The Four-Parameter Burr XII Distribution: Properties, Regression Model and Applications

Ahmed Z. Afify[a], Gauss M. Cordeiro[b], Edwin M. M. Ortega[c],
Haitham M. Yousof[a], Nadeem Shafique Butt[d]
[a]*Department of Statistics, Mathematics and Insurance, Benha University, Egypt*
[b]*Departamento de Estatística, Universidade Federal de Pernambuco, Brazil*
[c]*Departamento de Ciências Exatas, Universidade de São Paulo, 13418-900 Piracicaba, SP, Brazil*
[d]*Department of Family and Community Medicine, KAU, Jeddah, Saudi Arabia*

Abstract

This paper introduces a new four-parameter lifetime model called the Weibull Burr XII distribution. The new model has the advantage of being capable of modeling various shapes of aging and failure criteria. We derive some of its structural properties including ordinary and incomplete moments, quantile and generating functions, probability weighted moments and order statistics. The new density function can be expressed as a linear mixture of Burr XII densities. We propose a log-linear regression model using a new distribution so-called the log-Weibull Burr XII distribution. The maximum likelihood method is used to estimate the model parameters. Simulation results to assess the performance of the maximum likelihood estimation are discussed. We prove empirically the importance and flexibility of the new model in modeling various types of data.

Keywords: Burr XII, Maximum Likelihood, Moments, Order Statistics, Weibull G-Family.

1 Introduction

The statistical literature contains hundreds of continuous univariate distributions which have several applications from finance, economics, environmental, biomedical sciences and engineering, among others. These applications have shown that data sets following the well-known models are more often the exception rather than the reality. So, a significant progress has been made towards the generalization of some classical distributions and their successful applications in several areas.

The Burr-XII (BXII) distribution originally proposed by Burr (1942) has many applications in different areas including reliability, failure time modeling and acceptance sampling plans. Shao (2004) extended the three-parameter BXII distribution and used it to model extreme events with applications to flood frequency. Tadikamalla (1980) studied the BXII model and its related models, namely: Pareto II (Lomax), log-logistic, compound Weibull gamma and Weibull exponential distributions.

Recently, many authors constructed generalizations of the BXII distribution. For example, Paranaíba *et al.* (2011) proposed the beta BXII, Paranaíba *et al.* (2013) studied the Kumaraswamy BXII, Gomes *et al.* (2015) proposed the McDonald BXII, Mead (2014) introduced the beta exponentiated BXII, Al-Saiarie *et al.* (2014) studied the Marshall-Olkin extended BXII and Mead and Afify (2016) investigated the Kumaraswamy exponentiated BXII distributions.

The cumulative distribution function (cdf) and probability density function (pdf) of the two parameter BXII distribution are given by (for $x > 0$)

$$G(x;\alpha,\beta) = 1 - (1 + x^{\alpha})^{-\beta} \qquad \text{and} \qquad g(x;\alpha,\beta) = \alpha\beta x^{\alpha-1} (1 + x^{\alpha})^{-\beta-1}, \tag{1}$$

respectively, where α and β are positive shape parameters.

The aim of this paper is to define and study a new lifetime model called the *Weibull Burr XII* (WBXII) distribution. Its main feature is that two additional shape parameters are inserted in equation (1) to provide more flexibility for the generated model. Using the Weibull-G (W-G)

ACCEPTED MANUSCRIPT

family of distributions (Bourguignon *et al*., 2014), we construct the four-parameter WBXII model and give a comprehensive description of some of its mathematical properties. In fact, the WBXII model can provide better fits than at least eight other models, each one having the same number of parameters.

Further, the WBXII model due to its flexibility in accommodating all forms of the hazard rate function (hrf) (see Figure 2) seems to be an important distribution that can be used to serve as an alternative model to other lifetime distributions available in the literature for modeling positive real data in many areas. We prove that the WBXII distribution is capable of modelling various shapes of data using two different data sets. It can provide better fits to these data sets.

Let $g(x;\xi)$ and $G(x;\xi)$ denote the density and cumulative functions of the baseline model with parameter vector ξ and consider the Weibull cdf $\Pi(x) = 1 - \exp(-ax^b)$ (for $x > 0$) with positive parameters a and b. Based on this density, Bourguignon *et al*. (2014) replaced the argument x by $G(x;\xi)/\overline{G}(x;\xi)$, where $\overline{G}(x;\xi)$ is the reliability function and defined the cdf of their W-G family by

$$F(x;a,b,\xi) = a\,b\int_0^{\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]} t^{b-1}\,e^{-at^b}dt = 1 - \exp\left\{-a\left[\frac{G\,(x;\xi)}{\overline{G}(x;\xi)}\right]^b\right\}. \tag{2}$$

The corresponding pdf of (2) is given by

$$f(x;a,b,\xi) = ab\,g(x;\xi)\left[\frac{G(x;\xi)^{b-1}}{\overline{G}(x;\xi)^{b+1}}\right]\exp\left\{-a\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^b\right\}, \tag{3}$$

where a and b are two additional positive shape parameters. In general a random variable X with pdf (3) is denoted by X ~W-G(a,b,ξ). If $b = 1$, the W-G class reduces to the exponential-G (Ex-G) family.

To this end, we use equations (1) and (2) to obtain the four-parameter WBXII cdf (for $x \geq 0$)

$$F(x;\alpha,\beta,a,b) = 1 - \exp\left\{-a\left[(1+x^\alpha)^\beta - 1\right]^b\right\}. \tag{4}$$

The corresponding pdf of (4) is given by

$$f(x;\alpha,\beta,a,b) = \alpha\,\beta\,a\,b\,x^{\alpha-1}\,(1+x^{\alpha})^{\beta b-1}\left[1-(1+x^{\alpha})^{-\beta}\right]^{b-1}\times \exp\left\{-a\left[(1+x^{\alpha})^{\beta}-1\right]^{b}\right\}, \qquad (5)$$

where α,β,a and b are positive shape parameters. Henceforth, we denote a random variable X having pdf (5) by $X\ \sim$WBXII(α,β,a,b). The WBXII model reduces to the exponential BXII distribution when $b=1$. For $\alpha=1$ and $\beta=1$, we obtain the Weibull Lomax and Weibull log-logistic distributions, respectively. The case $b=\alpha=1$ refers to the exponential Lomax distribution and the case $b=\beta=1$ refers to the exponential log-logistic distribution. For $a=b=\alpha=1$ and $a=b=\beta=1$, we have the standard Lomax and standard log-logistic distributions, respectively.

The survival function (sf), hrf and cumulative hazard rate function (chrf) of X are, respectively, given by

$$S(x;\alpha,\beta,a,b)=\exp\left\{-a\left[(1+x^{\alpha})^{\beta}-1\right]^{b}\right\},$$

$$h(x;\alpha,\beta,a,b)=\alpha\,\beta\,a\,b\,x^{\alpha-1}\,(1+x^{\alpha})^{\beta b-1}\left[1-(1+x^{\alpha})^{-\beta}\right]^{b-1}$$

and

$$H(x;\alpha,\beta,a,b)=a\left[(1+x^{\alpha})^{\beta}-1\right]^{b}.$$

The remainder of the paper is organized as follows: in Section 2, we provide some plots for the pdf and hrf of the WBXII model and derive useful mixture representations for the pdf and cdf. We obtain in Section 3 some mathematical properties of the WBXII distribution including ordinary and incomplete moments, quantile and generating functions, moments of the residual, reversed residual life and probability weighted moments (PWMs), order statistics and their moments. The maximum likelihood estimates (MLEs) of the model parameters are determined in Section 4, as well as simulation results to assess the performance of the MLEs are discussed. In Section 5, we propose the log-Weibull BXII (LWBXII) regression model and estimate the parameters by the method of maximum likelihood. In Section 6, the WBXII distribution is applied to two real data sets to illustrate its potentiality. Finally, in Section 7, we provide some concluding remarks.

2 Plots and Linear Representation

In this section, we provide some plots of the pdf and hrf of the WBXII model to show its flexibility. Figure 1 displays some plots of the WBXII density for some parameter values α, β, a and b. Plots of the hrf of the WBXII model for selected parameter values are given in Figure 2, where the hrf can be bathtub, upside down bathtub (unimodal), increasing, decreasing or constant.

2.1 Linear Representation

The WBXII density function (5) can be expressed as

$$\begin{aligned} f(x) &= \alpha\beta\, a\, b\, x^{\alpha-1}\,(1+x^{\alpha})^{-\beta-1}\,\frac{\left[1-(1+x^{\alpha})^{-\beta}\right]^{b-1}}{\left[(1+x^{\alpha})^{-\beta}\right]^{b+1}}\times \\ & \underbrace{\exp\left\{-a\frac{\left[1-(1+x^{\alpha})^{-\beta}\right]^{b}}{\left[(1+x^{\alpha})^{-\beta}\right]^{b}}\right\}}_{A}. \end{aligned} \tag{6}$$

By expanding A, we can write

$$A=\sum_{k=0}^{\infty}\frac{(-1)^k\,a^k}{k!}\frac{\left[1-(1+x^{\alpha})^{-\beta}\right]^{bk}}{\left[(1+x^{\alpha})^{-\beta}\right]^{bk}}.$$

Inserting this expansion in (6) and, after some algebra, we have

$$f(x)=\alpha\,\beta\,b\,x^{\alpha-1}\,(1+x^{\alpha})^{-\beta-1}\sum_{k=0}^{\infty}\frac{(-1)^k a^{k+1}\left[1-(1+x^{\alpha})^{-\beta}\right]^{b(k+1)-1}}{k!\left[(1+x^{\alpha})^{-\beta}\right]^{b(k+1)+1}}.$$

Consider the power series

$$(1-z)^{-q}=\sum_{j=0}^{\infty}\frac{\Gamma(q+j)}{j!\,\Gamma(q)}\,z^j, \tag{7}$$

which holds for $|z|<1$ and $q>0$ real non-integer.

After applying the power series (7) to $\left[(1+x^{\alpha})^{-\beta}\right]^{-[b(k+1)+1]}$, the last equation can be rewritten as

$$f(x) = \alpha\,\beta\,x^{\alpha-1}\,(1+x^{\alpha})^{-\beta-1}\sum_{k,j=0}^{\infty}\frac{(-1)^k\,b a^{k+1}\Gamma\,(b\,(k+1)+j+1)}{k!\,j!\,\Gamma\,(b\,(k+1)+1)}\times \left[1-(1+x^{\alpha})^{-\beta}\right]^{b(k+1)+j-1}. \quad (8)$$

Next, consider the generalized binomial series given by

$$(1-z)^{\alpha-1} = \sum_{r=0}^{\infty}\frac{(-1)^r\,\Gamma\,(\alpha)}{r!\,\Gamma\,(\alpha-r)}\,z^r, \quad (9)$$

which holds for $|z| < 1$ and $\alpha > 0$ real non-integer.

Applying the generalized binomial (9) to the last term of (8) and after some simplifications, the WBXII density (8) can be expressed as a linear mixture of the BXII densities as

$$f(x) = \sum_{r=0}^{\infty}\upsilon_r\;\; g(x;\alpha,\beta\,(r+1)), \quad (10)$$

where

$$\upsilon_r = \upsilon_r\,(a,b) = \sum_{k,j=0}^{\infty}\frac{(-1)^{k+r}\,b a^{k+1}\Gamma\,(b\,(k+1)+j)\,\Gamma\,(b\,(k+1)+j+1)}{k!\,j!\;(r+1)!\Gamma\,(b\,(k+1)+1)\,\Gamma\,(b\,(k+1)+j-r)}$$

and $g(x;\alpha,\beta\,(r+1))$ is the BXII density with parameters α and $\beta\,(r+1)$.

Let W be a random variable having the BXII distribution (1) with parameters α and β. For $r < \alpha\beta$, the rth ordinary and incomplete moments of W are, respectively, given by

$$\mu'_r = \beta\,B\left(\beta-\frac{r}{\alpha},\frac{r}{\alpha}+1\right) \qquad \text{and} \qquad \varphi_r(z) = \beta\,B\left(z^{\alpha};\beta-\frac{r}{\alpha},\frac{r}{\alpha}+1\right),$$

where $B(a,b) = \int_0^{\infty} t^{a-1}\,(1+t)^{-(a+b)}dt$ and $B(z;a,b) = \int_0^{z} t^{a-1}\,(1+t)^{-(a+b)}dt$ are the beta and the incomplete beta functions of the second type, respectively. So, several structural properties of the WBXII model can be obtained from (10) and those properties of the BXII distribution.

Similarly, the cdf (4) of X can be expressed in the mixture form

$$F(x) = \sum_{r=0}^{\infty} \upsilon_r \; G(x;\alpha,\beta\,(r+1)),$$

where $G(x;\alpha,\beta(r+1))$ is the BXII cdf with parameters α and $\beta(r+1)$.

3 The WBXII Properties

We investigate mathematical properties of the WBXII distribution including ordinary and incomplete moments, quantile and generating functions and PWMs. It is better to obtain some structural properties of the WBXII distribution by establishing algebraic expansions than computing those directly by numerical integration of its density function.

3.1 Ordinary Moments

The nth ordinary moment of X is given by

$$\mu_n' = E(X^n) = \sum_{r=0}^{\infty} \upsilon_r \int_0^{\infty} x^n \; g(x;\alpha,\beta\,(r+1))dx.$$

For $n < \alpha\beta$, we obtain

$$\mu_n' = E(X^n) = \sum_{r=0}^{\infty} \upsilon_r \,\beta\,(r+1)\; B\left(\beta\,(r+1) - \frac{n}{\alpha}, \frac{n}{\alpha} + 1\right). \tag{11}$$

Setting $n = 1$ in (11), we have the mean of X.

The sth central moment (M_s) and cumulants (κ_s) of X, are, respectively, given by

$$M_s = E(X - \mu_1')^s = \sum_{i=0}^{s} (-1)^i \binom{s}{i} (\mu_1')^s \,\mu_{s-i}'$$

and

$$\kappa_s = \mu_s' - \sum_{i=0}^{s-1} \binom{s-1}{i-1} \kappa_r \,\mu_{s-r}',$$

where $\kappa_1 = \mu_1'$. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. The effects of the parameters a and b on the mean, variance, skewness and kurtosis for given values of α and β are displayed in Figures 3 and 4, respectively.

3.2 Quantile and Generating Functions

The quantile function (qf) of X is obtained by inverting (4) as

$$x_q = \left\{ \left[1 + \left(\frac{-1}{a} \log(1-q) \right)^{\frac{1}{b}} \right]^{\frac{1}{\beta}} - 1 \right\}^{\frac{1}{\alpha}}, \; 0 < q < 1. \tag{12}$$

By setting $q = 0.5$ in (12) gives the median of X. Simulating the WBXII random variable is straightforward. If U is a uniform variate on the unit interval $(0, 1)$, then the random variable $X = x_q$ at $q = U$ follows (5).

The moment generating function (mgf) of X, say $M_X(t) = E[\exp(tX)]$, can be obtained from (9) as

$$M_X(t) = \sum_{r=0}^{\infty} \upsilon_r \, M_{r+1}(t),$$

where $M_{r+1}(t)$ is the mgf of the BXII distribution with parameters α, $\beta(r+1)$. Paranaíba *et al.* (2011) provided a simple representation for the mgf of the three-parameter BXII distribution. In a similar manner, we provide another representation for the mgf, say $M(t)$, of the BXII(α,β) model. For $t < 0$, we can write

$$M(t) = \alpha\beta \int_0^{\infty} \exp(yt) \, y^{\alpha-1} (1 + y^{\alpha})^{-\beta-1} \, dy.$$

Next, we require the Meijer G-function defined by

$$G_{p,q}^{m,n}\left(x \middle| \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma\left(b_j + t\right) \prod_{j=1}^{n} \Gamma\left(1 - a_j - t\right)}{\prod_{j=n+1}^{p} \Gamma\left(a_j + t\right) \prod_{j=m+1}^{p} \Gamma\left(1 - b_j - t\right)} x^{-t} dt,$$

where $i = \sqrt{-1}$ is the complex unit and L denotes an integration path (Gradshteyn and Ryzhik, 2000, Section 9.3). The Meijer G-function contains as particular cases many integrals with elementary and special functions (Prudnikov *et al.*, 1986). We now assume that $\alpha = m/\beta$, where m and β are positive integers. This condition is not restrictive since every positive real number can be approximated by a rational number.

We have the following result, which holds for m and k positive integers, $\mu > -1$ and $p > 0$ (Prudnikov *et al.*, 1992, p. 21),

$$\begin{aligned} I\left(p, \mu, \frac{m}{\beta}, \nu\right) &= \int_0^\infty \exp(-px)\, x^\mu \left(1 + x^{\frac{m}{\beta}}\right)^\nu dx \\ &= V G^{\beta,\beta+m}_{\beta+m,\beta}\left(\frac{m^m}{p^m} \Big| \begin{array}{c} \triangle(m,-\mu), \triangle(\beta,\nu+1) \\ \triangle(\beta,0) \end{array}\right), \end{aligned}$$

where $V = \frac{\beta^{-\nu} m^{\mu+\frac{1}{2}}}{(2\pi)^{\frac{m-1}{2}}\Gamma(-\nu) p^{\mu+1}}$ and $\triangle(\beta, a) = \frac{a}{\beta}, \frac{a+1}{\beta}, ..., \frac{a+\beta}{\beta}$. We can write (for $t < 0$)

$$M(t) = mI\left(-t, \frac{m}{\beta} - 1, \frac{m}{\beta}, -\beta - 1\right).$$

Hence, the mgf of X can be expressed as

$$M_X(t) = m \sum_{r=0}^{\infty} \upsilon_r\, I\left(-t, \frac{m}{\beta(r+1)} - 1, \frac{m}{\beta(r+1)}, -\beta(r+1) - 1\right).$$

3.3 Incomplete Moments

The sth incomplete moment, say $\varphi_s(t)$, of the WBXII distribution is given by $\varphi_s(t) = \int_0^t x^s f(x)dx$. We can write from equation (10)

$$\varphi_s(t) = \sum_{r=0}^{\infty} \upsilon_r \int_0^t x^s\, g(x; \alpha, \beta(r+1))dx,$$

and then using the lower incomplete gamma function, we obtain (for $s < \alpha\beta$)

$$\varphi_s(t) = \sum_{r=0}^{\infty} \upsilon_r\, \beta(r+1)\; B\left(t^\alpha; \beta(r+1) - \frac{s}{\alpha}, \frac{s}{\alpha} + 1\right).$$

The first incomplete moment of X, denoted by $\varphi_1(t)$, is simply determined from the above equation by setting $s = 1$.

The first incomplete moment has important applications related to the Bonferroni and Lorenz curves and the mean residual life and the mean waiting time. Furthermore, the amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. The mean deviations, about the mean and about the median of X, depend on $\varphi_1(t)$.

3.4 Residual and Reversed Residual Life Functions

The nth moment of the residual life, denoted by $m_n(t) = E[(X - t)^n \mid X > t],\ n = 1, 2, \ldots,$ uniquely determine $F(x)$ (see Navarro *et al*., 1998). The nth moment of the residual life of X is given by

$$m_n(t) = \frac{1}{1 - F(t)} \int_t^{\infty} (x - t)^n dF(x).$$

Then, we can write

$$m_n(t) = \frac{1}{R(t)} \sum_{i=0}^{n} \sum_{r=0}^{\infty} \frac{(-1)^{n-i}\, n!\, t^{n-i}}{i!\Gamma(n - i + 1)} \upsilon_r\, \beta\,(r + 1)\, B\left(t^{\alpha}; \beta\,(r + 1) - \frac{i}{\alpha}, \frac{i}{\alpha} + 1\right).$$

Another interesting function is the mean residual life (MRL) function or the life expectation at age x defined by $m_1(x) = E\,[(X - x) \mid X > x]$, which represents the expected additional life length for a unit which is alive at age x. The MRL of the WBXII distribution can be obtained by setting $n = 1$ in the last equation.

Navarro *et al*. (1998) proved that the nth moment of the reversed residual life, say $M_n(t) = E\,[(t - X)^n \mid X \le t]$ for $t > 0$ and $n = 1, 2, \ldots,$ uniquely determines $F(x)$.

Then, $M_n(t)$ is defined by

$$M_n(t) = \frac{1}{F(t)} \int_0^{t} (t - x)^n dF(x).$$

The nth moment of the reversed residual life of X

$$M_n(t) = \frac{1}{F(t)} \sum_{i=0}^{n} \sum_{r=0}^{\infty} \frac{(-1)^i n!}{i!(n-i)!} \upsilon_r \beta (r+1) \, B\left(t^{\alpha}; \beta (r+1) - \frac{i}{\alpha}, \frac{i}{\alpha} + 1\right).$$

The mean inactivity time (MIT) or mean waiting time (MWT), also called the mean reversed residual life function, say $M_1(t) = E[(t - X) \mid X \leq t]$, represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0, x)$. The MIT of X can be obtained by setting $n = 1$ in the above equation.

3.5 Probability Weighted Moments

The PWMs are expectations of certain functions of a random variable and they can be defined for any random variable whose ordinary moments exist.

The (s, r)th PWM of X following the WBXII model, $\rho_{s,r}$, is defined by

$$\rho_{s,r} = E\left\{X^s \, F(X)^r\right\} = \int_{-\infty}^{\infty} x^s \, F(x)^r \, f(x) \, dx.$$

Using (4) we can write

$$\begin{aligned} F(x)^r &= \left(1 - \exp\left\{-a \frac{\left[1 - (1 + x^{\alpha})^{-\beta}\right]^b}{\left[(1 + x^{\alpha})^{-\beta}\right]^b}\right\}\right)^r \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i \, \Gamma(r+1)}{i! \Gamma(r-i+1)} \exp\left\{-ia \frac{\left[1 - (1 + x^{\alpha})^{-\beta}\right]^b}{\left[(1 + x^{\alpha})^{-\beta}\right]^b}\right\}. \end{aligned}$$

Using (5) and the above equation, and after some simplifications, we obtain

$$\begin{aligned} f(x) \, F(x)^r &= \alpha\beta x^{\alpha-1} \sum_{i,k,j,m=0}^{\infty} \frac{(-1)^{i+k+m} \, b a^{k+1}}{i! k! j! m!} (1 + x^{\alpha})^{-\beta(m+1)-1} \\ &\quad \times \frac{(i+1)^k \, \Gamma(r+1) \, \Gamma(b(k+1) + j + 1) \, \Gamma(b(k+1) + j)}{\Gamma(r-i+1) \, \Gamma(b(k+1) + 1) \, \Gamma(b(k+1) + j - m)}. \end{aligned}$$

ACCEPTED MANUSCRIPT

After some algebra, we have

$$f(x)\;F(x)^r = \sum_{m=0}^{\infty} \Upsilon_m\, g(x;\alpha,\beta\,(m+1)),$$

where

$$\Upsilon_m = \frac{b\,(-1)^m}{(m+1)!} \sum_{i,k,j=0}^{\infty} \frac{(-1)^{i+k}\,a^{k+1}\,(i+1)^k\,\Gamma\,(r+1)}{i!k!j!\Gamma\,(r-i+1)} \times \frac{\Gamma\,(b\,(k+1)+j+1)\,\Gamma\,(b\,(k+1)+j)}{\Gamma\,(b\,(k+1)+1)\,\Gamma\,(b\,(k+1)+j-m)}.$$

Then, the (s,r)th PWM of X can be expressed as

$$\rho_{s,r} = \sum_{m=0}^{\infty} \beta\,(m+1)\,\Upsilon_m\;B\Big(\beta(m+1)-\frac{s}{\alpha},\frac{s}{\alpha}+1\Big).$$

3.6 Order Statistics

Let $X_1,\ldots,X_n$ be a random sample of size n from the WBXII distribution and $X_{(1)},\ldots,X_{(n)}$ be the corresponding order statistics. Then, the pdf of the ith order statistic $X_{i:n}$, say $f_{i:n}(x)$, is given by

$$f_{i:n}(x) = \frac{f(x)}{\mathrm{B}(i,n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-1}{j} F(x)^{i+j-1}. \tag{13}$$

We can write

$$F(x)^{i+j-1} = \sum_{w=0}^{\infty} \frac{(-1)^w\,\Gamma\,(i+j)}{i!\Gamma\,(i+j-w)} \exp\left\{-aw\frac{\left[1-(1+x^{\alpha})^{-\beta}\right]^b}{\left[(1+x^{\alpha})^{-\beta}\right]^b}\right\}. \tag{14}$$

By inserting (5) and (14) in equation (13), we obtain

$$f_{i:n}(x) = \sum_{m=0}^{\infty} t_m\, g(x;\alpha,\beta\,(m+1))\,, \tag{15}$$

where

$$t_m = \frac{b\ (-1)^m}{(m+1)!} \sum_{w,k,l=0}^{\infty} \sum_{j=0}^{n-i} \frac{(-1)^{w+k+j}\ a^{k+1}\ (w+1)^k \binom{n-1}{j}}{w!\ k!\ l!\ \mathrm{B}(i, n-i+1)} \times$$
$$\frac{\Gamma(i+j)\,\Gamma(b(k+1)+l+1)\,\Gamma(b(k+1)+l)}{\Gamma(i+j-w)\,\Gamma(b(k+1)+1)\,\Gamma(b(k+1)+l-m)}$$

and $g(x;\alpha,\beta(m+1))$ denotes the BXII density function with parameters α and $\beta(m+1)$. Thus, the density function of the WBXII order statistics is a linear mixture of two-parameter BXII densities. Based on equation (15), we can obtain some structural properties of $X_{i:n}$ from those BXII properties.

The qth moment of $X_{i:n}$ is given by

$$E(X_{i:n}^q) = \sum_{m=0}^{\infty} t_m \beta(m+1)\ B\left(\beta(m+1) - \frac{q}{\alpha}, \frac{q}{\alpha} + 1\right). \tag{16}$$

The L-moments are analogous to the ordinary moments and can be estimated by linear combinations of the order statistics. Then, using the moments in equation (16), we can derive explicit expressions for the L-moments of X as infinite weighted linear combinations of the means of suitable WBXII distributions. They are defined by ($s \geq 1$)

$$\lambda_s = \frac{1}{s} \sum_{d=0}^{s-1} (-1)^d \binom{s-1}{d} E(X_{s-d:s}),\ \ s \geq 1.$$

The first four L-moments, say λ_1, λ_2, λ_3 and λ_4 are, respectively, given by

$$\lambda_1 = E(X_{1:1}),\ \lambda_2 = \frac{1}{2} E(X_{2:2} - X_{1:2}),$$
$$\lambda_3 = \frac{1}{3} E(X_{3:3} - 2X_{2:3} + X_{1:3}) \text{ and}$$
$$\lambda_4 = \frac{1}{4} E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}).$$

We can obtain the λ's for X from equation (16) with $q = 1$.

4 Maximum Likelihood Estimation

We consider the estimation of the unknown parameters of the WBXII model from complete samples only by maximum likelihood. The MLEs of the parameters of the WBXII (α, β, a, b) model is now discussed. Let $x_1, \ldots, x_n$ be a random sample of this distribution with parameter vector $\theta = (\alpha, \beta, a, b)^\intercal$.

The log-likelihood function for θ, say $\ell = \ell(\theta)$, is given by

$$\begin{aligned}\ell &= n\log\alpha + n\log\beta + n\log a + n\log b + (\alpha-1)\sum_{i=1}^{n}\log x_i \\ &\quad + (\beta b - 1)\sum_{i=1}^{n}\log(1+x_i^{\alpha}) + (b-1)\sum_{i=1}^{n}\log s_i - a\sum_{i=1}^{n} s_i^{b},\end{aligned} \tag{17}$$

where $s_i = \left[1 - \left(1 + x_i^{\alpha}\right)^{-\beta}\right]$.

The last equation can be maximized either by using the different programs like `R` (`optim` function), `SAS` (`PROC NLMIXED`) or by solving the nonlinear likelihood equations obtained by differentiating (17).

The score vector elements, $\mathbf{U}(\theta) = \frac{\partial\ell}{\partial\theta} = \left(\frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\beta}, \frac{\partial\ell}{\partial a}, \frac{\partial\ell}{\partial b}\right)^\intercal$, are given by

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log x_i + (\beta b - 1)\sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i}{\left(1+x_i^{\alpha}\right)} + (b-1)\sum_{i=1}^{n}\frac{p_i}{s_i} - ab\sum_{i=1}^{n} p_i s_i^{b-1},$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} + b\sum_{i=1}^{n}\log(1+x_i^{\alpha}) + (b-1)\sum_{i=1}^{n}\frac{\log\left(1+x_i^{\alpha}\right)}{s_i\left(1+x_i^{\alpha}\right)^{\beta}} - ab\sum_{i=1}^{n}\frac{s_i^{b-1}\log\left(1+x_i^{\alpha}\right)}{\left(1+x_i^{\alpha}\right)^{\beta}},$$

$$\frac{\partial\ell}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n} s_i^{b} \quad\text{and}\quad \frac{\partial\ell}{\partial b} = \frac{n}{b} + \beta\sum_{i=1}^{n}\log(1+x_i^{\alpha}) + \sum_{i=1}^{n}\log s_i - a\sum_{i=1}^{n} s_i^{b}\log s_i,$$

respectively, where $p_i = \beta x_i^{\alpha}\left(1+x_i^{\alpha}\right)^{-\beta-1}\log(x_i)$.

We can obtain the estimates of the unknown parameters by setting the score vector to zero, $\mathbf{U}(\widehat{\theta}) = \mathbf{0}$. By solving these equations simultaneously gives the MLEs $\widehat{\alpha}, \widehat{\beta}, \widehat{a}$ and $\widehat{b}$. For the WBXII distribution all the second order derivatives exist.

The interval estimation of the model parameters requires the 4×4 observed information matrix $J(\theta) = \{J_{ij}\}$ for $i, j = \alpha, \beta, a, b$. The multivariate normal $N_4(0, J(\widehat{\theta})^{-1})$ distribution, under standard regularity conditions, can be used to provide approximate confidence intervals for the unknown parameters, where $J(\widehat{\theta})$ is the total observed information matrix evaluated at $\widehat{\theta}$. Then, approximate $100(1-\delta)\%$ confidence intervals for α, β, a and b can be determined by:

$\widehat{\alpha} \pm z_{\delta/2}\sqrt{\widehat{J_{\alpha\alpha}}}$, $\widehat{\beta} \pm z_{\delta/2}\sqrt{\widehat{J_{\beta\beta}}}$, $\widehat{a} \pm z_{\delta/2}\sqrt{\widehat{J_{aa}}}$ and $\widehat{b} \pm z_{\delta/2}\sqrt{\widehat{J_{bb}}}$, where $z_{\delta/2}$ is the upper δth percentile of the standard normal model.

4.1 Simulation Study

In this section, we assess the performance of the MLEs of the WBXII parameters using Monte Carlo simulations. For different combinations of α, β, a and b, samples of sizes $n = 100$, 200, 500 and 1000 are generated from the WBXII model. We repeat the simulation $k = 1,000$ times and evaluate the MLEs and their standard errors (in parentheses). The empirical results are given in Table 1. It is evident that the estimates are quite stable and close to the true values of the parameters for these sample sizes. Additionally, as the sample size increases, the biases and the standard errors of the MLEs decrease as expected.

5 The Log-Weibull Burr XII Regression Model with Censored Data

Henceforth, X is a random variable following the WBXII density function (3) and Y is defined by $Y = \log(\gamma X)$, where $\gamma > 0$ is a new parameter. It is easy to verify that the density function of Y

obtained by replacing $\alpha = 1/\sigma$ and $\gamma = \exp(\mu)$ reduces to

$$\begin{aligned} f(y) &= \frac{a\,b\,\beta}{\sigma}\exp\left(\frac{y-\mu}{\sigma}\right)\left[1+\exp\left(\frac{y-\mu}{\sigma}\right)\right]^{\beta\,b-1}\left\{1-\left[1+\exp\left(\frac{y-\mu}{\sigma}\right)\right]^{-\beta}\right\}^{b-1}\times \\ & \exp\left\{-a\left[\left\{1+\exp\left(\frac{y-\mu}{\sigma}\right)\right\}^{\beta}-1\right]^{b}\right\}, \quad -\infty < y < \infty, \end{aligned} \tag{18}$$

where $a > 0$, $b > 0$, $\beta > 0$, $\mu \in \mathbf{R}$ and $\sigma > 0$. We refer to equation (18) as the new *log-Weibull Burr XII* (LWBXII) distribution, say $Y \sim \mathrm{LWBXII}(a, b, \beta, \sigma, \mu)$, where μ is the location parameter, σ is the dispersion parameter and a, b and β are shape parameters. Thus,

$$\text{if} \quad X \sim \mathrm{WBXII}(a, b, \beta, \alpha) \quad \text{then} \quad Y = \log(\gamma\, X) \sim \mathrm{LWBXII}(a, b, \beta, \sigma, \mu).$$

In Figure 5, we plot this density function for selected values of the parameters a, b and β showing that the LWBXII density could be very flexible for modeling its kurtosis, skewness and bimodal forms.

The corresponding survival function is

$$S(y) = \exp\left\{-a\left[\left\{1+\exp\left(\frac{y-\mu}{\sigma}\right)\right\}^{\beta}-1\right]^{b}\right\}. \tag{19}$$

The random variable $Z = (Y - \mu)/\sigma$ has density function

$$\begin{aligned} f(z) &= a\,b\,\beta\exp(z)\,[1+\exp(z)]^{\beta\,b-1}\left\{1-[1+\exp(z)]^{-\beta}\right\}^{b-1}\times \\ & \exp\left\{-a\left[\{1+\exp(z)\}^{\beta}-1\right]^{b}\right\}. \end{aligned} \tag{20}$$

In many practical applications, the lifetimes are affected by explanatory variables such as the cholesterol level, blood pressure, weight and many others. Parametric models to estimate univariate survival functions and for censored data regression problems are widely used. A parametric model that provides a good fit to lifetime data tends to yield more precise estimates of the quantities of interest. Based on the LWBXII density, we propose a linear location-scale regression model linking the response variable y_i and the explanatory variable vector $\mathbf{x}_i^T = (x_{i1}, \ldots, x_{ip})$ as follows

$$y_i = \mathbf{v}_i^T \boldsymbol{\tau} + \sigma z_i, \;\; i = 1, \ldots, n, \tag{21}$$

where the random error z_i has density function (20), $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_p)^T$, $\sigma > 0$, $a > 0$, $\beta > 0$ and $\beta > 0$ are unknown parameters. The parameter $\mu_i = \mathbf{v}_i^T \boldsymbol{\tau}$ is the location of y_i. The location parameter vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)^T$ is given by a linear model $\boldsymbol{\mu} = \mathbf{V}\boldsymbol{\tau}$, where $\mathbf{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)^T$ is a known model matrix. The LWBXII model (21) opens new possibilities for fitting many different types of data. It contains as special models the following new regression models:

- **Log-exponential Burr XII (LEBXII) regression model**

  For $b = 1$, the survival function is

  $$S(y) = \exp\left\{-a\left[\left\{1 + \exp\left(\frac{y-\mu}{\sigma}\right)\right\}^{\beta} - 1\right]\right\}.$$

- **Log-Weibull Lomax (LWLomax) regression model**

  For $\sigma = 1$, the survival function is

  $$S(y) = \exp\left\{-a\left[\{1 + \exp(y-\mu)\}^{\beta} - 1\right]^{b}\right\}.$$

- **Log-Weibull log-logistic (LWLLogistic) distribution**

  For $\beta = 1$, the survival function becomes

  $$S(y) \;\; = \;\; \exp\left\{-a \exp\left[b\left(\frac{y-\mu}{\sigma}\right)\right]\right\}.$$

Consider a sample $(y_1, \mathbf{v}_1), \ldots, (y_n, \mathbf{v}_n)$ of n independent observations, where each random response is defined by $y_i = \min\{\log(x_i), \log(c_i)\}$. We assume non-informative censoring such that the observed lifetimes and censoring times are independent. Let F and C be the sets of individuals for which y_i is the log-lifetime or log-censoring, respectively. Conventional likelihood estimation techniques can be applied here. The log-likelihood function for the vector of parameters $\boldsymbol{\theta} = (a, b, \beta, \sigma, \boldsymbol{\tau}^T)^T$ from model (21) has the form $l(\boldsymbol{\theta}) = \sum_{i\in F} l_i(\boldsymbol{\theta}) + \sum_{i\in C} l_i^{(c)}(\boldsymbol{\theta})$, where $l_i(\boldsymbol{\theta}) = \log[f(y_i)]$, $l_i^{(c)}(\boldsymbol{\theta}) = \log[S(y_i)]$, $f(y_i)$ is the density (18) and $S(y_i)$ is the survival function (19) of Y_i. The total log-likelihood function for $\boldsymbol{\theta}$ reduces to

$$\begin{aligned} l(\boldsymbol{\theta}) \;=\;& r\log\left(\frac{a\,b\,\beta}{\sigma}\right) + \sum_{i\in F} z_i + (\beta\, b - 1)\sum_{i\in F}\log[1+\exp(z_i)] \\ & +(b-1)\sum_{i\in F}\log\{1-[1+\exp(z_i)]^{-\beta}\} - a\sum_{i\in F}\{[1+\exp(z_i)]^{\beta}-1\}^{b} \\ & -a\sum_{i\in C}\{[1+\exp(z_i)]^{\beta}-1\}^{b}, \qquad (22)\end{aligned}$$

where $z_i = y_i - \mathbf{v}_i^T\boldsymbol{\tau}/\sigma$ and r is the number of uncensored observations (failures). The MLE $\widehat{\boldsymbol{\theta}}$ of the vector of unknown parameters can be determined by maximizing the log-likelihood (22). We use the NLMixed procedure in SAS to calculate the estimate $\widehat{\boldsymbol{\theta}}$. Initial values for $\boldsymbol{\beta}$ and σ are taken from the fit of the log-Weibull regression model. The fit of the LWBXII regression model gives the estimated survival function for y_i

$$S(y_i; \hat{a}, \hat{b}, \hat{\beta}, \hat{\sigma}, \widehat{\boldsymbol{\tau}}^T) = \exp\left\{-\hat{a}\left[\left\{1+\exp\left(\frac{y_i - \mathbf{v}_i^T\widehat{\boldsymbol{\tau}}}{\hat{\sigma}}\right)\right\}^{\hat{\beta}} - 1\right]^{\hat{b}}\right\}.$$

Under conditions that are fulfilled for the parameter vector θ in the interior of the parameter space but not on the boundary, the asymptotic distribution of $(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ is multivariate normal $N_{p+4}(0, K(\boldsymbol{\theta})^{-1})$, where $K(\boldsymbol{\theta})$ is the information matrix. The asymptotic covariance matrix $K(\boldsymbol{\theta})^{-1}$ of $\widehat{\boldsymbol{\theta}}$ can be approximated by the inverse of the $(p+4)\times(p+4)$ observed information matrix $-\ddot{\mathbf{L}}(\boldsymbol{\theta})$. The elements of the observed information matrix $-\ddot{\mathbf{L}}(\boldsymbol{\theta})$, namely $-\mathbf{L}_{aa}$, $-\mathbf{L}_{ab}$, $-\mathbf{L}_{a\beta}$, $-\mathbf{L}_{a\sigma}$, $-\mathbf{L}_{a\tau_j}$, $-\mathbf{L}_{bb}$, $-\mathbf{L}_{b\beta}$, $-\mathbf{L}_{b\sigma}$, $-\mathbf{L}_{b\tau_j}$, $-\mathbf{L}_{\beta\beta}$, $-\mathbf{L}_{\beta\sigma}$, $-\mathbf{L}_{\beta\tau_j}$, $-\mathbf{L}_{\sigma\sigma}$, $-\mathbf{L}_{\sigma\tau_j}$ and $-\mathbf{L}_{\tau_j\tau_{j'}}$, for $j, j' = 1, \ldots, p$, can be evaluated numerically. The approximate multivariate normal distribution $N_{p+4}(0, -\ddot{\mathbf{L}}(\boldsymbol{\theta})^{-1})$ for $\widehat{\boldsymbol{\theta}}$ can be used in the classical way to construct approximate confidence intervals for some parameters in $\boldsymbol{\theta}$.

We can use the likelihood ratio (LR) statistic for comparing some special models with the

LWBXII model. We consider the partition $\boldsymbol{\theta} = (\boldsymbol{\theta}_1^T, \boldsymbol{\theta}_2^T)^T$, where $\boldsymbol{\theta}_1$ is a subset of parameters of interest and $\boldsymbol{\theta}_2$ is a subset of remaining parameters. The LR statistic for testing the null hypothesis $H_0 : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_1^{(0)}$ versus the alternative hypothesis $H_1 : \boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_1^{(0)}$ is given by $w = 2\{\ell(\widehat{\boldsymbol{\theta}}) - \ell(\widetilde{\boldsymbol{\theta}})\}$, where $\widetilde{\boldsymbol{\theta}}$ and $\widehat{\boldsymbol{\theta}}$ are the estimates under the null and alternative hypotheses, respectively. The statistic w is asymptotically (as $n \to \infty$) distributed as χ_k^2, where k is the dimension of the subset of parameters $\boldsymbol{\theta}_1$ of interest.

6 Applications

6.1 Application 1: Glass Fibre Data

The importance and flexibility of the WBXII distribution are illustrated by means of a real data set. It consists of 63 observations of the strengths of 1.5 cm glass fibres (the units of measurement are not given) originally obtained by workers at the UK National Physical Laboratory (see, Smith and Naylor, 1987).

For the glass fibre data, we shall compare the fits of the WBXII and BXII distributions and the following competitive non-nested models: Kumaraswamy exponentiated Burr XII (KwEBXII) (Mead and Afify, 2016), transmuted complementary Weibull geometric (TCWG) (Afify *et al.*, 2014), exponentiated transmuted generalized Rayleigh (ETGR) (Afify *et al.*, 2015), transmuted Marshall-Olkin Fréchet (TMOFr) (Afify *et al.*, 2015), beta exponentiated Burr XII (BEBXII) (Mead, 2014), transmuted exponentiated generalized Weibull (TExGW) (Yousof *et al.* 2015), Weibull Fréchet (WFr) (Afify *et al.*, 2016), Weibull Lomax (WL) (Tahir *et al.*, 2015), and beta Burr XII (BBXII) (Paranaíba *et al.*, 2014) with corresponding densities (for $x > 0$) given in Appendix A.

In order to compare the distributions, we consider the following criteria: the $-2\widehat{\ell}$ (maximized log-likelihood), *AIC* (Akaike information criterion), *CAIC* (consistent Akaike information criterion), BIC (Bayesian information criterion) and $HQIC$ (Hannan-Quinn information criterion). Also, we apply formal goodness-of-fit tests in order to verify which distribution fits better to these data. In particular, we consider the Cramér-von Mises (W^*) and Anderson-Darling (A^*)statistics. The W^* and A^* statistics are described in details in Chen and Balakrishnan (1995). The model with minimum values for these statistics could be chosen as the best model to fit the data.

Table 2 lists the values of the MLEs and their corresponding standard errors (in parentheses) of the model parameters. These results are obtained using the MATHCAD PROGRAM.

In Table 3, we compare the WBXII model with the KwEBXII, TCWG, ETGR, TMOFr, BEBXII, TEXGW, BBXII and BXII distributions. We note that the WBXII model gives the lowest values for the AIC, BIC, $CAIC$, $HQIC$, W^* and A^* statistics among all fitted models. So, the WBXII model could be chosen as the best model to explain the current data. It is clear from the plots in Figure 6 that the WBXII density provides a better fit to the histogram of the data. The plots in Figure 7 support the fitted WBXII distribution than the other nested and non-nested models.

6.2 Application 2: Regression Model

In this section, we consider a data set provided by the Instituto de Saúde Coletiva - Universidade Federal da Bahia. These data were designed to evaluate the effect of vitamin A supplementation on recurrent diarrheal episodes in small children (see Barreto *et al.*, 1994). Censoring times are random, and we aim to verify the treatment effect in time until the first occurrence of diarrheal episodes. This can be done by means of an appropriate regression model with censored data. The data from a randomized community trial that was designed to evaluate the effect of vitamin A supplementation on diarrheal episodes in 1,207 pre-school children, aged 6-48 months at the baseline, who were assigned to receive either placebo or vitamin A in a small city in the Northeast of Brazil from December 1990 to December 1991. The vitamin A dosage was 100,000 IU for children younger than 12 months and 200,000 IU for older children, which is the highest dosage guideline established by the World Health Organization (WHO) for the prevention of vitamin A deficiency.

The total time is defined as the time from the first dose of vitamin A until the occurrence of an episode of diarrhea. An episode of diarrhea is defined as a sequence of days with diarrhea and a day with diarrhea is defined when 3 or more liquid or semi-liquid motions are reported in a 24-hour period. The information on the occurrence of diarrhea collected at each visit corresponds to a recall period of 48-72 hours. The number of liquid and semi-liquid motions per 24 hours is recorded.

The covariates considered in the models are:

- v_{i1}: age at baseline (in months);
- v_{i2}: treatment (0 = placebo, 1 = vitamin A);
- v_{i3}: gender (0 = girl, 1 = boy).

Next, we present results by fitting the model

$$y_i = \tau_0 + \tau_1 v_{i1} + \tau_2 v_{i2} + \tau_3 v_{i3} + \sigma z_i,$$

where variable Y_i follows the LWBXII distribution given in (18), $i = 1, 2, \ldots, 1207$. The MLEs of the model parameters are evaluated using the NLMixed procedure in SAS. Iterative maximization of the logarithm of the likelihood function (22) starts with initial values for $\boldsymbol{\tau}$ and σ, which are taken from the fit of the log-Weibull regression model.

We note from the fitted LWBXII regression model that x_1 is significant at 1% and that there is a significant difference between the age for the survival times.

A summary of the values of the *AIC*, *BIC* and *CAIC* to compare the LWBXII, LEBXII, LWLomax and LWLogistic regression models is given in Table 5. The LWBXII and LEBXII regression models outperform the LWLomax and LWLogistic models irrespective of the criteria and then they can be used effectively in the analysis of these data.

A comparison of the proposed regression model with some of its sub-models using LR statistics is addressed in Table 6. The figures in this table, specially the p-values, indicate that the new LWBXII and LEBXII regression models yield better fits to the current data than its two null models.

7 Conclusions

In this paper, we propose a new four-parameter model called the Weibull Burr XII (WBXII) distribution, which extends the Burr XII (BXII) distribution. The WBXII density function can be expressed as a linear mixture of BXII densities. We derive explicit expressions for some of its mathematical and statistical quantities including the ordinary and incomplete moments, cumulants, quantile and generating functions and probability weighted moments. We also obtain the density function of the order statistics and their moments. We discuss maximum likelihood estimation. The proposed distribution provides better fits than some other nested and non-nested models by using two real data sets. We hope that the proposed model will attract wider applications in areas such as survival and lifetime data, meteorology, hydrology, engineering and others.

Appendix A:

In this appendix we provide the densities used in the applications.

- KwEBXII distribution

$$f(x) = \frac{a\,b\,c\,\theta\,\beta x^{c-1}}{(1+x^c)^{\theta+1}} \left[1-(1+x^c)^{-\theta}\right]^{a\beta-1} \left\{1-\left[1-(1+x^c)^{-\theta}\right]^{a\beta}\right\}^{b-1}.$$

- TCWG distribution

$$\begin{aligned} f(x) &= \alpha\,\beta\,\gamma\,(\gamma x)^{\beta-1}\, \mathrm{e}^{-(\gamma x)^\beta} \left[\alpha+(1-\alpha)\,\mathrm{e}^{-(\gamma x)^\beta}\right]^{-3} \times \\ & \left[\alpha\,(1-\lambda)-(\alpha-\alpha\lambda-\lambda-1)\,\mathrm{e}^{-(\gamma x)^\beta}\right]. \end{aligned}$$

- ETGR distribution

$$\begin{aligned} f(x) &= 2\,\alpha\,\delta\,\beta^2\,x\,\mathrm{e}^{-(\beta x)^2} \left[1-\mathrm{e}^{-(\beta x)^2}\right]^{\alpha\delta-1} \left\{1+\lambda-2\lambda\left[1-\mathrm{e}^{-(\beta x)^2}\right]^{\alpha}\right\} \times \\ & \left\{1+\lambda-\lambda\left[1-\mathrm{e}^{-(\beta x)^2}\right]^{\alpha}\right\}^{\delta-1}. \end{aligned}$$

- TMOFr distribution

$$f(x) = \frac{a\,\beta\,\alpha^\beta x^{-\beta-1} \mathrm{e}^{-\left(\frac{\alpha}{x}\right)^\beta}}{\left[a+(1-a)\,\mathrm{e}^{-\left(\frac{\alpha}{x}\right)^\beta}\right]^2} \left[1+\lambda-\frac{2\lambda \mathrm{e}^{-\left(\frac{\alpha}{x}\right)^\beta}}{a+(1-a)\,\mathrm{e}^{-\left(\frac{\alpha}{x}\right)^\beta}}\right].$$

- BEBXII distribution

$$f(x) = \frac{c\,\theta\,\beta}{B(a,b)} x^{c-1}\,(1+x^c)^{-\theta-1} \left[1-(1+x^c)^{-\theta}\right]^{a\beta-1} \left\{1-\left[1-(1+x^c)^{-\theta}\right]^{\beta}\right\}^{b-1}.$$

- TExGW distribution

$$f(x) = a\,b\,\beta\,\alpha^\beta x^{\beta-1} \mathrm{e}^{-a(\alpha x)^\beta} \left[1-\mathrm{e}^{-a(\alpha x)^\beta}\right]^{b-1} \left\{1+\lambda-2\lambda\left[1-\mathrm{e}^{-a(\alpha x)^\beta}\right]^{b}\right\}.$$

- WFr distribution

$$f(x) = a\, b\, \beta\, \alpha^{\beta} x^{-\beta-1} e^{-b\left(\frac{\alpha}{x}\right)^{\beta}} \left\{1 - e^{-\left(\frac{\alpha}{x}\right)^{\beta}}\right\}^{-b-1} \times e^{-a\left\{e^{-\left(\frac{\alpha}{x}\right)^{\beta}}-1\right\}^{-b}}.$$

- WL distribution

$$f(x) = \frac{a\, b\, \alpha}{\beta} \left(1 + \frac{x}{\beta}\right)^{b\alpha-1} \left[1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}\right]^{b-1} e^{-a\left\{\left[\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right]\right\}^{b}}.$$

- BBXII distribution

$$f(x) = \frac{c\, \theta\, \beta^{-c}}{B(a,b)} x^{c-1} \left[1 + \left(\frac{x}{\beta}\right)^{c}\right]^{-\theta b-1} \left\{1 - \left[1 + \left(\frac{x}{\beta}\right)^{c}\right]^{-\theta}\right\}^{a-1}.$$

The parameters of the above densities are all positive real numbers except the parameter λ, where $|\lambda| \leq 1$ and $x > 0$.

References

[1] Afify, A. Z., Hamedani, G. G., Ghosh, I. and Mead, M. E. (2015). The transmuted Marshall-Olkin Fréchet distribution: properties and applications. International Journal of Statistics and Probability, **4**, 132-184.

[2] Afify, A. Z., Nofal, Z. M. and Butt, N. S. (2014). Transmuted complementary Weibull geometric distribution. Pakistan Journal of Statistics and Operation Research, **10**, 435-454.

[3] Afify, A. Z., Nofal, Z. M. and Ebraheim, A. N. (2015). Exponentiated transmuted generalized Rayleigh distribution: A new four parameter Rayleigh distribution. Pakistan Journal of Statistics and Operation Research, **11**, 115-134

[4] Afify, A. Z., Yousof, H. M., Cordeiro, G. M., Ortega, E. M. M. and Nofal, Z. M. (2016). The Weibull Fréchet distribution and its applications. Journal of Applied Statistics, **43**, 2608-2626.

[5] Al-Saiari, A. Y., Baharith, L. A. and Mousa, S. A. (2014). Marshall-Olkin extended Burr type XII distribution. International Journal of Statistics and Probability, **3**, 78-84.

[6] Barreto, M. L., Farenzena, G. G., Fiaccone, R. L., Santos, L. M. P., Assis, A. M. O., Araújo, M. P. N. and Santos, P. A. B. (1994). Effect of vitamin A supplementation on diarrhoea and acute lower-respiratory-tract infections in young children in Brazil. The Lancet, **344**, 228-231.

[7] Bourguignon, M., Silva, R. B. and Cordeiro, G. M. (2014). The Weibull-G family of probability distributions. Journal of Data Science, **12**, 53-68.

[8] Burr, I. W. (1942). Cumulative frequency functions. Annals of Mathematical Statistics, **13**, 215-232.

[9] Chen, G. and Balakrishnan, N. (1995). A general purpose approximate goodness-of-fit test. Journal of Quality Technology, **27**, 154 -161.

[10] Gomes, A. E., da-Silva, C. Q. and Cordeiro, G. M. (2015). Two extended Burr models: Theory and practice. Communication in Statistics Theory - Methods, **44**, 1706-1734.

[11] Gradshteyn, I. S. and Ryzhik, I. M. (2000). Table of Integrals, Series and Products (sixth edition).San Diego: Academic Press.

[12] Mead, M. E. (2014). The beta exponentiated Burr XII distribution. Journal of Statistics: Advances in Theory and Applications, **12**, 53-73.

[13] Mead, M. E. and Afify, A. Z. (2016). On five parameter Burr XII distribution: properties and applications. South African Statistical Journal. Forthcomig.

[14] Navarro, J. Franco, M. and Ruiz, J. M. (1998). Characterization through moments of the residual life and conditional spacing. The Indian Journal of Statistics, **60**, Series A, 36-48.

[15] Paranaíba, P. F., Ortega, E. M. M., Cordeiro, G. M. and de Pascoa, M. (2013). The Kumaraswamy Burr XII distribution: theory and practice. Journal of Statistical Computation and Simulation, **83**, 2117-2143.

[16] Paranaíba, P. F., Ortega, E. M. M., Cordeiro, G. M. and Pescim, R. R. (2011). The beta Burr XII distribution with application to lifetime data. Computation Statistics and Data Analysis, **55**, 1118-1136.

[17] Prudnikov, A. P., Brychkov, Y. A. and Marichev, O. I. (1986). Integrals and Series, **1**. Gordon and Breach Science Publishers, Amsterdam.

[18] Prudnikov, A. P., Brychkov, Y. A. and Marichev, O. I. (1992). Integrals and Series, **4**. Gordon and Breach Science Publishers, Amsterdam.

ACCEPTED MANUSCRIPT

[19] Shao, Q. (2004). Notes on maximum likelihood estimation for the three-parameter Burr XII distribution . Computational Statistics and Data Analysis, **45**, 675-687.

[20] Smith, R. L. and Naylor, J. C. (1987). A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution. Applied Statistics, **36**, 358-369.

[21] Tadikamalla, P. R. (1980). A look at the Burr and related distributions, International Statistical Review, **48**, 337-344.

[22] Tahir, M. H., Cordeiro, G. M., Mansoor, M. and Zubair, M. (2015). The Weibull-Lomax distribution: properties and applications. Hacettepe Journal of Mathematics and Statistics. Forthcoming.

[23] Yousof, H. M., Afify, A. Z., Alizadeh, M., Butt, N. S., Hamedani, G. G. and Ali, M. M. (2015). The transmuted exponentiated generalized-G family of distributions. Pakistan Journal of Statistics and Operation Research, **11**, 441-464.

ACCEPTED MANUSCRIPT

Table 1: MLEs and standard errors for various parameter values.

Actual Values				Sample	Estimated Values (Standard Deviations)			
α	β	a	b	size (n)	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{a}$	$\widehat{b}$
2.5	4	2.5	5	100	2.52178	4.019508	2.511852	5.131948
					(0.748566)	(1.023036)	(0.520946)	(0.73552)
				200	2.55434	4.029435	2.456489	4.962408
					(0.394184)	(0.475746)	(0.402375)	(0.557122)
				500	2.474571	4.027779	2.510952	5.01093
					(0.298284)	(0.462672)	(0.31505)	(0.195476)
				1000	2.521237	3.995163	2.502051	5.012125
					(0.093599)	(0.203402)	(0.207569)	(0.093944)
3	3	0.5	0.5	100	3.126232	2.895511	0.527005	0.479079
					(0.722604)	(0.924394)	(0.290221)	(0.305075)
				200	2.891608	3.124661	0.488041	0.500701
					(0.711989)	(0.884334)	(0.189888)	(0.195327)
				500	2.970981	2.934284	0.495323	0.508823
					(0.708819)	(0.771858)	(0.161157	(0.183254)
				1000	3.005697	2.978301	0.50048	0.499082
					(0.657257)	(0.49657)	(0.106829)	(0.10835)
2	5	1.5	1.75	100	2.142251	4.93574	1.511454	1.765754
					(1.027794)	(1.362116)	(0.470984)	(0.780493)
				200	2.046017	4.904674	1.504422	1.72069
					(0.505309)	(1.042812)	(0.41307)	(0.586415)
				500	1.981871	4.899376	1.48027	1.760829
					(0.236126)	(0.517487)	(0.235757)	(0.239855)
				1000	1.997151	4.999491	1.498517	1.759312
					(0.113938)	(0.239913)	(0.104453)	(0.102966)

Table 2: MLEs of the parameters from fitted models to the strengths of 1.5 cm glass fibre data and the corresponding SEs (given in parentheses).

Model	α	β	a	b	
WBXII	1.6077	2.7409	0.0026	1.8888	
	(0.3760)	(1.0100)	(0.0032)	(0.7680)	
WL	17.5336	110.7104	581.4052	5.1752	
	102.1130	(659.3920)	(28.2900)	(0.2010)	
WFr	0.3865	0.2436	1.4762	16.8561	
	(0.7990)	(0.2850)	(4.7820)	(20.4850)	
	a	b	c	θ	β
KwEBXII	4.022	137.8974	1.0241	1.3285	4.0102
	(24.1410)	(115.5110)	(0.6650)	(1.2970)	(26.0651)
	α	β	γ	λ	
TCWG	55.4366	7.9096	0.3904	0.0862	
	(59.0080)	(0.8670)	(0.0470)	(0.3780)	
	α	β	λ	δ	
ETGR	14.1641	0.9867	0.0009	0.3872	
	(10.9710)	(0.0540)	(0.0180)	(0.2760)	
	α	β	a	λ	
TMOFr	0.6500	6.8744	376.268	0.1499	
	(0.0490)	(0.5960)	(246.8320)	(0.3020)	
	a	b	c	θ	β
BBXII	26.1629	14.7050	0.9271	5.5864	8.2620
	(14.5880)	(12.8850)	(0.2130)	(5.2150)	(8.1320)
	a	b	c	θ	β
BEBXII	26.5651	23.3641	0.8777	1.2975	1.6224
	(26.4000)	(21.1450)	(0.5920)	(1.1120)	(0.9540)
	α	β	a	b	λ
TExGW	2.4230	0.5009	3.5578	647.4932	0.2361
	(7.1840)	(0.0950)	(5.3180)	(497.8990)	(0.2130)
	α	β			
BXII	7.4821	0.3207			
	(1.2850)	(0.0650)			

Table 3: The goodness of fit criteria for strengths of 1.5 cm glass fibre data.

Model	$-2\widehat{\ell}$	AIC	BIC	$HQIC$	$CAIC$	W^*	A^*
WBXII	28.607	36.607	45.18	39.979	37.297	0.19257	1.05507
WL	29.868	37.868	46.441	41.24	38.558	0.24429	1.31348
WFr	31.001	39.001	47.574	42.373	39.691	0.27786	1.48538
KwEBXII	39.041	49.041	59.757	53.255	50.093	0.43694	2.3495
TCWG	44.541	52.541	61.114	55.913	53.231	0.50241	2.72575
ETGR	47.858	55.858	64.43	59.229	56.547	0.54315	3.0567
TMOFr	48.46	56.46	65.032	59.831	57.149	0.56541	3.10166
BBXII	51.71	61.71	72.426	65.925	62.763	0.64538	3.50125
BEBXII	57.044	67.044	77.76	71.259	68.097	0.71739	3.91975
TExGW	76.435	86.435	97.151	90.65	87.488	0.43694	2.3495
BXII	97.442	101.442	105.729	103.128	101.642	1.17788	7.36685

ACCEPTED MANUSCRIPT

Table 4: MLEs of the parameters from the LWBXII regression model fitted to the vitamin A data set, the corresponding SEs (given in parentheses), p-values in [.].

Model	a	b	β	σ	τ_0	τ_1	τ_2	τ_3
LWBXII	0.4472	0.6699	0.1966	0.2111	2.0220	0.0223	0.0948	0.0363
	(0.1718)	(0.2901)	(0.0768)	(0.0639)	(0.2508)	(0.0027)	(0.0579)	(0.0600)
					[<0.001]	[<0.001]	[0.1016]	[0.5452]
LEBXII	0.7936	1	0.1231	0.2700	1.7892	0.0224	0.0950	0.0489
	(0.2723)		(0.0212)	(0.0305)	(0.1311)	(0.0027)	(0.0586)	(0.0588)
					[<0.001]	[<0.001]	[0.1061]	[0.4051]
LWLomax	41.1704	2.5441	0.0785	1	0.7760	0.0315	0.1389	0.0427
	(5.0098)	(0.2548)	(0.0154)		(0.3903)	(0.0027)	(0.0637)	(0.0638)
					[0.0470]	[<0.001]	[0.0295]	[0.5042]
LWLogistic	0.0184	0.3101	1	0.2807	-0.2161	0.0324	0.1465	0.0199
	(0.0393)	(0.6093)		(0.5513)	(1.9295)	(0.0025)	(0.0597)	(0.0597)
					[0.9108]	[<0.001]	[0.0142]	[0.7376]

Table 5: $-2\hat{l}$, $AIC, CAIC, BIC$ and $HQIC$ statistics for comparing the LWBXII, LEBXII, LWLomax and LWLogistic regression models.

Model	$-2\hat{l}$	AIC	$CAIC$	BIC	$HQIC$
LWBXII	3207.7	3219.7	3219.8	3260.5	3235.1
LEBXII	3204.4	3218.4	3218.5	3254.1	3231.8
LWLomax	3296.0	3283.0	3283.1	3318.7	3296.4
LWLogistic	3396.8	3410.8	3410.9	3446.4	3324.2

ACCEPTED MANUSCRIPT

Table 6: LR statistics for the vitamin A data.

Comparison	Hypotheses	LR statistic	p-value
LWBXII vs LEBXII	$H_0 : b = 1$ vs $H_1 : H_0$ is false	0.7	0.4027
LWBXII vs LWLomax	$H_0 : \sigma = 1$ vs $H_1 : H_0$ is false	65.3	<0.0001
LWBXII vs LWLogistic	$H_0 : \beta = 1$ vs $H_1 : H_0$ is false	193.1	<0.0001

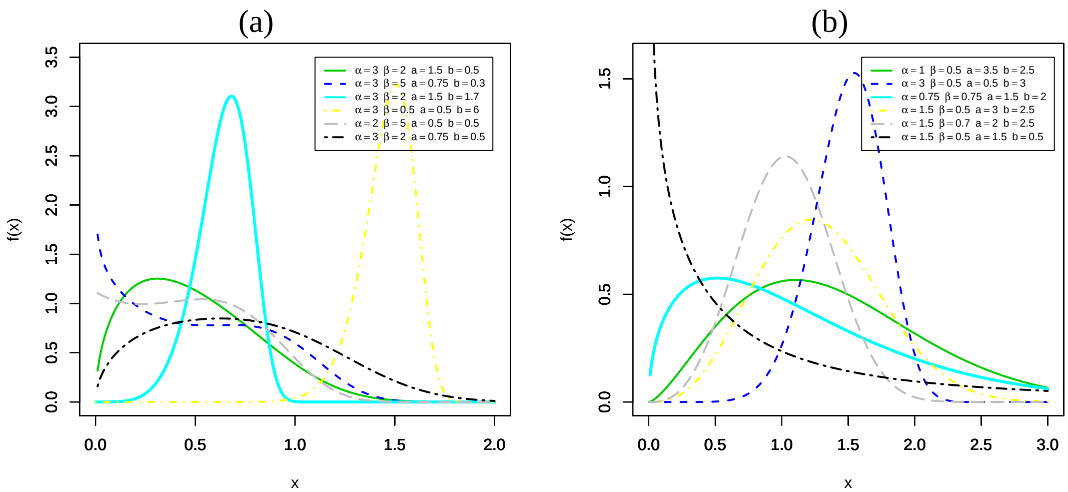


Figure 1: Plots of the WBXII pdf for some parameter values.

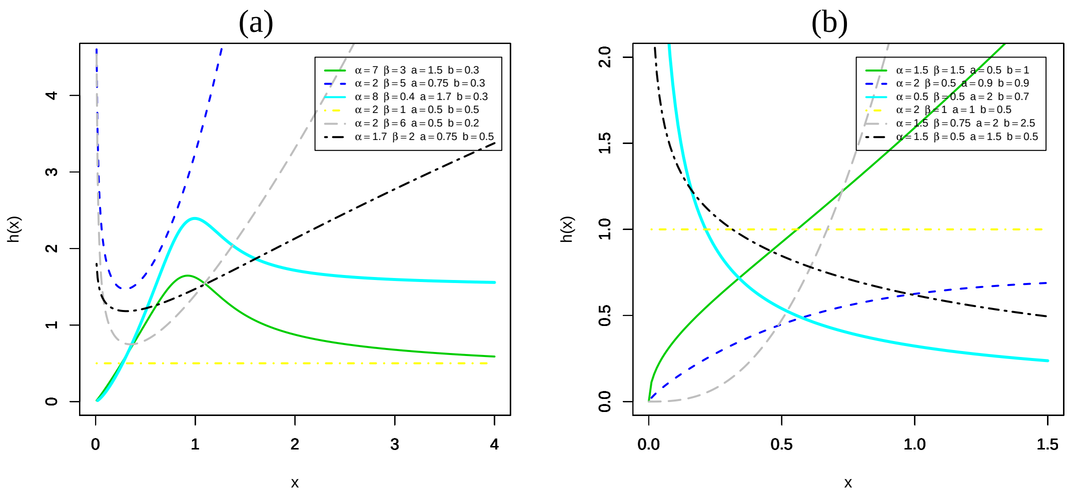


Figure 2: Plots of the WBXII hrf for some parameter values.

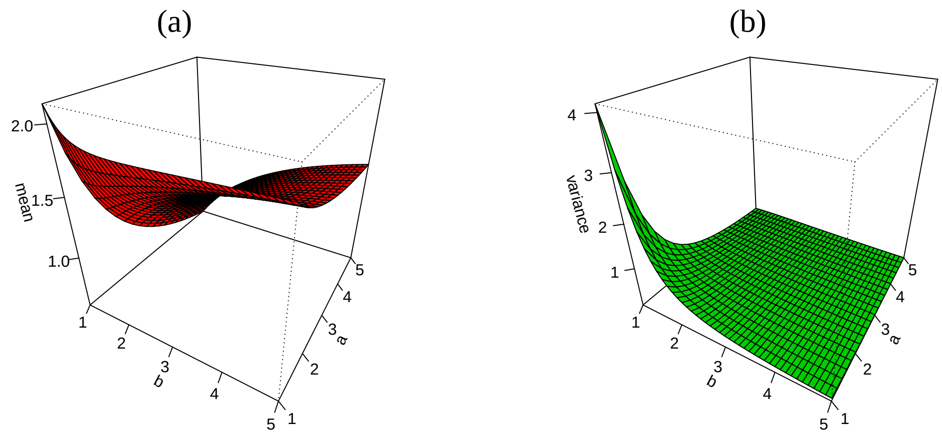


Figure 3: Plots of mean and variance of the WBXII model.

ACCEPTED MANUSCRIPT

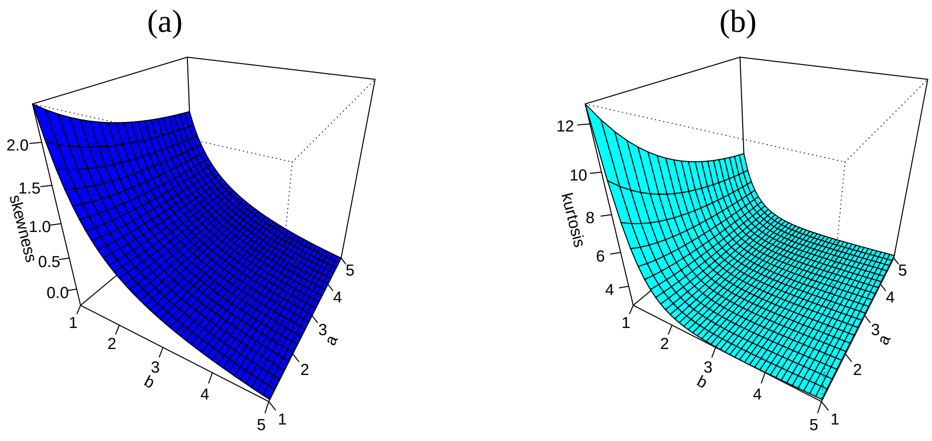
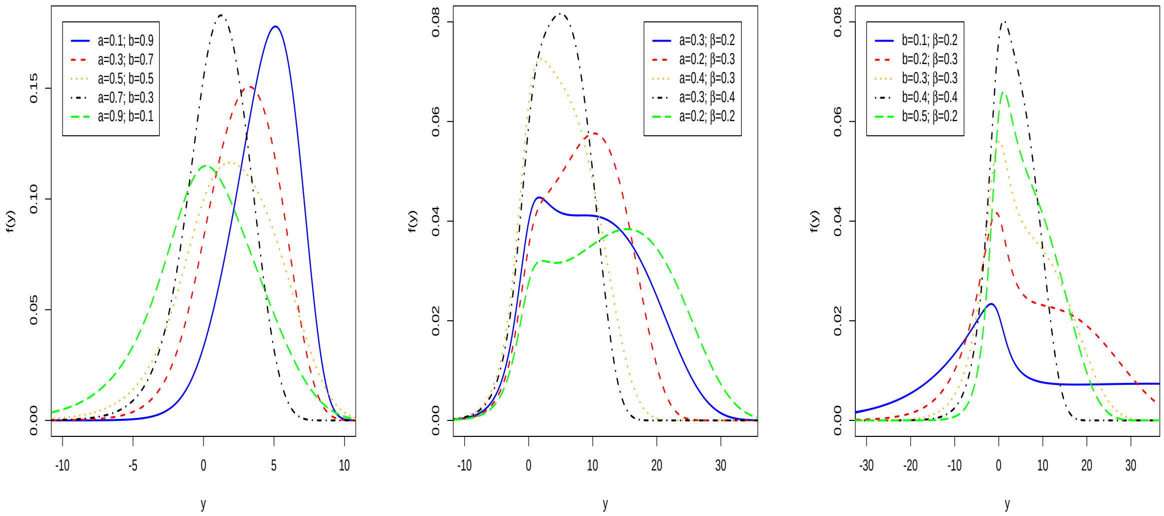


Figure 4: Plots of skewness and kurtosis of the WBXII model.



(a)

(b) (c)

Figure 5: The LWBXII density curves: (a) For some values of a and b with $\beta = 0.5$, $\mu = 0.0$ and $\sigma = 1.0$. (b) For some values of a and β with $b = 0.5$, $\mu = 0.0$ and $\sigma = 1.0$. (c) For some values of b and β with $a = 0.5$, $\mu = 0.0$ and $\sigma = 1.0$.

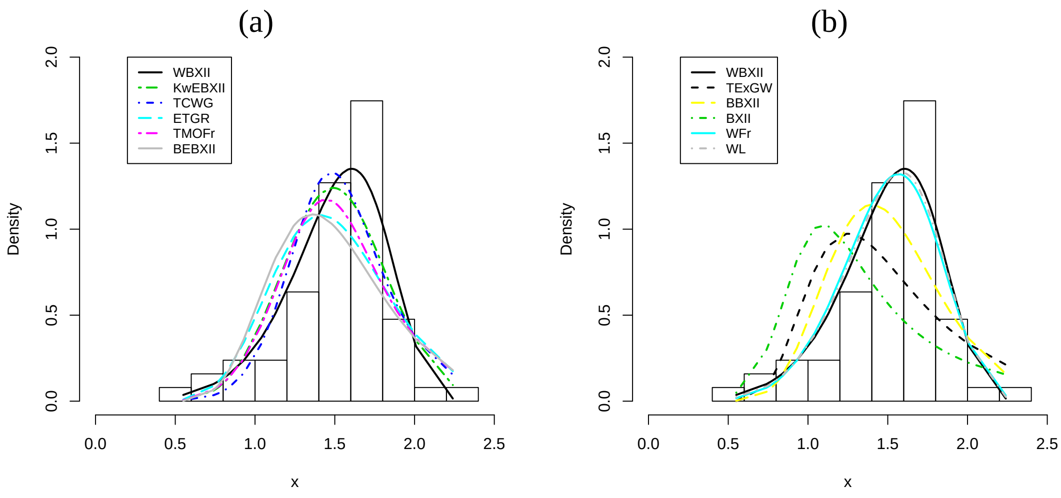


Figure 6: Estimated pdfs for the WBXII model and other competitive models.

ACCEPTED MANUSCRIPT

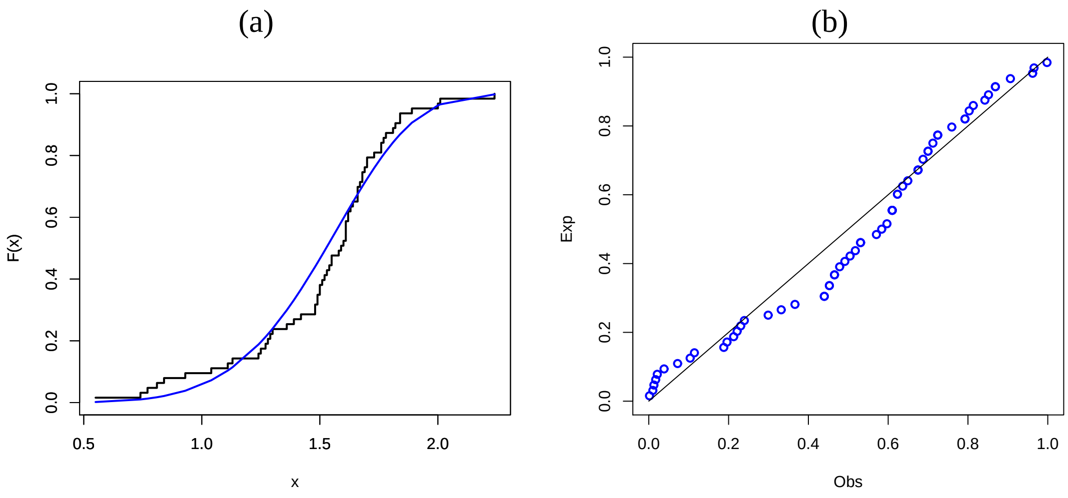


Figure 7: (a) The estimated cdf of the WBXII model. (b) QQ-plot of the WBXII model.