The Generalized Odd Weibull Generated Family of Distributions: Statistical Properties and Applications

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Abstract

In this work, we propose a new class of lifetime distributions calledthe generalized odd Weibull generatedfamily. It can provide better fits than some of the well known lifetime models and this fact represents a good characterization of this family. Some of its mathematical properties are derived. The maximum likelihood method is used for estimating the model parameters. We study the behaviour of the estimators by means of two Monte Carlo simulations. The importance of the family illustrated by means of two applications to real data sets.

Keywords: Generated distribution; Maximum likelihood; Moment; Quantile function; Simulation.

1. Introduction

Recently, some attempts have been made to define new families of distributions that extend well-known distributions and at the same time provide great flexibility in modelling data in practice. So, several classes by adding one or more parameters to generate new distributions have been proposed in the statistical literature. Some well-known generators are Gupta et al. (1998) who proposed the exponentiated-G class, which consists of raising the cumulative distribution function (cdf) to a positive power parameter. Many other classes cited by Marshall and Olkin (1997), Eugene et al. (2002), Cordeiro et al. (2013), Alzaatreh et al. (2013), Yousof et al. (2015), Merovc et al. (2016), Yousof et al. (2016), Alizadeh et al. (2016a,b), Afify et al.(2016a,b,c,d), Aryal and Yousof (2017), Korkmaz and Genç (2017), Hamedani et al. (2017), Brito et al. (2017), Alizadeh et al. (2017b), Cordeiro et al. (2017a,b), Yousof et al. (2017a,b,c,d), Nofal et al. (2017), Hamedani et al. (2017), Hamedani et al. (2018), Yousof et al. (2018), Korkmaz et al. (2018), among others. Let $r(t)$ be the probability density function (pdf) of a random variable $T \in d, e]$ for $-\infty \leq d < e < \infty$ and let $W[G(x)]$ be a function of the cdf of a random variable X such that $W[G(x)]$ satisfies the following conditions:

$$\begin{cases} (i) & W[G(x)] \in d, e], \\ (ii) & W[G(x)] \text{is differentiable and monotonically non-decreasing, and} \\ (iii) & W[G(x)] \to d \text{ as } x \to -\infty \text{ and } W[G(x)] \to d \text{ as } x \to \infty. \end{cases} \quad (1)$$

Alzaatreh et al. (2013) defined the T-X family of distributions by

$$F(x) = \int_d^{W[G(x)]} r(t)\,dt, \quad (2)$$

where $W[G(x)]$ satisfies the conditions (1). The pdf corresponding to (2) is given by

$$f(x) = \left\{\frac{d}{dx} W[G(x)]\right\} r\{W[G(x)]\} \quad (3)$$

In this paper, we propose a new wider class of continuous distributions called the Generalized Odd Weibull Generated ("GOWG-G" for short) family by taking $W[G(x)] = \frac{G(x;\phi)^\alpha}{1-G(x;\phi)^\alpha}$ and $r(t) = \beta t^{\beta-1} e^{-t^\beta}$, $t > 0, \alpha > 0, \beta > 0$. Its cumulative distribution function (cdf) is given by

$$F(x;\alpha,\beta,\phi) = \int_0^{\frac{G(x;\phi)^\alpha}{1-G(x;\phi)^\alpha}} \beta t^{\beta-1} e^{-t^\beta} = 1 - \exp\left\{-\left[\frac{G(x;\phi)^\alpha}{1-G(x;\phi)^\alpha}\right]^\beta\right\}. \quad (4)$$

The corresponding pdf is given by

$$f(x;\alpha,\beta,\phi) = \frac{\alpha\beta g(x;\phi)\, G(x;\phi)^{\alpha\beta-1}}{[1-G(x;\phi)^\alpha]^{\beta+1}} \exp\left\{-\left[\frac{G(x;\phi)^\alpha}{1-G(x;\phi)^\alpha}\right]^\beta\right\}, \quad (5)$$

where $g(x;\phi)$ is the baseline pdf. Hereafter, a random variable X with density function (5) is denoted by $X \sim$GOWG-G(α,β,ϕ). Further, we can omit sometimes the dependence on the vector ϕ of the parameters and write simply $G(x) = G(x;\phi)$ and $g(x) = g(x;\phi)$.

The hazard rate function (hrf) of X becomes

$$h(x;\alpha,\beta,\phi) = \frac{\alpha\beta g(x)\, G(x)^{\alpha\beta-1}}{[1-G(x)^\alpha]^{\beta+1}}. \quad (6)$$

An interpretation of the GOW family (4) can be given as follows. Let T be a random variable describing a stochastic system by the cdf $G(x)^\alpha$ (for $\alpha > 0$). Then, if the random variable X represents the odds ratio, the risk that the system following the lifetime T will be not working at time x is given by $G(x)^\alpha/[1-G(x)^\alpha]$. Suppose that we are interested in modeling the randomness of the odds ratio using a Weibull model with cdf $R(t) = 1 - \exp(-t^\beta)$ (for $t > 0$). Then, the cdf of X can be written as

$$Pr(X \leq x) = R\left[\frac{G(x)^\alpha}{1-G(x)^\alpha}\right],$$

which is exactly equal to the family (4).

If $U \sim u(0,1)$ then the solution of nonlinear equation

$$x_u = G^{-1}\left([-\log(1-u)]^{\frac{1}{\alpha\beta}}\left\{1 + [-\log(1-u)]^{\frac{1}{\beta}}\right\}^{-\frac{1}{\alpha}}\right). \quad (7)$$

By using taylor expansion and generalized binomial expansion we can demonstrate that the pdf (5) of X has the expansion

$$f(x;\alpha,\beta,\phi)=\sum_{i,j=0}^{\infty} w_{i,j} h_{\alpha\beta(i+1)+\alpha j}(x), \tag{8}$$

where

$$w_{i,j}=\alpha\beta(i!)^{-1}(-1)^{i+j}[\alpha\beta(i+1)+\alpha j]^{-1}\binom{-\beta(i+1)-1}{j},$$

and $h_\delta(x)=\delta G^{\delta-1}(x)g(x)$ is the pdf of the Exp-G distribution with power parameter δ. The corresponding GOWG-G cdf is obtained by integrating (8)

$$F(x;\alpha,\beta,\phi)=\sum_{i,j=0}^{\infty} w_{i,j} H_{\alpha\beta(i+1)+\alpha j+1}(x), \tag{9}$$

where $H_\delta(x)=G^\delta(x)$ denotes the exponentiated-G ("Exp-G" for short) cumulative distribution. Equation (7) reveals that the GOWG-G density function is a linear combination of Exp-G density functions. Thus, some mathematical properties of the new model can be derived from those properties of the Exp-G distribution. For example, the ordinary and incomplete moments and moment generating function (mgf) of X can be obtained from those quantities of the Exp-G distribution. Let $a=\inf\{x|G(x)>0\}$, the asymptotics of equations (4), (5) and (6) as $x\to a$ are given by

$$F(x)\sim G(x)^{\alpha\beta} as x\to a,$$
$$f(x)\sim\alpha\beta g(x)\, G(x)^{\alpha\beta-1} as x\to a,$$
$$h(x)\sim\alpha\beta g(x)\, G(x)^{\alpha\beta-1} as x\to a.$$

The asymptotics of equations (4), (5) and (6) as $x\to\infty$ are given by

$$1-F(x)\sim\exp\{-[\alpha\bar{G}(x)]^{-\beta}\} as x\to\infty$$
$$f(x)\sim\beta\alpha^{-\beta}g(x)\bar{G}(x)^{-(\beta+1)}\exp\{-[\alpha\bar{G}(x)]^{-\beta}\} as x\to\infty,$$
$$h(x)\sim\beta\alpha^{-\beta}g(x)\bar{G}(x)^{-(\beta+1)} as x\to\infty.$$

The rest of the paper is organized as follows. In Section 2, we derive some of the mathematical properties of the new family. Maximum likelihood estimation for the model parameters under uncensored data is addressed in Section 3. Two simulation studies are performed in Section 4 to assess the performance of the maximum likelihood estimations. In Section 5, potentiality of the proposed class is illustrated by means of two real data sets. Finally, Section 6 provides some concluding remarkes.

2. Some statistical properties

2.1 General properties

The r^{th} moment of X, say μ'_r, follows from (8) as

$$\mu'_r=E(X^r)=\sum_{i,j=0}^{\infty} w_{i,j}E\left(Z^r_{\alpha\beta(i+1)+\alpha j}\right). \tag{10}$$

Henceforth, Z_ζ denotes the Exp-G distribution with power parameter ζ. For $\zeta>0$, we have $E(Z_\zeta^r)=\zeta\int_{-\infty}^{\infty}x^r g(x;\varphi)G(x;\varphi)^{\zeta-1}dx$, which can be computed numerically in terms of the baseline quantile function (qf) $Q_G(u;\varphi)=G^{-1}(u;\varphi)$ as $E(Z_\zeta^n)=\zeta\int_0^1 Q_G(u;\varphi)^n u^{\zeta-1}du$. The variance, skewness, and kurtosis measures can now be calculated using the well known relations. The n^{th} central moment of X, say M_n, is given by

$$M_n = E(X-\mu_1')^n = \sum_{i,j=0}^{\infty}\sum_{r=0}^{n} l_{i,j,r} E\left(Y_{\alpha\beta(i+1)+\alpha j}^r\right),$$

where $l_{i,j,r} = w_{i,j}(-1)^{n-r}\binom{n}{r}(\mu_r')^{n-r}$. The s^{th} incomplete moment, say $\varphi_s(t)$, of X can be expressed from (8) as

$$\varphi_s(t) = \int_{-\infty}^{t} x^s f(x)dx = \sum_{i,j=0}^{\infty} w_{i,j}\int_{-\infty}^{t} x^s h_{\alpha\beta(i+1)+\alpha j}(x)dx. \quad (11)$$

The mgf $M_X(t) = E(e^{tX})$ of X. can be derived from equation (8) as $M_X(t) = \sum_{i,j=0}^{\infty} w_{i,j} M_{\alpha\beta(i+1)+\alpha j}(t)$, where $M_\zeta(t)$ is the mgf of Z_ζ. Hence, $M_X(t)$ can be determined from the Exp-G generating function.

2.2 Moments of the residual and reversed residual lifes

The n^{th} moment of the residual life, say $z_n(t) = E[(X-t)^n|X>t], n = 1,2,...$, uniquely determines $F(x)$ (see Navarro et al., 1998). The n^{th} moment of the residual life of X is given by $z_n(t) = [1-F(t)]^{-1}\int_t^{\infty}(x-t)^n dF(x)$. Therefore,

$$z_n(t) = [1-F(t)]^{-1}\sum_{i,j=0}^{\infty}\sum_{r=0}^{n} \Upsilon_{i,j,r}\int_t^{\infty} x^r h_{\alpha\beta(i+1)+\alpha j}(x)dx, \quad (12)$$

where $\Upsilon_{i,j,r} = w_{i,j}\binom{n}{r}(-t)^{n-r}$. The n^{th} moment of the reversed residual life, say $M_n(t) = E[(t-X)^n|X \le t]$ for $t > 0$ and $n = 1,2,...$ uniquely determines $F(x)$ (Navarro et al., 1998). We obtain $Z_n(t) = [F(t)]^{-1}\int_0^t (t-x)^n dF(x)$. Therefore, the n^{th} moment of the reversed residual life of X becomes

$$Z_n(t) = [F(t)]^{-1}\sum_{i,j,k=0}^{\infty}\sum_{r=0}^{n} \Upsilon_{i,j,r}^{*}\int_0^t x^r h_{\alpha\beta(i+1)+\alpha j}(x)dx, \quad (13)$$

where $\Upsilon_{i,j,r}^{*} = w_{i,j}(-1)^r\binom{n}{r}t^{n-r}$. The mean inactivity time (MIT) or mean waiting time (MWT) also called the mean reversed residual life function is given by $Z_1(t) = E[(t-X)|X \le t]$, and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0, t)$.

2.3 Entropies

An entropy is a measure of variation or uncertainty of a random variable X. Two popular entropy measures are the Rényi and Shannon entropies (Shannon, 1948; Renyi, 1961). The Rényi entropy of a random variable with pdf $f(x)$ is defined as

$$I_R(\gamma) = \frac{1}{1-\gamma}\log\left[\int_0^{\infty} f^{\gamma}(x)dx\right],$$

for $\gamma > 0$ and $\gamma \neq 1$. The Shannon entropy of a random variable X is defined by $E\{-\log[f(X)]\}$. It is the special case of the Rényi entropy when $\gamma \uparrow 1$. Direct calculation yields

$$E\{-\log[f(X)]\} = -\log(\alpha\beta) - E\{\log[g(X;\phi)]\} + (1-\alpha\beta)E\{\log[G(X;\phi)]\}$$
$$+(1+\beta)E\{\log[1-G(X;\phi)^{\alpha}]\} + E\left\{\left[\frac{G(X;\phi)^{\alpha}}{1-G(X;\phi)^{\alpha}}\right]^{\beta}\right\}.$$

First we define and compute

$$A(a_1, a_2; \alpha, \beta) = \int_0^1 u^{a_1}(1-u^{\alpha})^{-a_2}\exp\left[-\left(\frac{u^{\alpha}}{1-u^{\alpha}}\right)^{\beta}\right]du.$$

Using generalized binomial expansion and taylor expansion , we obtain

$$A(a_1, a_2; \alpha, \beta) = \sum_{i,j=0}^{\infty} (-1)^{i+j} (i!)^{-1} (a_1 + \alpha\beta i + \alpha j + 1)^{-1} \binom{-(a_2 + \beta i)}{j}.$$

Proposition 1 Let X be a random variable with pdf (5). Then

$$E\{\log[G(X)]\} = \alpha\beta \frac{\partial}{\partial t} A(\alpha\beta + t - 1, \beta + 1; \alpha, \beta)|_{t=0}$$

$$E\{\log[1 - G(X)^\alpha]\} = \alpha\beta \frac{\partial}{\partial t} A(\alpha\beta - 1, \beta + 1 - t; \alpha, \beta)|_{t=0}$$

$$E\left\{\left[\frac{G(X)^\alpha}{1 - G(X)^\alpha}\right]^\beta\right\} = \alpha\beta A(2\alpha\beta - 1, 2\beta + 1; \alpha, \beta)$$

The simplest formula for the entropy of X is given by

$$\begin{aligned} &E\{-\log[f(X)]\} = -\log(\alpha\beta) - E\{\log[g(X; \phi)]\} \\ &+(1 - \alpha\beta)\alpha\beta \frac{\partial}{\partial t} A(\alpha\beta + t - 1, \beta + 1; \alpha, \beta)|_{t=0} \\ &+(\beta + 1)\alpha\beta \frac{\partial}{\partial t} A(\alpha\beta - 1, \beta + 1 - t; \alpha, \beta)|_{t=0} \\ &+\alpha\beta A(2\alpha\beta - 1, 2\beta + 1; \alpha, \beta) \end{aligned}$$

After some algebraic developments, we obtain an alternative expression for

$$I_R(\gamma) I_R(\gamma) = \frac{\gamma}{1-\gamma} \log(\alpha\beta) + \frac{1}{1-\gamma} \log\left[\sum_{i,j=0}^{\infty} w_{i,j}^* I(\alpha, \beta, \gamma, i, j)\right]$$

where

$$I(\alpha, \beta, \gamma, i, j) = \int_0^\infty g(x)^\gamma G(x)^{\gamma(\alpha\beta - 1) + \alpha\beta i + \alpha j} dx$$

and

$$w_{i,j}^* = (i!)^{-1} (-1)^{i+j} \gamma i \binom{-[\gamma(\beta + 1) + \beta i]}{j}.$$

2.4 Order statistics

Suppose $X_1, X_2, \dots, X_n$ is a random sample from the F-G distribution. Let $X_{i:n}$ denote the ith order statistic. The pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = K f(x)\, F^{i-1}(x)\, \{1 - F(x)\}^{n-i} =$$

$$K \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} f(x)\, F(x)^{j+i-1}$$

where $K = n!/[(i-1)!\,(n-i)!]$. Then, the density function of the $X_{i:n}$ can be expressed as

$$f_{i:n} = \sum_{r,s=0}^{\infty} t_{i,j} h_{\alpha\beta(r+1)+\alpha s}(x) \qquad (14)$$

where

$$t_{i,j} = K\alpha\beta \sum_{j=0}^{n-i} \sum_{k=0}^{i+j-1} \frac{(-1)^{j+k+r+s} (k+1)^r}{r! [\alpha\beta(r+1) + \alpha s]} \binom{n-i}{j} \binom{-\beta(r+1) - 1}{s}$$

With using this expansion we can easily obtain moments,generating function and incomplete moment of order statistics from any G. Equation (14) reveals that the pdf of the GOWG-G order statistic can be expressed as a linear combination of the Exp-G densities. Therefore, some statistical and mathematical properties of these order statistics can be obtained by using this result. Analogous to the ordinary moments we can get the **L**-moments but it can be estimated by the linear combinations of order statistics in (14). They exist as long the mean of the distribution exists, even if some higher moments may not exist, and are relatively robust to the effects of outliers. Based upon the moments in Equation (14), we can derive explicit expressions for the **L**-moments of X as infinite weighted linear combinations of the means of suitable GOWG-G order statistics. They are linear functions of expected order statistics defined by

$$\lambda_r = \frac{1}{r}\sum_{d=0}^{r-1} (-1)^d \binom{r-1}{d} E(X_{r-d:r}), r \geq 1.$$

2.5 Probability weighted moments

Generally, the probability weighted moments (PWMs) method can be used for estimating parameters of a distribution whose inverse form cannot be expressed explicitly. The PWMs are expectations of certain functions of a random variable and they can be defined for any random variable whose ordinary moments exist. They have low variance and no severe bias and can compare favorably with estimators obtained by the maximum likelihood method. The $(s,r)^{th}$ PWM of X following the GOW-G family of distribution, say $\rho_{s,r}$, is formally defined by

$$\rho_{s,r} = E\{X^s F(X)^r\} = \int_{-\infty}^{\infty} x^s F(X)^r f(x) dx.$$

From Equation (4) and (5) , we can write

$$f(x)F(x)^r = \sum_{m,l=0}^{\infty} \Omega_{m,l} h_{\alpha\beta(m+1)+\alpha l}(x),$$

where

$$\Omega_{m,l} \ = \sum_{k=0}^{r} \frac{\alpha\beta(-1)^{k+m+l} (k+1)^m}{m![\alpha\beta(m+1)+\alpha l]} \binom{-\beta(m+1)-1}{l}.$$

Finally, the $(s,r)^{th}$ PWMs of X can be obtained from an infinite linear combination of Exp-G moments given by

$$\rho_{s,r} = \sum_{k,m=0}^{\infty} \Omega_{k,m} E\left(Y_{\alpha\beta(m+1)+\alpha l}^s\right).$$

3. Special GOWG models

In here, we obtain the new two extended models based on the new family. We also note that GOWG-G family reduces to odd Weibull-G (OW-G) family, introduced by Bourguignon et al. (2014), for $\alpha = 1$.

3.1 The GOWG-normal

We define the GOWG-normal (GOWG-N) distribution from (5) by taking $G(x;\mu,\sigma) = \Phi\left(\frac{x-\mu}{\sigma}\right)$ and $g(x;\mu,\sigma) = \sigma^{-1}\phi\left(\frac{x-\mu}{\sigma}\right)$ with $\xi = (\mu,\sigma)$, where $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cdf of the standard normal distribution, respectively, where $x \in \Re, \mathbf{\Theta} = (\alpha,\beta,\mu,\sigma), \mu \in \Re$ and $\alpha,\beta,\sigma > 0$. We plot this pdf and its hrf in Figure 1. From Figure 1, we see that the pdf shapes of the GOWG-N are left skewed and bi-modal. Also, its hrf are increasing or firstly unimodal and then increasing.

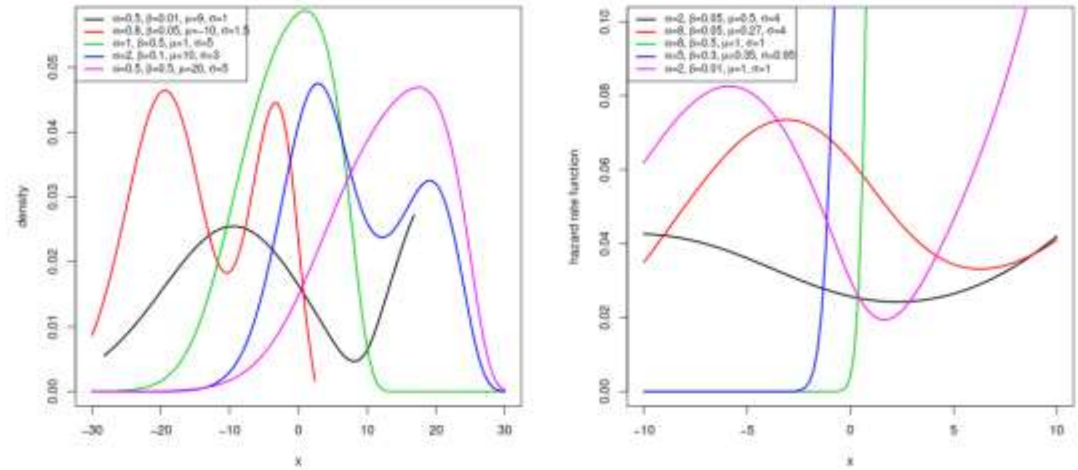


Figure 1: Plots of the pdf and hrf of the GOWG-N distribution

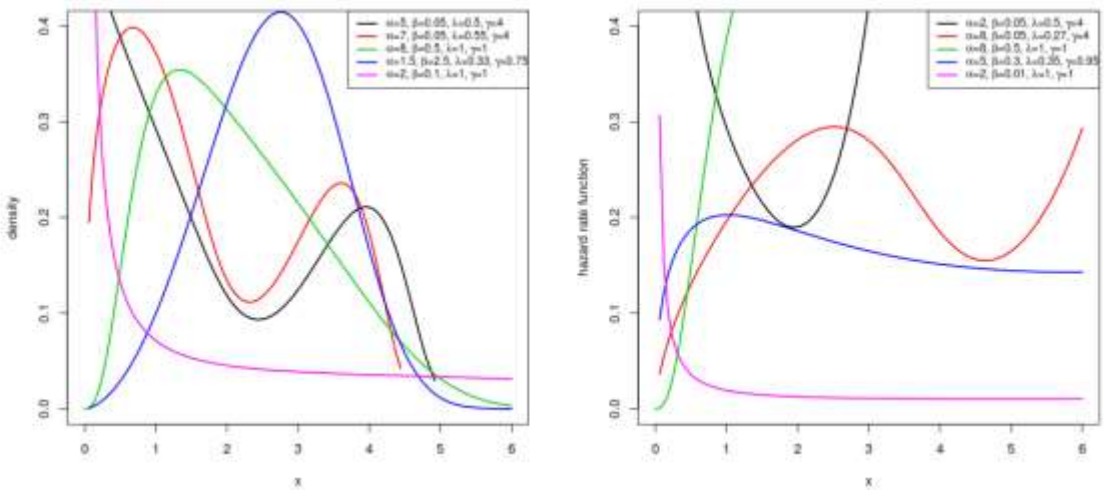


Figure 2: Plots of the pdf and hrf of the GOWG-W distribution

3.2 The GOWG-Weibull

Consider the cdf $G(x) = 1 - \exp[-(\lambda x)^{\gamma}]$ of the Weibull distribution with scale $\lambda > 0$ and shape $\gamma > 0$. The pdf of the GOWG-Weibull (GOWG-W) model (for $x > 0$) follows from (5). Some plots of the GOWG-W pdf and hrf for selected parameter values are displayed in Figure 2. Figure 2 reveals that the GOWG-W density can be concave down, right skewed or bi-modal. The hrf of the XG-W model can be increasing, decreasing, bathtub or unimodal then bathtub.

4. Estimation

Several approaches for parameter estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. The MLEs enjoy desirable properties and can be used for constructing confidence intervals and also for test statistics. The normal approximation for these estimators in large samples can be easily handled either analytically or numerically. Here, we consider the estimation of the unknown parameters of the new family from complete samples only by maximum likelihood. Let $x_1, \ldots, x_n$ be a random sample from the GOWG-G distribution with a $(q +$

2) $\times 1$ parameter vector $\Theta = (\alpha, \beta, \phi)^{\text{ú}}$, where ϕ is a $q \times 1$ baseline parameter vector. The log-likelihood function for Φ is given by

$$\ell(\Theta) = n\log\alpha + n\log\beta + \sum_{i=0}^{n} \log g(x_i;\phi) + (\alpha\beta - 1)\sum_{i=0}^{n} \log G(x_i;\phi)$$

$$-(\beta+1)\sum_{i=0}^{n} \log s_i - \sum_{i=0}^{n} p_i \qquad (15)$$

where $s_i = 1 - G(x;\phi)^{\alpha}$and $p_i = \left[\frac{G(x;\phi)^{\alpha}}{s_i}\right]^{\beta}$. Equation of (15) can be maximized either directly by using the R (optim function), SAS (PROC NLMIXED) or Ox program (sub-routine MaxBFGS) or by solving the nonlinear likelihood equations obtained by differentiating (15). The score vector components, say $\mathbf{U}(\Theta) = \frac{\partial \ell}{\partial \Theta} = (\frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \phi_r})^{\text{ú}} = \left(U_a, U_b, U_{\phi_r}\right)^{\text{T}}$, are given by

$$U_{\alpha} = \frac{n}{\alpha} + \beta\sum_{i=0}^{n} \log G(x_i;\phi) - (\beta+1)\sum_{i=0}^{n} \frac{z_i}{s_i} - \sum_{i=0}^{n} \frac{d_i}{p_i},$$

$$U_{\beta} = \frac{n}{\beta} + \alpha\sum_{i=0}^{n} \log G(x_i;\phi) - \sum_{i=0}^{n} \log s_i - \sum_{i=0}^{n} q_i$$

and (for $r = 1, \ldots, q$)

$$U_{\phi_r} = \sum_{i=0}^{n} \frac{g_r'(x_i;\phi)}{g(x_i;\phi)} + (\alpha\beta - 1)\sum_{i=0}^{n} \frac{G_r'(x_i;\phi)}{G(x_i;\phi)} - (\beta+1)\sum_{i=0}^{n} \frac{t_i}{s_i} - \sum_{i=0}^{n} a_i.$$

Where

$$g_r'(x_i;\phi) = \frac{\partial g(x_i;\phi)}{\partial \phi_r}, G_r'(x_i;\phi) = \frac{\partial G(x_i;\phi)}{\partial \phi_r}, z_i = -\frac{\log G(x;\phi)}{G(x;\phi)^{-\alpha}}, t_i = -\alpha\frac{g(x;\phi)}{G(x;\phi)^{1-\alpha}},$$

$$d_i = -\beta\frac{z_i[s_i + G(x;\phi)^{\alpha}]}{s_i^2\left[\frac{G(x;\phi)^{\alpha}}{s_i}\right]^{1-\beta}}, q_i = \frac{\log\left[\frac{G(x;\phi)^{\alpha}}{s_i}\right]}{\left[\frac{G(x;\phi)^{\alpha}}{s_i}\right]^{-\beta}} \text{and} a_i = -\beta\frac{[s_i z_i + t_i G(x;\phi)^{\alpha}]}{s_i^2\left[\frac{G(x;\phi)^{\alpha}}{s_i}\right]^{1-\beta}}.$$

Setting the nonlinear system of equations $U_{\alpha} = U_{\beta} = 0$ and $U_{\phi_k} = \mathbf{0}$ and solving them simultaneously yields the MLEs $\widehat{\Theta} = (\hat{\alpha}, \hat{\beta}, \hat{\phi}^{\text{ú}})^{\text{ú}}$. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize $\ell(\Theta)$. For interval estimation of the parameters, we can evaluate numerically the elements of the $(q+2) \times (q+2)$ observed information matrix $J(\Theta) = (-\frac{\partial^2 \ell}{\partial \theta_r \theta_s})$. Under standard regularity conditions when $n \to \infty$, the distribution of $\widehat{\Theta}$ can be approximated by a multivariate normal $N_p(0, J(\widehat{\Theta})^{-1})$ distribution to construct approximate confidence intervals for the parameters. Here, $J(\widehat{\Theta})$ is the total observed information matrix evaluated at $\widehat{\Theta}$. The method of the re-sampling bootstrap can be used for correcting the biases of the MLEs of the model parameters. Good interval estimates may also be obtained using the bootstrap percentile method.

5. Simulation studies

In this Section, we perform the two simulation studies by using the GOWG-N and GOWG-W distributions to see the performance of MLEs of these distribution. The random numbers generation is obtained by inverse of their cdfs. Inverse process and results of MLEs were obtained using optim-CG routine in the R programme.In the first simulation study, we obtain the graphical results. We generate $N = 1000$ samples of size $n = 20, 25, \ldots, 1000$ from GOWG-N distribution with true parameters values $\alpha = 5$, $\beta =$

0.5, $\mu = 0$ and $\sigma = 1$. In this simulation study, we empirically calculate the mean, standard deviations (sd), bias and mean square error (MSE) of the MLEs. The bias and MSE are calculated by (for $h = \alpha, \beta, \mu, \sigma$)

$$\widehat{Bias}_h = \frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{h}_i - h\right),$$

and

$$\widehat{MSE}_h = \frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{h}_i - h\right)^2$$

respectively. We give results of this simulation study in Figure 3.

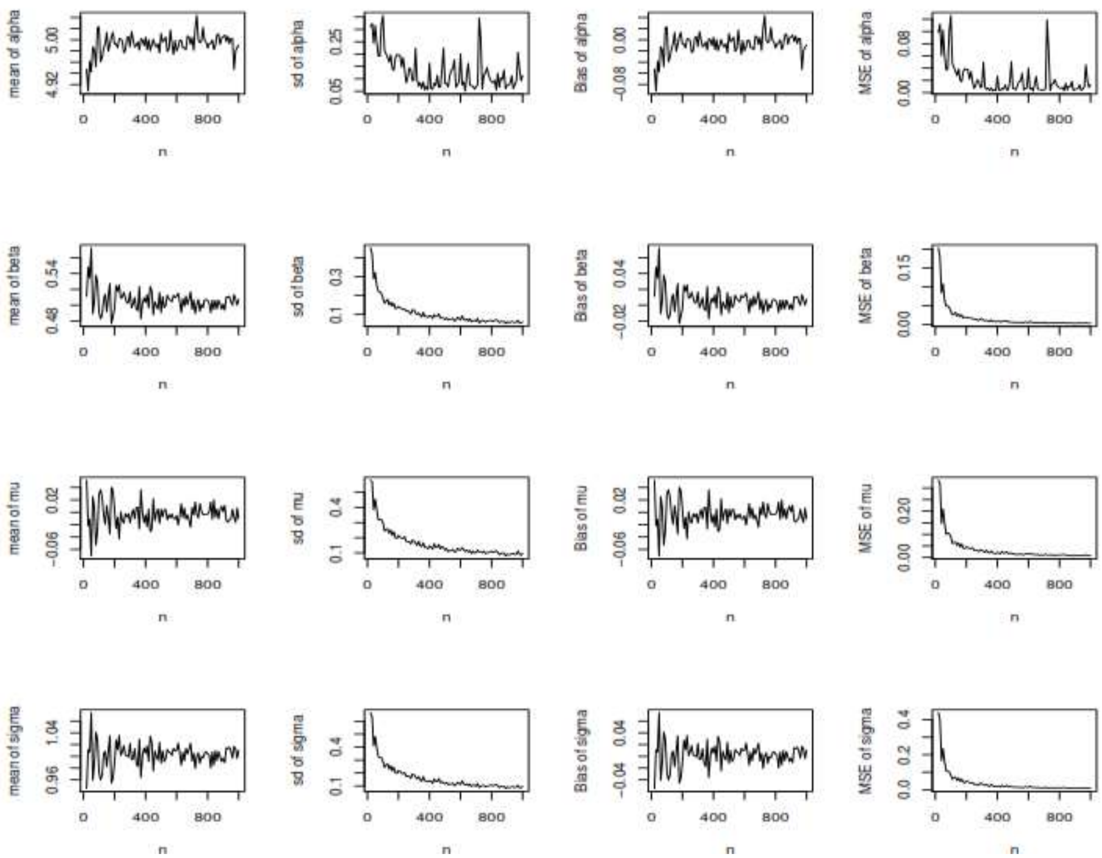


Figure 3: Simulation results of the special GOWG-N distribution

In the second simulation study, we generate 1,000 samples of sizes 20,50 and 100 from selected GOWG-W distributions. For this simulation study, we obtain the empirical means and sd's of the parameters.The results of this simulation study are reported in Table 1.

From Figure 3, we observe that when the sample size increases, the empirical means approach to true parameter value. At the same time, the all biases and MSEs approach to 0. The standard deviations decrease in all the cases, while sample size increases. Table 1 shows that when the sample size increases, the empirical means approach to true parameter value and the sds decreases in all the cases as expected.

Table 1: Empirical means and standard deviations (in parentheses) for the special GOWG-W distributions.

Parameters	n = 20				n = 50				n = 100			
$\alpha,\beta,\lambda,\gamma$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\gamma}$
1,0.5,0.5,1	**1.2829**	**0.4870**	**0.6737**	**1.5813**	**1.0661**	**0.5525**	**0.5375**	**1.2106**	**1.0543**	**0.5333**	**0.5037**	**1.1291**
	(0.5810)	(0.4088)	(0.4648)	(0.8087)	(0.4869)	(0.3391)	(0.2900)	(0.4497)	(0.4321)	(0.2496)	(0.2094)	(0.3417)
0.5,0.5,0.5,0.5	**0.5298**	**0.5416**	**0.7052**	**0.6926**	**0.5039**	**0.5028**	**0.5477**	**0.6203**	**0.4987**	**0.5010**	**0.5424**	**0.5573**
	(0.2500)	(0.3362)	(0.3142)	(0.3521)	(0.1456)	(0.1684)	(0.2589)	(0.2441)	(0.1341)	(0.1486)	(0.2090)	(0.1439)
2,2,2,2	**2.0867**	**1.9811**	**2.0195**	**2.1969**	**2.0237**	**1.9962**	**2.0035**	**2.0859**	**2.0059**	**1.9940**	**1.9991**	**2.0619**
	(0.3678)	(0.3205)	(0.1522)	(0.4265)	(0.2801)	(0.1928)	(0.1263)	(0.2974)	(0.3114)	(0.1482)	(0.1292)	(0.2396)
3,0.1,2,0.1	**3.0476**	**0.0875**	**1.9928**	**0.1080**	**3.0079**	**0.1032**	**2.0040**	**0.1028**	**3.0040**	**0.1037**	**1.9977**	**0.0997**
	(0.2549)	(0.1178)	(0.0439)	(0.0272)	(0.2012)	(0.0292)	(0.0307)	(0.0129)	(0.2259)	(0.0233)	(0.0364)	(0.0088)
1,2,3,4	**1.2094**	**1.9888**	**3.1249**	**4.0625**	**1.0836**	**2.0246**	**3.0462**	**4.0314**	**1.0704**	**1.9896**	**3.0443**	**4.0272**
	(0.3132)	(0.2964)	(0.1723)	(0.5783)	(0.2908)	(0.2022)	(0.1714)	(0.3843)	(0.2098)	(0.1740)	(0.1235)	(0.3399)
4,3,2,1	**4.0056**	**3.0048**	**1.9574**	**1.0749**	**4.0060**	**3.0138**	**1.9756**	**1.0291**	**4.0068**	**3.0091**	**1.9891**	**1.0165**
	(0.1986)	(0.1664)	(0.1789)	(0.1707)	(0.0525)	(0.0705)	(0.1032)	(0.0893)	(0.0449)	(0.0514)	(0.0759)	(0.0635)

6. Real data applications

In this section, we illustrate the flexibility of the GOWG-N and GOWG-W models via two data sets. We compare these models with several extensions and generalizations of the normal and Weibull distributions in the literature. To determine the best model, we also computed the estimated log-likelihood values $\hat{\ell}$, Kolmogorov-Smirnov (KS), Cramervon Mises (W^*) and Anderson-Darling (A^*) goodness of-fit statistics for distribution models. We note that the statistics W^* and A^* are described in detail in Chen and Balakrishnan (1995). In general, it can be chosen as the best model which has the smaller the values of the K-S, W^* and A^* statistics and the larger the values of $\hat{\ell}$ and p-values. All computations are performed by the maxLik routine in the R programme. The details are the followings.

6.1 Windshield data set

As first example, we consider the data studied by Murthy et al. (2004) representing the failure times for a particular windshield device. This data set has been analyzied by Brito et al. (2017) and Korkmaz et al. (2017). We compare the GOWG-N model with Gompertz-normal (Gom-N) model (Alizadeh, et al., 2017a), odd exponentiated half logistic normal (OEHL-N) model (Afify, et al., 2017e), odd Lindley normal (OL-N) model (da-Silva et al., 2017), odd log logistic normal (Braga et al., 2016) model and odd Weibull normal (OW-N) (Bourguignon et al., 2014) model. We give MLEs of parameters, $\hat{\ell}$, A^*,W^* and KS goodness-of-fits statistics in Table 2 for this data. Table 2 shows that the GOWG-N model has the smallest values of the A^* andKS statistics, and has the biggest $\hat{\ell}$ value among the fitted models. Hence, it could be chosen as the best model.

Table 2: MLEs of the model parameters for the windshield data, the corresponding standard errors (given in parentheses), A^*,W^* and KS statistics for the applications models

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\mu}$	$\hat{\sigma}$	$-\hat{\ell}$	A^*	W^*	KS
GOWG-N	**1.3603**	**0.1410**	**3.0043**	**0.4152**	124.1260	0.5471	0.1040	0.0819
	(1.1740)	(0.0739)	(0.4694)	(0.1978)				
Gom-N	**0.0288**	**0.0768**	**0.7449**	**0.5327**	127.1258	0.5586	0.0825	0.0849
	(0.0.0145)	(0.0139)	(0.0001)	(0.0001)				
EHOLL-N	**0.0983**	**0.2114**	**3.9034**	**0.4847**	125.6029	1.5225	0.3170	0.1183
	(0.0173)	(1.001E-07)	(0.0001)	(1.23E-6)				
OL-N		**7.3048**	**5.2721**	**1.9995**	129.8938	1.2223	0.2098	0.1194
		(4.8310)	(0.9752)	(0.2310)				
OLL-N	**0.4519**		**2.6262**	**0.6025**	127.0619	0.6748	0.1246	0.0950
	(0.2321)		(0.1267)	(0.2179)				
OW-N		**0.1321**	**3.1654**	**0.3635**	124.1876	0.7466	0.1525	0.0940
		(0.0663)	(0.0874)	(0.1106)				

The the plots of the fitted pdfs and cdfs for models are shown in Figure 4. Also, Figure 5 displays the probability-probability (P-P) plots for the models. From these plots, we can conclude that the GOWG-N distribution is suitable to this data set. The GOWG-N model captures the data as bimodal.

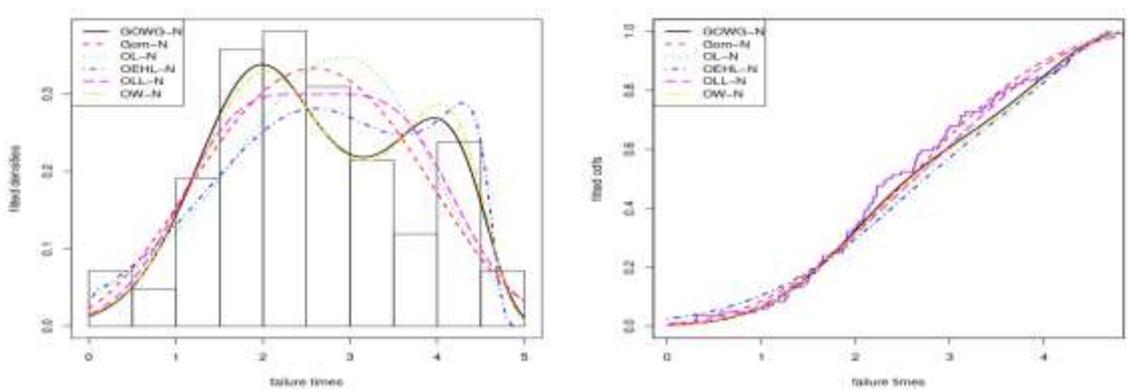
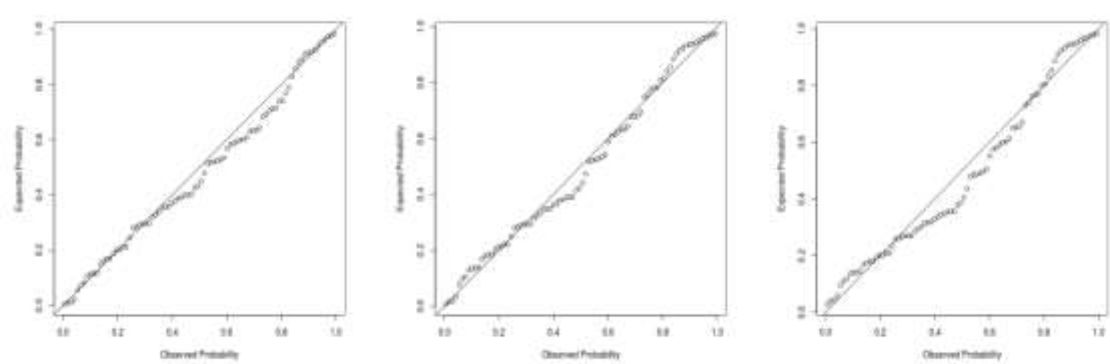
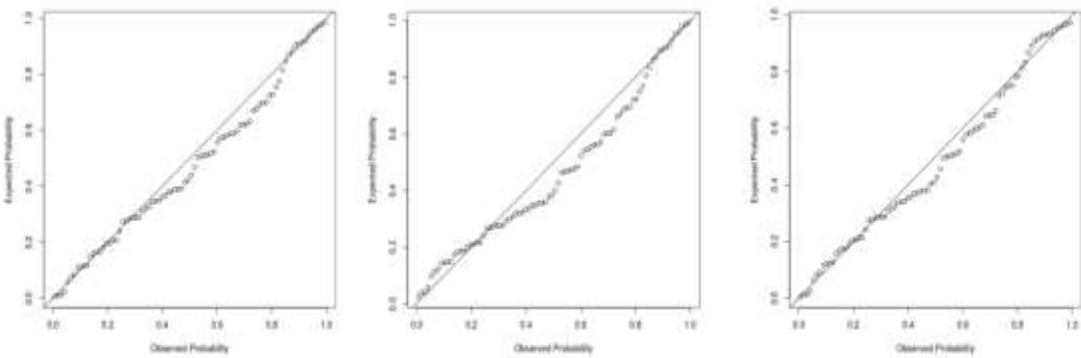


Figure 4: The fitted pdfs (left) and cdfs (right)for the first data set



(a) GWOG-N (b) Gom-N (c) OL-N



(d) OW-N (e) OEHL-N (f) OLL-N

Figure 5: P-P plots for the first data set data

6.2 Failure times data set

The following data set represents the failure times (in minutes) for a sample of 15 electronic components in an accelerated life test studied by Lawless (2003) and Mir Mostafaee et al. (2016). The data are: 1.4, 5.1, 6.3, 10.8, 12.1, 18.5, 19.7, 22.2, 23, 30.6, 37.3, 46.3, 53.9, 59.8, 66.2. We compare the GOWG-W model with Gompertz-Weibull (Gom-W) model (Alizadeh, et al., 2017a), odd exponentiated half logistic Weibull (OEHL-W) model (Afify, et al., 2017e), Kumaraswamy Weibull (Kw-W) model (Cordeiro et al., 2010), odd log logistic Weibull (Cruz et al., 2016) model and odd Weibull Weibull (OW-W) (Bourguignon et al., 2014) model. We give MLEs of parameters, $\hat{\ell}$, A^*,W^* and KS goodness-of-fits statistics in Table 3 for this data. Table 3 shows that the GOWG-W model has the smallest values of the A^*, W^* and KS statistics, and has the biggest $\hat{\ell}$ value among the fitted models. Hence, it could be chosen as the best model.

Table 3: MLEs of the model parameters for the failure times data, the corresponding standard errors (given in parentheses), A^*,W^* and KS statistics for the applications models

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\gamma}$	$-\hat{\ell}$	A^*	W^*	KS
GOWG-W	**7.5748**	**0.0400**	**0.0346**	**4.2841**	62.4511	0.1505	0.0230	0.0943
	(2.5198)	(0.0101)	(0.0002)	(0.0190)				
Gom-W	**0.2385**	**0.1067**	**0.2158**	**0.8163**	63.7255	0.1776	0.0273	0.1307
	(0.6532)	(0.2076)	(0.4971)	(0.4930)				
Kw-W	**0.0834**	**0.9595**	**0.0156**	**9.5058**	62.9480	0.2528	0.0497	0.1693
	(0.0279)	(0.3159)	(0.0001)	(0.0016)				
EHOLL-W	**0.0776**	**1.0598**	**0.0153**	**9.1687**	62.8915	0.2324	0.0396	0.1484
	(0.0202)	(0.0008)	(0.0007)	(0.0007)				
OLL-W	**0.4382**		**0.0321**	**2.6259**	63.4573	0.2235	0.0429	0.1606
	(0.2465)		(0.0066)	(1.2559)				
OW-W		**0.2163**	**0.02259**	**4.4546**	63.0820	0.3683	0.0781	0.1857
		(0.0536)	(0.0019)	(0.0491)				

The the plots of the fitted pdfs and cdfs for models are shown in Figure 6. Also, Figure 7 displays the P-P plots for the models. From these plots, we can conclude that the GOWG-

W distribution is suitable to this data set. The GOWG-W model also captures the data as bimodal.

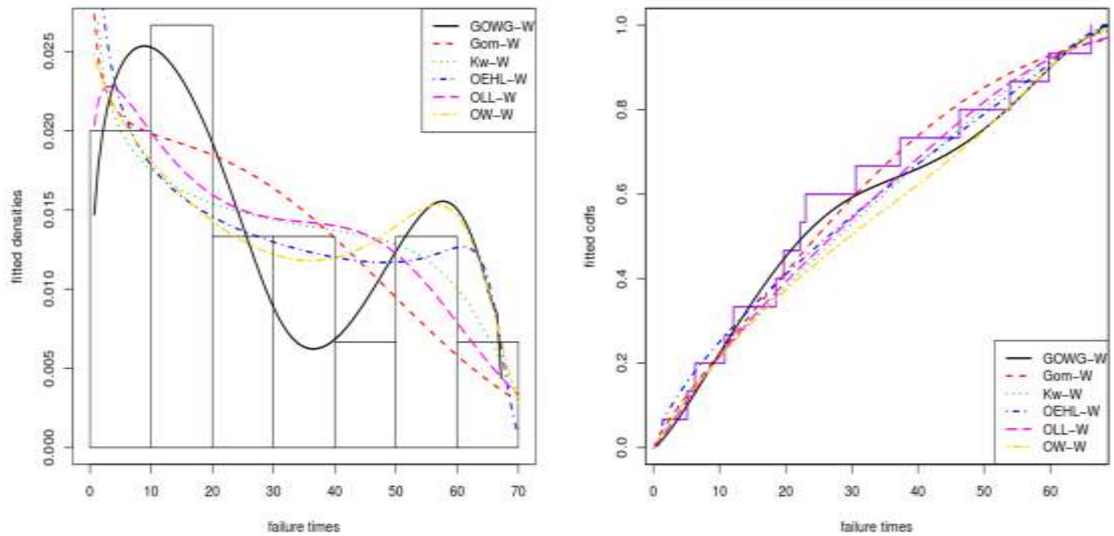
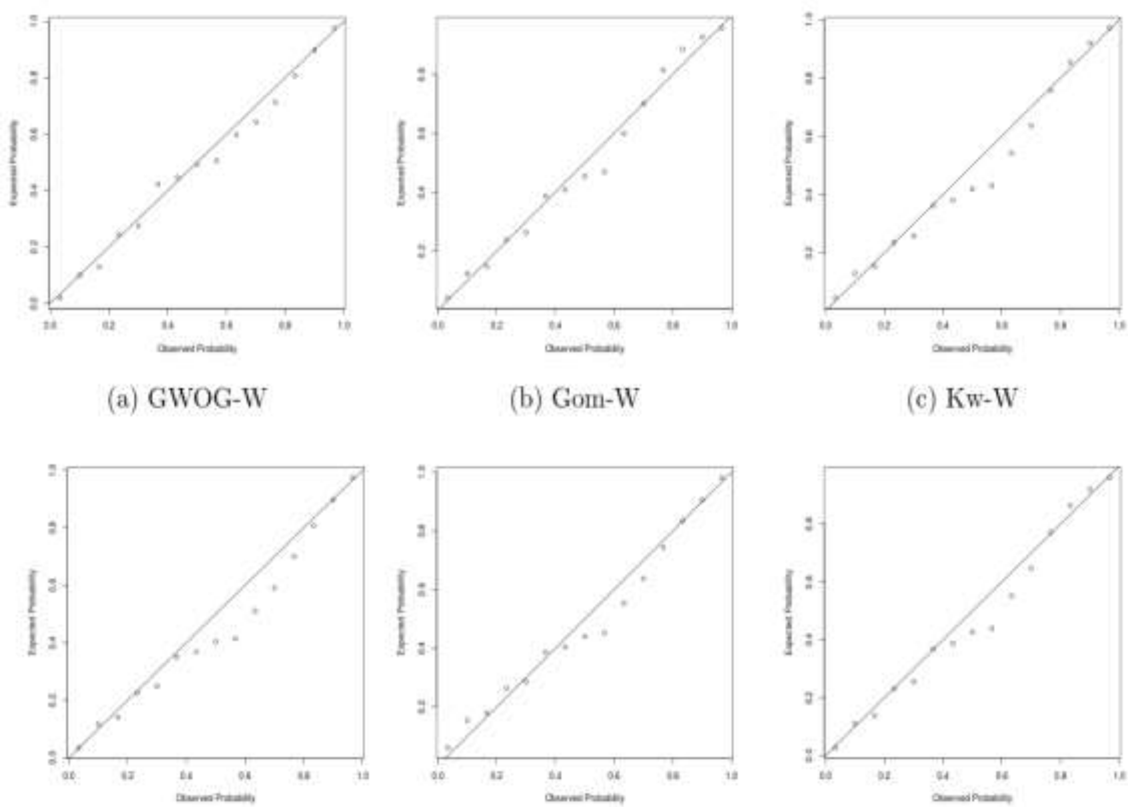


Figure 6: The fitted pdfs (left) and cdfs (right)for the second data set



(a) GWOG-W (b) Gom-W (c) Kw-W

(d) OW-W (e) OEHL-W (f) OLL-W

Figure 7: P-P plots for the second data set data

7. Conclusions

In this work, we propose a new class of lifetime distributions calledthe generalized odd Weibull generatedfamily. It can provide better fits than some of the well known lifetime models and this fact represents a good characterization of this family. Some of its mathematical properties are derived. The maximum likelihood method is used for estimating the model parameters. We study the behaviour of the estimators by means of two Monte Carlo simulations. The importance of the family illustrated by means of two applications to real data sets.

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