, β, a, b, c)	1.438	0.583	83.720	14.428	3.460	108.80	0.119	0.779
	(1.447)	(0.211)	(78.890)	(15.870)	(9.663)			
MBW(α, β, a, b, c)	1.765	1.426	36.336	3.361	3.096	109.10	0.124	0.811
	(1.097)	(1.488)	(4.439)	(6.695)	(4.714)			
ETGR$(\alpha, \beta, \lambda, \gamma)$	2.121	0.698	0.320	7.790		113.40	0.207	1.340
	(0.315)	(0.040)	(0.228)	(1.727)				

Table 3 provides the MLEs of the model parameters, their corresponding standard errors (SEs) and the values of $-2\widehat{\ell}$, W^* and A^*. The plots of the fitted Kw–CWG pdf and other fitted pdfs defined before, for the current data, are displayed in Figure 6. Figure 7 displays the estimated cdf and estimated survival function of the Kw–CWG distribution. The QQ-plots of the fitted models are given in Figure 8. They reveal that the Kw–CWG, Kw–CRG and Kw–CEG distributions provide the best fits and they can be considered very competitive models to other distributions with positive support.

6.2 Application 2: Serum reversal data-censored

Aids is a pathology that mobilizes its sufferers because of the implications for their interpersonal relationships and reproduction. Therapeutic advances have enabled seropositive women to bear children safely. Here, we analyze a data set on the time to serum reversal of 148 children exposed to HIV by vertical transmission, born at Hospital das Clínicas (Ribeirão Preto School of Medicine) from 1995 to 2001, where the mothers were not treated (Silva, 2004; Perdoná, 2006). Vertical HIV transmission can occur during gestation in around 35% of cases, during labor and birth itself in some 65% of cases, or during breast feeding, varying from 7% to 22% of cases. Serum reversal or serological reversal can occur in children of HIV-contaminated mothers. It is the process by which HIV antibodies disappear from

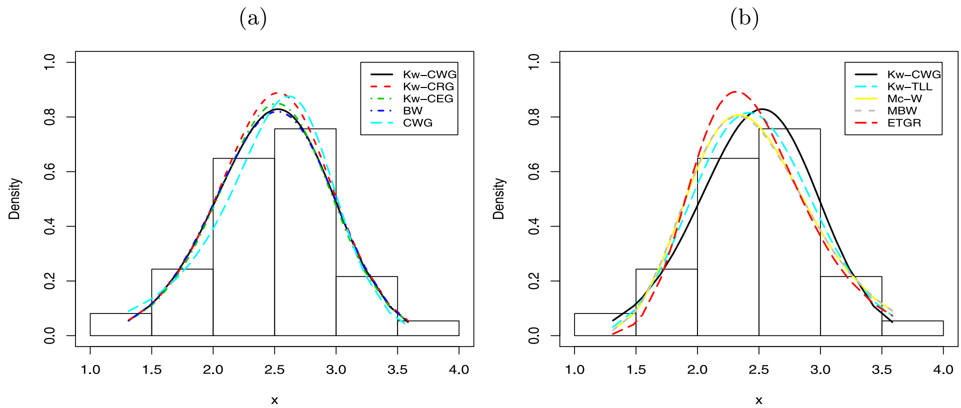

Figure 6 *The estimated Kw–CWG pdf and other estimated pdfs*. (a) *The estimated Kw–CWG*, *Kw–CRG*, *Kw–CEG*, *BW and CWG densities*. (b) *The estimated Kw–CWG*, *Kw–TLL*, *Mc–W*, *MBW and ETGR densities*.

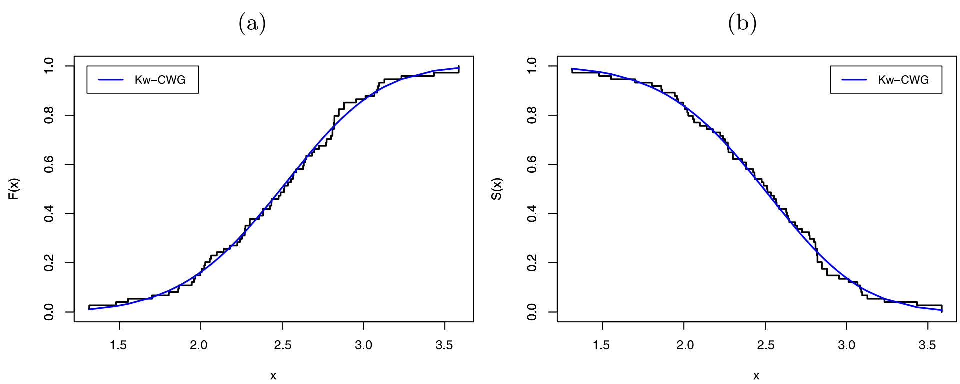

Figure 7 (a) *The estimated cdf of the Kw–CWG model*. (b) *The estimated survival function of the Kw–CWG model*.

the blood in an individual who tested positive for HIV infection. As the months pass, the maternal antibodies are eliminated and the child ceases to be HIV positive. The exposed newborns were monitored until definition of their serological condition, after administration of Zidovudin (AZT) in the first 24 hours and for the following 6 weeks. We assume that the lifetimes are independently distributed, and also independent from the censoring mechanism.

In order to compare the distributions, we consider some goodness-of-fit measures including the Akaike information criterion (*AIC*), Bayesian information criterion (*BIC*), Hannan–Quinn information criterion (*HQIC*), consistent Akaike information criterion (*CAIC*) and $-2\widehat{\ell}$. These goodness-of-fit statistics evaluated at

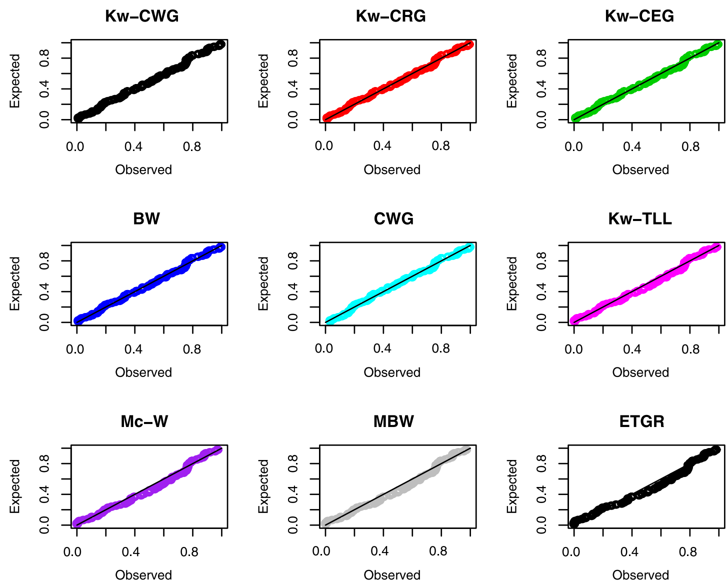

Figure 8 *QQ-plots of the Kw–CWG distribution and other competitive distributions.*

Table 4 *Goodness of fit statistic for the serum reversal data*

Model	$-2\hat{l}$	AIC	CAIC	BIC	HQIC
Kw–CWG	765.4	775.4	775.8	790.4	781.5
Kw–CEG	771.6	779.6	779.8	791.6	784.5
Kw–W	775.5	783.5	783.8	795.5	788.4
GCWG	775.7	783.7	784.0	795.7	788.6
ECWG	766.9	774.9	775.2	786.9	779.8
GW	790.2	796.2	796.4	805.2	799.9
Weibull	804.0	808.0	808.1	814.0	810.4

the MLEs are given by

$$AIC = -2\widehat{\ell} + 2k, \qquad CAIC = -2\widehat{\ell} + 2kn/(n-k-1),$$
$$HQIC = -2\widehat{\ell} + 2k\log\bigl(\log(n)\bigr) \quad \text{and} \quad BIC = -2\widehat{\ell} + k\log(n).$$

where k is the number of parameters and n is the sample size.

Tables 4 and 5 provide the values of the $-2\hat{l}$, AIC, BIC, CAIC and HQIC statistics and the MLEs (and their SEs in parentheses) of the parameters, respectively. These results indicate that the Kw-CWG and ECWG models have the lowest $-2\hat{l}$, AIC, BIC, CAIC and HQIC values among those of all fitted models, and then they could be chosen as the best models.

Table 5 *MLEs of the model parameters and the corresponding SEs (given in parentheses) for the serum reversal data*

Model	α	β	γ	a	b
Kw–CWG	0.0003 (0.0005)	6.6656 (0.2783)	0.0036 (0.0001)	0.2604 (0.0460)	3.0844 (1.9301)
Kw–CEG	1e-8 (1e-9)	1	0.0466 (0.0039)	0.2899 (0.0562)	2.0234 (1.3789)
Kw–W	1	6.7550 (2.1887)	0.0039 (0.0004)	0.2358 (0.0756)	0.1769 (0.1037)
GCWG	0.00008 (0.0001)	1.2577 (0.5792)	0.0092 (0.0080)	1	297.85 (0.0008)
ECWG	0.0054 (0.0052)	6.6560 (2.5641)	0.0036 (0.00009)	0.2218 (0.0307)	1
GW	1	6.3607 (0.2782)	0.0028 (0.0001)	0.3670 (0.0462)	1
Weibull	1	3.1132 (0.3250)	0.0033 (0.0001)	1	1

Table 6 *LR statistics for the serum reversal data*

Model	Hypotheses	Statistics w	P-value
Kw–CWG vs Kw–CEG	$H_0 : \beta = 1$ vs $H_1 : H_0$ is false	6.2	0.0128
Kw–CWG vs Kw–W	$H_0 : \alpha = 1$ vs $H_1 : H_0$ is false	10.1	0.0015
Kw–CWG vs GCWG	$H_0 : a = 1$ vs $H_1 : H_0$ is false	10.3	0.0013
Kw–CWG vs ECWG	$H_0 : b = 1$ vs $H_1 : H_0$ is false	1.5	0.2207
Kw–CWG vs GW	$H_0 : b = \alpha = 1$ vs $H_1 : H_0$ is false	24.8	<0.0001
Kw–CWG vs Weibull	$H_0 : a = b = \alpha = 1$ vs $H_1 : H_0$ is false	38.6	<0.0001

A comparison of the proposed distribution with some of its sub-models using LR statistics is addressed in Table 6. The numbers in this table, specially the p-values, suggest that the new Kw–CWG and ECWG models yield better fits to these data than the other three distributions.

In order to assess if the model is appropriate, plots of the estimated survival functions of the Kw–CWG, Kw–CEG, Kw–W, GCWG, ECWG, GW and Weibull distributions and the empirical survival function are displayed in Figures 9 and 10. We conclude that the Kw–CWG and ECWG models provide good fits to these data.

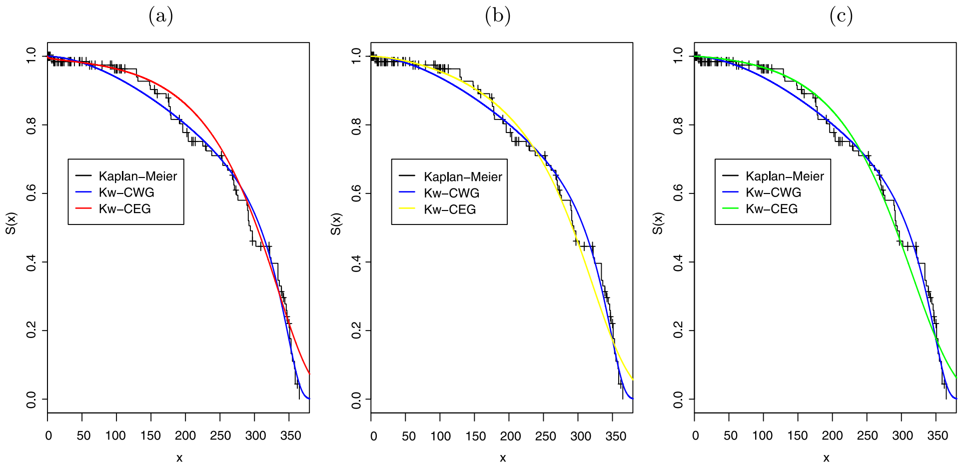

Figure 9 *Estimated survival function for the fitted Kw–CWG distribution and some other models and the empirical survival function for the serum reversal data.* (a) *Kw–CWG vs Kw–CEG.* (b) *Kw–CWG vs Kw–W.* (c) *Kw–CWG vs GCWG.*

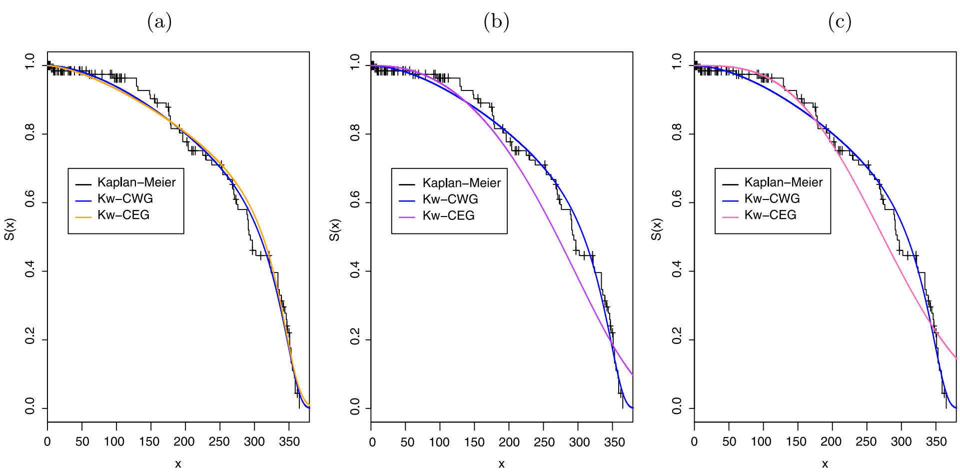

Figure 10 *Estimated survival function for the fitted Kw–CWG distribution and some other models and the empirical survival function for the serum reversal data.* (a) *Kw–CWG vs ECWG.* (b) *Kw–CWG vs GW.* (c) *Kw–CWG vs Weibull.*

7 Concluding remarks

In this paper, we propose a new five-parameter model, named the *Kumaraswamy complementary Weibull geometric* (Kw–CWG) distribution, which extends the complementary Weibull geometric (CWG) distribution introduced by Tojeiro et al. (2014). The Kw–CWG model is motivated by the wide use of the Weibull distribution in practice and also for the fact that the generalization provides more flexibility to analyze positive real data. We provide some of its mathematical and statistical properties. The Kw–CWG density function can be expressed as a mixture of Weibull densities. We derive explicit expressions for the ordinary moments, generating function, Rényi entropy, mean residual life and mean inactivity time. We obtain the density function of the order statistics and their moments. We investigate the maximum likelihood estimation of the model parameters. We also provide some simulation results to assess the performance of the proposed model. Two applications to real data illustrate that the proposed distribution provides consistently better fits than other nested and non-nested models. We hope that the new model will attract wider applications in areas such as engineering, survival and lifetime data, hydrology, economics (income inequality) and others.

Appendix: Mixture representation

We derive mixture representations for the pdf and cdf of X. In order to obtain a simple form for the Kw–CWG pdf, we expand (6) using the power series

$$(1-z)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{j!\Gamma(b-j)} z^j, \qquad |z| < 1, b > 0. \tag{16}$$

Using expansion (16) in equation (6) and after some algebra, the pdf of X can be expressed as

$$f(x) = \beta\gamma ab(\gamma x)^{\beta-1} \exp[-(\gamma x)^\beta] \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{j!\Gamma(b-j)} \alpha^{a(j+1)} \times \{1-\exp[-(\gamma x)^\beta]\}^{a(j+1)-1} \{\alpha + (1-\alpha)\exp[-(\gamma x)^\beta]\}^{-[a(j+1)+1]}.$$

For $|z| < 1, b > 0$, the power series holds

$$(1-z)^{-b} = \sum_{j=0}^{\infty} \frac{\Gamma(b+j)}{j!\Gamma(b)} z^j. \tag{17}$$

By applying (17) in the expression $\{\alpha + (1-\alpha)\exp[-(\gamma x)^\beta]\}^{-[a(j+1)+1]}$, we obtain

$$f(x) = \beta\gamma ab\alpha^{-1}(\gamma x)^{\beta-1} \sum_{j,k=0}^{\infty} \frac{(-1)^j \Gamma(b)\Gamma(aj+a+k+1)}{j!k!\Gamma(b-j)\Gamma(aj+a+1)} \times \left(1-\frac{1}{\alpha}\right)^k \exp[-(k+1)(\gamma x)^\beta]\{1-\exp[-(\gamma x)^\beta]\}^{a(j+1)-1}.$$

By using (16) in the last binomial term, the pdf of X becomes

$$f(x) = \beta(k+i+1)\gamma^{\beta}x^{\beta-1}\sum_{k,i=0}^{\infty} s_{k,i}\exp[-(k+i+1)(\gamma x)^{\beta}], \tag{18}$$

where $s_{k,i}$ is a constant given by

$$s_{k,i} = \sum_{j=0}^{\infty} \frac{(-1)^{j+i} a\Gamma(b+1)\Gamma(aj+a)\Gamma(aj+a+k+1)}{j!k!i!\alpha(k+i+1)\Gamma(b-j)\Gamma(aj+a+1)\Gamma(aj+a-i)}\left(1-\frac{1}{\alpha}\right)^{k}.$$

Equation (18) can be rewritten as

$$f(x) = \sum_{k,i=0}^{\infty} s_{k,i}h_{k+i+1}(x), \tag{19}$$

where $h_{k+i+1}(x)$ is the Weibull pdf with shape parameter β and scale parameter $\gamma(k+i+1)^{1/\beta}$.

Equation (19) reveals that the Kw–CWG density function can be written as a linear mixture of Weibull densities. So, several of its structural properties can be obtained from those of the Weibull distribution.

By integrating (19), we obtain

$$F(x) = \sum_{k,i=0}^{\infty} s_{k,i}H_{k+i+1}(x), \tag{20}$$

where $H_{k+i+1}(x)$ is the Weibull cdf with shape parameter β and scale parameter $\gamma(k+i+1)^{1/\beta}$.

Equations (19) and (20) are the main results of this Appendix. They can be used to study some structural properties of the proposed family.

References

Adamidis, K. and Loukas, S. (1998). A lifetime distribution with decreasing failure rate. *Statistics and Probability Letters* **39**, 35–42. MR1649319

Afify, A. Z., Nofal, Z. M. and Butt, N. S. (2014). Transmuted complementary Weibull geometric distribution. *Pakistan Journal of Statistics and Operation Research* **10**, 435–454. MR3302433

Afify, A. Z., Nofal, Z. M. and Ebraheim, A. N. (2015). Exponentiated transmuted generalized Rayleigh distribution: A new four parameter Rayleigh distribution. *Pakistan Journal of Statistics and Operation Research* **11**, 115–134. MR3638751

Afify, A. Z., Cordeiro, G. M., Yousof, H. M., Alzaatreh, A. and Nofal, Z. M. (2016). The Kumaraswamy transmuted-*G* family of distributions: Properties and applications. *Journal of Data Science* **14**, 245–270.

Barreto-Souza, W., de Morais, A. L. and Cordeiro, G. M. (2011). The Weibull-geometric distribution. *Journal of Statistical Computation and Simulation* **81**, 645–657. MR2788571

Chen, G. and Balakrishnan, N. (1995). A general purpose approximate goodness-of-fit test. *Journal of Quality Technology* **27**, 154–161.

Cordeiro, G. M. and de Castro, M. (2011). A new family of generalized distributions. *Journal of Statistical Computation and Simulation* **81**, 883–898. MR2806932

Cordeiro, G. M., Ortega, E. M. M. and Nadarajah, S. (2010). The Kumaraswamy Weibull distribution with application to failure data. *Journal of The Franklin Institute* **347**, 1399–1429. MR2720934

Cordeiro, G. M., Nadarajah, S. and Ortega, E. M. M. (2013). General results for the beta Weibull distribution. *Journal of Statistical Computation and Simulation* **83**, 1082–1114. MR3169222

Cordeiro, G. M., Hashimoto, E. M. and Ortega, E. M. M. (2014). The McDonald Weibull model. *Statistics: A Journal of Theoretical and Applied Statistics* **48**, 256–278. MR3175769

Eugene, N., Lee, C. and Famoye, F. (2002). Beta-normal distribution and its applications. *Communications in Statistics—Theory and Methods* **31**, 497–512. MR1902307

Gomes, A. E., da-Silva, C. Q., Cordeiro, G. M. and Ortega, E. M. M. (2014). A new lifetime model: The Kumaraswamy generalized Rayleigh distribution. *Journal of Statistical Computation and Simulation* **84**, 290–309. MR3169327

Guess, F. and Proschan, F. (1988). Mean residual life, theory and applications. *Handbook of Statistics* **7**, 215–224.

Gupta, R. D. and Kundu, D. (1999). Generalized exponential distributions: Theory and methods. *Australian and New Zealand Journal of Statistics* **41**, 173–188. MR1705342

Gupta, R. D. and Kundu, D. (2001). Exponentiated exponential family: An alternative to gamma and Weibull distributions. *Biometrical Journal* **43**, 117–130. MR1847768

Khan, M. N. (2015). The modified beta Weibull distribution. *Hacettepe Journal of Mathematics and Statistics* **44**, 1553–1568.

Kundu, D. and Raqab, M. Z. (2005). Generalized Rayleigh distribution: Different methods of estimations. *Computational Statistics and Data Analysis* **49**, 187–200. MR2129172

Kundu, D. and Raqab, M. Z. (2009). Estimation of $R = P(Y < X)$ for three parameter Weibull distribution. *Statistics and Probability Letters* **79**, 1839–1846. MR2560482

Lai, C. D. and Xie, M. (2006). *Stochastic Ageing and Dependence for Reliability*. *Springer Science and Business Media*. Berlin: Springer. MR2223811

Lee, C., Famoye, F. and Olumolade, O. (2007). Beta-Weibull distribution: Some properties and applications to censored data. *Journal of Modern Applied Statistical Methods* **6**, 173–186.

Louzada, F., Roman, M. and Cancho, V. G. (2011). The complementary exponential geometric distribution: Model, properties, and a comparison with its counterpart. *Computational Statistics and Data Analysis* **55**, 2516–2524. MR2787009

Louzada, F., Marchi, V. and Carpenter, J. (2013). The complementary exponentiated exponential geometric lifetime distribution. *Journal of Probability and Statistics* **2013**, Article ID 502159. MR3037955

Mudholkar, G. S., Srivastava, D. K. and Kollia, G. D. (1996). A generalization of the Weibull distribution with application to the analysis of survival data. *Journal of the American Statistical Association* **91**, 1575–1583. MR1439097

Nadarajah, S., Cordeiro, G. M. and Ortega, E. M. M. (2013). The exponentiated Weibull distribution: A survey. *Statistical Papers* **54**, 839–877. MR3072904

Nassar, M. M. and Eissa, F. H. (2003). On the exponentiated Weibull distribution. *Communications in Statistics—Theory and Methods* **32**, 1317–1336. MR1985853

Nofal, Z. M., Afify, A. Z., Yousof, H. M. and Cordeiro, G. M. (2017). The generalized transmuted-G family of distributions. *Communications in Statistics—Theory and Methods* **46**, 4119–4136.

Perdoná, G. S. C. (2006). Modelos de Riscos Aplicados a Análise de Sobrevivência. Doctoral thesis, Institute of Computer Science and Mathematics, University of São Paulo, Brasil (in Portuguese).

Rayleigh, J. W. S. (1880). On the resultant of a large number of vibration of the same pitch and arbitrary phase. *Philosophical Magazine, 5th Series* **10**, 73–78.

Rinne, H. (2009). *The Weibull Distribution: A Handbook*. London: CRC Press. MR2477856

Silva, A. N. F. (2004). Estudo evolutivo das crianças expostas ao HIV e notificadas pelo núcleo de vigilância epidemiológica do HCFMRP-USP. M.Sc. thesis, University of São. Paulo, Brasil (in Portuguese).

Tojeiro, C., Louzada, F., Roman, M. and Borges, P. (2014). The complementary Weibull geometric distribution. *Journal of Statistical Computation and Simulation* **84**, 1345–1362. MR3169397

Weibull, W. (1951). A statistical distribution function of wide applicability. *Journal of Applied Mechanics, Transactions, ASME* **18**, 293–297.

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