The McDonald's Inverse Weibull Distribution

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Abstract

We have proposed a new Inverse Weibull distribution by using the generalized Beta distribution of McDonald (1984). Basic properties of the proposed distribution has been studied. Parameter estimation has been discussed alongside an illustrative example.

Keywords: Inverse Weibull Distribution, McDonald Distribution, Generalized Beta Function.

1. Introduction

The Beta function defined as:

$$B(a,b)=\int_0^1 w^{a-1}(1-w)^{b-1}\,dw;$$

has attracted lot of attention in statistics. The function has been the basis of popular Beta distribution with density function:

$$f_B(x;a,b)=\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1};0<x<1;a,b\in \mathrm{R}^+; \qquad (1)$$

and distribution function:

$$F_B(x)=I_x(a,b)=\frac{B_x(a,b)}{B(a,b)}=\frac{1}{B(a,b)}\int_0^x w^{a-1}(1-w)^{b-1}\,dw; \qquad (2)$$

where $I_x(a,b)$ is incomplete Beta function ratio. The distribution function of Beta random variable; given in (2) has played very key role in generalizing the probability distributions. Eugene, Lee, & Famoye(2002) has used (2) to define a generalized class of distributions for any parent distribution having cdf $G(x)$ as:

$$F_{B-G}(x)=\frac{1}{B(a,b)}\int_0^{G(x)} w^{a-1}(1-w)^{b-1}\,dw. \qquad (3)$$

The density function corresponding to (3) is:

$$f_{B-G}\left(x\right)=\frac{1}{B\left(a,b\right)}g\left(x\right)\left[G\left(x\right)\right]^{a-1}\left[1-G\left(x\right)\right]^{b-1}; \tag{4}$$

where $g\left(x\right)=G'\left(x\right)$. Eugene, et al.(2002) has defined the Beta-Normal distribution by using $G\left(x\right)$ as cdf of Normal distribution in (4). Nadarajah & Gupta(2004) has defined the Beta-Frechet distribution on the basis of (4).

Kumaraswamy(1980) has introduced another distribution on $\left[0,1\right]$ that can be used as an alternate to the Beta distribution. The proposed distribution has been named as the Kumaraswamy distribution and has the density function as:

$$f_K\left(x;a,b\right)=abx^{a-1}\left(1-x^a\right)^{b-1};0<x<1;a,b\in\mathrm{R}^+. \tag{5}$$

Jones(2009) has studied the properties of (5) and has shown that the distribution can be used as an alternate of Beta distribution. The density (5) has also provided basis for generalization of distribution on the lines of (4). Cordeiro & de Castro(2011) have used (5) to propose the Kumaraswamy generalized distributions having density function as:

$$f_{K-G}\left(x\right)=abg\left(x\right)\left[G\left(x\right)\right]^{a-1}\left[1-G^a\left(x\right)\right]^{b-1}; \tag{6}$$

where $G\left(x\right)$ is distribution function of any available probability distribution. Cordeiro, Ortega, & Nadarajah(2010) have used distribution function of Weibull distribution in (6) as $G\left(x\right)$ to propose the Kumaraswamy Weibull (Kum-W) distribution.

McDonald(1984) has proposed another method of generalizing the probability distributions; based upon the generalized Beta distribution given as;

$$f_M\left(x;a,b,c\right)=\frac{c}{B\left(ac^{-1},b\right)}x^{a-1}\left(1-x^c\right)^{b-1}\;0<x<1;a,b,c\in\mathrm{R}^+; \tag{7}$$

and has defined defined the McDonald generalized distributions as:

$$f_{M-G}\left(x\right)=\frac{c}{B\left(ac^{-1},b\right)}g\left(x\right)\left[G\left(x\right)\right]^{a-1}\left[1-G^c\left(x\right)\right]^{b-1}. \tag{8}$$

Cordeiro & Lemonte(2012) have used the distribution function of Inverse Beta distribution in (8) to propose the McDonald Inverted Beta distribution.

In this paper we have proposed the McDonald Inverse Weibull distribution by using the distribution function of Inverse Weibull distribution in (8). The distribution with its common properties is defined in the following section.

2. The McDonald Inverse Weibull Distribution

Suppose that the random variable X has an Inverse Weibull distribution with density and distribution function given as:

$$g_{IW}\left(x;\alpha,\beta\right)=\frac{\alpha\beta}{x^{\beta+1}}\exp\left(-\frac{\alpha}{x^\beta}\right);x,\alpha,\beta\in\mathrm{R}^+;$$

and

$$G_{IW}(x)=\exp\left(-\frac{\alpha}{x^{\beta}}\right);x,\alpha,\beta\in\mathrm{R}^{+}. \tag{9}$$

The density function of Inverse Weibull distribution will be denoted by $IW(\alpha,\beta)$. We use (9) in (8) to propose the McDonald Inverse Weibull distribution as below:

$$\begin{aligned} f_{M-IW}(x) &= \frac{c}{B(ac^{-1},b)}\frac{\alpha\beta}{x^{\beta+1}}\exp\left(-\frac{\alpha}{x^{\beta}}\right)\left[\exp\left(-\frac{\alpha}{x^{\beta}}\right)\right]^{a-1} \\ &\quad\times\left[1-\left\{\exp\left(-\frac{\alpha}{x^{\beta}}\right)\right\}^{c}\right]^{b-1} \\ &= \frac{c}{B(ac^{-1},b)}\frac{\alpha\beta}{x^{\beta+1}}\exp\left(-\frac{a\alpha}{x^{\beta}}\right)\left[1-\exp\left(-\frac{c\alpha}{x^{\beta}}\right)\right]^{b-1}. \end{aligned} \tag{10}$$

Now using the following series expansion:

$$(1+x)^{\lambda}=\sum_{m=0}^{\infty}\frac{\Gamma(\lambda+1)}{m!\Gamma(\lambda-m+1)}x^{m};$$

the density (10) can be written as:

$$f_{M-IW}(x) = \frac{c}{B(ac^{-1},b)}\frac{\alpha\beta}{x^{\beta+1}}\exp\left(-\frac{a\alpha}{x^{\beta}}\right)\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(b)}{m!\Gamma(b-m)}\exp\left(-\frac{cm\alpha}{x^{\beta}}\right) \tag{11}$$

$$\begin{aligned} &= \frac{c}{B(ac^{-1},b)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(b)}{m!\Gamma(b-m)}\frac{\alpha\beta}{x^{\beta+1}}\exp\left\{-\frac{\alpha(cm+a)}{x^{\beta}}\right\} \\ &= \frac{c}{B(ac^{-1},b)}\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(b)}{m!\Gamma(b-m)}\frac{\alpha(cm+a)\beta}{(cm+a)x^{\beta+1}}\exp\left\{-\frac{\alpha(cm+a)}{x^{\beta}}\right\} \\ &= \frac{c}{B(ac^{-1},b)}\sum_{m=0}^{\infty}w_{m}\frac{\alpha(cm+a)\beta}{x^{\beta+1}}\exp\left\{-\frac{\alpha(cm+a)}{x^{\beta}}\right\} \\ &= \frac{c}{B(ac^{-1},b)}\sum_{m=0}^{\infty}w_{m}IW\{\alpha(cm+a),\beta\}; \end{aligned} \tag{12}$$

where

$$w_{m}=w(a,b,c,m)=\frac{(-1)^{m}\Gamma(b)}{m!(cm+a)\Gamma(b-m)}.$$

We will denote the density (10) by $MIW\left(a,b,c,\alpha,\beta\right)$. From (12) we can readily see that the density function of $MIW\left(a,b,c,\alpha,\beta\right)$ is weighted sum of density of $IW\left\{\alpha\left(cm+a\right),\beta\right\}$. Plot of the density function for various choices of the parameters is given in Figure-1. The distribution function of $MIW\left(a,b,c,\alpha,\beta\right)$ is:

$$\begin{aligned} F_{M-IW}\left(x\right) &= \int_0^x f_{M-IW}\left(t\right)dt \\ &= \frac{c}{B\left(ac^{-1},b\right)}\sum_{m=0}^{\infty} w_m \int_0^x \frac{\alpha\left(cm+a\right)\beta}{t^{\beta+1}}\exp\left\{-\frac{\alpha\left(cm+a\right)}{t^{\beta}}\right\}dt \end{aligned}$$

Solving the integral:

$$\begin{aligned} F_{M-IW}\left(x\right) &= \frac{c}{B\left(ac^{-1},b\right)}\sum_{m=0}^{\infty} w_m \exp\left\{-\frac{\alpha\left(cm+a\right)}{x^{\beta}}\right\} \\ &= \frac{c\exp\left(-\frac{a\alpha}{x^{\beta}}\right)}{aB\left(ac^{-1},b\right)}\,{}_2F_1\left[1-b,ac^{-1};1+ac^{-1};\exp\left(-\frac{a\alpha}{x^{\beta}}\right)\right]. \end{aligned} \tag{13}$$

The hazard rate function is immediately written by using (10) and (12) as:

$$\begin{aligned} h_{M-IW}\left(x\right) &= \frac{f_{M-IW}\left(x\right)}{1-F_{M-IW}\left(x\right)} \\ &= \frac{\frac{c}{B\left(ac^{-1},b\right)}\frac{\alpha\beta}{x^{\beta+1}}\exp\left(-\frac{a\alpha}{x^{\beta}}\right)\left[1-\exp\left(-\frac{c\alpha}{x^{\beta}}\right)\right]^{b-1}}{1-\frac{c\exp\left(-\frac{a\alpha}{x^{\beta}}\right)}{aB\left(ac^{-1},b\right)}\,{}_2F_1\left[1-b,ac^{-1};1+ac^{-1};\exp\left(-\frac{a\alpha}{x^{\beta}}\right)\right]} \end{aligned} \tag{14}$$

The plot of probability density funciton (pdf) and hazard rate function (HF) is given in Appendix B and Appendix C respectively.

3. Generating Function and Moments

In this section we have obtained the Moment Generating Function of McDonald Inverse Weibull distribution. The moment generating function of $MIW\left(a,b,c,\alpha,\beta\right)$ is:

$$\begin{aligned} M_X\left(t\right) = E\left(e^{tx}\right) &= \int_0^{\infty} e^{tx} f_{M-IW}\left(x\right)dx \\ &= \int_0^{\infty} e^{tx}\frac{c}{B\left(ac^{-1},b\right)}\sum_{m=0}^{\infty} w_m \frac{\alpha\left(cm+a\right)\beta}{x^{\beta+1}}\exp\left\{-\frac{\alpha\left(cm+a\right)}{x^{\beta}}\right\}dx \\ &= \frac{c}{B\left(ac^{-1},b\right)}\sum_{m=0}^{\infty} w_m \int_0^{\infty} e^{tx}\frac{\alpha\left(cm+a\right)\beta}{x^{\beta+1}}\exp\left\{-\frac{\alpha\left(cm+a\right)}{x^{\beta}}\right\}dx. \end{aligned}$$

Now expanding e^{tx} in power series, the moment generating function is:

$$M_X(t) = \frac{c}{B(ac^{-1},b)}\sum_{m=0}^{\infty} w_m \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty} x^r \frac{\alpha(cm+a)\beta}{x^{\beta+1}} \times \exp\left\{-\frac{\alpha(cm+a)}{x^{\beta}}\right\}dx \tag{15}$$

$$= \sum_{r=0}^{\infty}\frac{t^r}{r!}\frac{c}{B(ac^{-1},b)}\sum_{m=0}^{\infty} w_m \{\alpha(cm+a)\}^{\frac{r}{\beta}}\Gamma\left(1-\frac{r}{\beta}\right).$$

Equating the coefficient of $\frac{t^r}{r!}$ in (15), the $r-th$ raw moment of $MIW(a,b,c,\alpha,\beta)$ is immediately written as:

$$\mu_r^{/} = \frac{c}{B(ac^{-1},b)}\sum_{m=0}^{\infty} w_m \{\alpha(cm+a)\}^{\frac{r}{\beta}}\Gamma\left(1-\frac{r}{\beta}\right); \tag{16}$$

which exist for $r<\beta$. The mean, variance, skewness and kurtosis can be obtained from (16).

4. Order Statistics

In this section we have presented the order statistics of $MIW(a,b,c,\alpha,\beta)$ distribution. The distribution of $k-th$ order statistics for a random sample of size n from distribution $F(x)$ is given as:

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} f(x)\left[F(x)\right]^{k-1}\left[1-F(x)\right]^{n-k}; \tag{17}$$

where $f(x)=F^{'}(x)$. Using the series expansio of last term of (17), the distribution of $k-th$ order statistics can be written as:

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} f(x)\sum_{h=1}^{n-k}(-1)^h\binom{n-k}{h}\left[F(x)\right]^{k-h-1}.$$

Now using the density and distribution function of $MIW(a,b,c,\alpha,\beta)$ in above equation, the distribution of $k-th$ order statistics for $MIW(a,b,c,\alpha,\beta)$ is:

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!}\frac{c}{B(ac^{-1},b)}\sum_{m=0}^{\infty} w_m \frac{\alpha(cm+a)\beta}{x^{\beta+1}} \times \exp\left\{-\frac{\alpha(cm+a)}{x^{\beta}}\right\}\sum_{h=1}^{n-k}(-1)^h\binom{n-k}{h} \times \frac{c}{B(ac^{-1},b)}\left[\sum_{m=0}^{\infty} w_m \exp\left\{-\frac{\alpha(cm+a)}{x^{\beta}}\right\}\right]^{k-h-1}$$

5. Estimation

In this section we have given the maximum likelihood estimation of parameters of $MIW(a,b,c,\alpha,\beta)$. The likelihood function based on a random sample of size n from the density is given as:

$$LF(x_1,...,x_n) =_{i=1}^{n} f(x_i).$$

The log of likelihood function is:

$$\ln(LF) = \sum_{i=1}^{n} \ln\{f(x_i)\}.$$

For $MIW(a,b,c,\alpha,\beta)$ we have the density given in (10). The log of density is:

$$\begin{aligned}\ln\{f_{M-IW}(x)\} &= \ln(c) - \ln\left\{B\left(ac^{-1},b\right)\right\} + \ln(\alpha) + \ln(\beta) - (\beta+1)\ln x - \frac{a\alpha}{x^{\beta}} \\ &\quad + (b-1)\ln\left\{1-\exp\left(-\frac{c\alpha}{x^{\beta}}\right)\right\}.\end{aligned}$$

The log of likelihood function is therefore:

$$\begin{aligned}\ln(LF) &= n\ln(c) - n\ln\left\{B\left(ac^{-1},b\right)\right\} + n\ln(\alpha) + n\ln(\beta) - (\beta+1)\sum_{i=1}^{n}\ln x_i \\ &\quad - \sum_{i=1}^{n}\frac{a\alpha}{x_i^{\beta}} + (b-1)\sum_{i=1}^{n}\ln\left\{1-\exp\left(-\frac{c\alpha}{x_i^{\beta}}\right)\right\}.\end{aligned} \tag{18}$$

The derivatives of log of likelihood function *w.r.t.* the parameters are given below:

$$\frac{\partial \ln(LF)}{\partial a} = \frac{n}{c}\left\{\psi\left(ac^{-1}+b\right) - \psi\left(ac^{-1}\right) - \frac{c}{n}\sum_{i=1}^{n}\frac{\alpha}{x_i^{\beta}}\right\}; \tag{19}$$

$$\frac{\partial \ln(LF)}{\partial b} = n\left\{\psi\left(ac^{-1}+b\right) - \psi(b) - \frac{1}{n}\sum_{i=1}^{n}\ln\left(1-\exp\left(-\frac{c\alpha}{x_i^{\beta}}\right)\right)\right\}; \tag{20}$$

$$\begin{aligned}\frac{\partial \ln(LF)}{\partial c} &= \frac{n}{c^2}\left\{c - a\psi\left(ac^{-1}+b\right) + a\psi\left(ac^{-1}\right) - \frac{c(b-1)}{n^2}\right. \\ &\quad \left.\times\sum_{i=1}^{n}\frac{\alpha x_i^{-\beta}\exp\left(-c\alpha x_i^{-\beta}\right)}{1-\exp\left(-c\alpha x_i^{-\beta}\right)}\right\};\end{aligned} \tag{21}$$

$$\frac{\partial \ln(LF)}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} a x_i^{-\beta} + (b-1)\sum_{i=1}^{n}\frac{c x_i^{-\beta}\exp\left(-c\alpha x_i^{-\beta}\right)}{1-\exp\left(-c\alpha x_i^{-\beta}\right)}; \tag{22}$$

$$\begin{aligned}\frac{\partial \ln\left(LF\right)}{\partial \beta} &= \frac{n}{\beta} - \sum_{i=1}^{n} \ln\left(x_i\right) + \sum_{i=1}^{n} a\alpha \ln\left(x_i\right) x_i^{-\beta} \\ &-\left(b-1\right)\sum_{i=1}^{n} \frac{c\alpha \ln\left(x_i\right) x_i^{-\beta} \exp\left(-c\alpha x_i^{-\beta}\right)}{1-\exp\left(-c\alpha x_i^{-\beta}\right)}.\end{aligned} \tag{23}$$

The maximum likelihood estimates of the parameters can be obtained by solving equations (19) through (23) numerically. The observed information matrix for $MIW\left(a,b,c,\alpha,\beta\right)$ is given as:

$$J\left(\mathbf{q}\right) = -\begin{bmatrix} J_{aa} & J_{ab} & J_{ac} & J_{a\alpha} & J_{a\beta} \\ & J_{bb} & J_{bc} & J_{b\alpha} & J_{b\beta} \\ & & J_{cc} & J_{c\alpha} & J_{c\beta} \\ & & & J_{\alpha\alpha} & J_{\alpha\beta} \\ & & & & J_{\beta\beta} \end{bmatrix}; \tag{24}$$

where $J_{\theta_j\theta_k} = \dfrac{\partial^2 \ln\left(LF\right)}{\partial\theta_j \partial\theta_k}$. The entries of observed information matrix are given in the appendix.

6. Application

SPSS buit-in dataset "Employee database" is used to obtain fit the McDonalds Invcerse Weibull Distribution. The ML estimates of the parameters of McDonald's Inverse Weibull Distribution and some other distributions are given in the table below. We have also given the AIC and negative of log likelihood function to decide about the suitability of the model.

	Model	Parameters (SE in Parenthesis)					-2 Log (LF)	AIC
		a	**b**	**c**	**Alpha**	**Beta**		
Salary	**M-IW**	138.96 (62.16)	16.92 (3.86)	1.64 (3.52)	123.01 (48.16)	0.73 (0.08)	9478.491	9488.491
	Inverse Weibull				485.72(83.30)	0.68(0.02)	10343.47	10347.47
	Weibull				584.47(59.32)	0.24(0.01)	11869.48	11873.48
	Gamma				2.10(0.08)	0.04(0.002)	11392.86	11396.86

From the table we can see that the McDonalds Inverse Weibull Distribution has least value of AIC as compared with the competing models. This indicates that our proposed McDonalds Inverse Weibull distribution fits the data reasonably well.

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Appendix A:

The entries of observed information metrix are given below:

$$J_{aa} = \frac{\partial^2 \ln(LF)}{\partial^2 a} = \frac{n}{c^2}\left\{\psi'\left(ac^{-1}+b\right)-\psi'\left(ac^{-1}\right)\right\}$$

$$J_{ab} = \frac{\partial^2 \ln(LF)}{\partial a \partial b} = \frac{n}{c}\psi'\left(ac^{-1}+b\right)$$

$$J_{ac} = \frac{\partial^2 \ln(LF)}{\partial a \partial c} = \frac{n}{c^3}\left[c\left\{\psi\left(ac^{-1}\right)-\psi\left(ac^{-1}+b\right)\right\}+a\left\{\psi'\left(ac^{-1}+b\right)-\psi'\left(ac^{-1}\right)\right\}\right]$$

$$J_{a\alpha} = \frac{\partial^2 \ln(LF)}{\partial a \partial \alpha} = -\sum_{i=1}^{n} x_i^{-\beta}$$

$$J_{a\beta} = \frac{\partial^2 \ln(LF)}{\partial a \partial \alpha} = \sum_{i=1}^{n} \ln(x_i)\, x_i^{-\beta}$$

$$J_{bb} = \frac{\partial^2 \ln(LF)}{\partial^2 b} = n\left\{\psi'\left(ac^{-1}+b\right)-\psi'(b)\right\}$$

$$J_{bc} = \frac{\partial^2 \ln(LF)}{\partial b \partial c} = -\frac{an}{c}\psi'\left(ac^{-1}+b\right)+\sum_{i=1}^{n}\frac{\alpha x_i^{-\beta}\exp\left(-c\alpha x_i^{-\beta}\right)}{1-\exp\left(-c\alpha x_i^{-\beta}\right)}$$

$$J_{b\alpha} = \frac{\partial^2 \ln(LF)}{\partial b \partial \alpha} = \sum_{i=1}^{n}\frac{c x_i^{-\beta}\exp\left(-c\alpha x_i^{-\beta}\right)}{1-\exp\left(-c\alpha x_i^{-\beta}\right)}$$

$$J_{b\beta} = \frac{\partial^2 \ln(LF)}{\partial b \partial \beta} = -\sum_{i=1}^{n}\frac{c\alpha \ln(x_i)\, x_i^{-\beta}\exp\left(-c\alpha x_i^{-\beta}\right)}{1-\exp\left(-c\alpha x_i^{-\beta}\right)}$$

$$J_{cc} = \frac{\partial^2 \ln(LF)}{\partial^2 c} = -\frac{n}{c^4}\left[c^2-2ca\left\{\psi\left(ac^{-1}+b\right)-\psi\left(ac^{-1}\right)\right\}\right.$$

$$\left.-a\left\{\psi'\left(ac^{-1}+b\right)-\psi'\left(ac^{-1}\right)\right\}-(b-1)\sum_{i=1}^{n}\frac{\alpha^2 x_i^{-2\beta}\exp\left(-c\alpha x_i^{-\beta}\right)}{\left\{1-\exp\left(-c\alpha x_i^{-\beta}\right)\right\}^2}\right]$$

$$J_{c\alpha} = \frac{\partial^2 \ln(LF)}{\partial c \partial \alpha} = (b-1)\left[\sum_{i=1}^{n}\frac{x_i^{-\beta}\exp\left(-c\alpha x_i^{-\beta}\right)}{1-\exp\left(-c\alpha x_i^{-\beta}\right)}-\sum_{i=1}^{n}\frac{c\alpha x_i^{-2\beta}\exp\left(-c\alpha x_i^{-\beta}\right)}{\left\{1-\exp\left(-c\alpha x_i^{-\beta}\right)\right\}^2}\right]$$

$$J_{c\beta} = \frac{\partial^2 \ln(LF)}{\partial c \partial \beta} = (b-1)\left[\sum_{i=1}^{n} \frac{c\alpha^2 \ln(x_i) x_i^{-2\beta} \exp(-c\alpha x_i^{-\beta})}{\left\{1-\exp(-c\alpha x_i^{-\beta})\right\}^2} - \sum_{i=1}^{n} \frac{\ln(x_i) x_i^{-\beta} \exp(-c\alpha x_i^{-\beta})}{1-\exp(-c\alpha x_i^{-\beta})}\right]$$

$$J_{\alpha\alpha} = \frac{\partial^2 \ln(LF)}{\partial^2 \alpha} = -\frac{n}{\alpha^2} - (b-1)\sum_{i=1}^{n} \frac{c^2 x_i^{-2\beta} \exp(-c\alpha x_i^{-\beta})}{\left\{1-\exp(-c\alpha x_i^{-\beta})\right\}^2}$$

$$J_{\alpha\beta} = \frac{\partial^2 \ln(LF)}{\partial \alpha \partial \beta} = a\sum_{i=1}^{n} \ln(x_i) x_i^{-\beta} + (b-1)\left[\sum_{i=1}^{n} \frac{c^2\alpha \ln(x_i) x_i^{-2\beta} \exp(-c\alpha x_i^{-\beta})}{\left\{1-\exp(-c\alpha x_i^{-\beta})\right\}^2} - \sum_{i=1}^{n} \frac{c\ln(x_i) x_i^{-\beta} \exp(-c\alpha x_i^{-\beta})}{1-\exp(-c\alpha x_i^{-\beta})}\right]$$

$$J_{\beta\beta} = \frac{\partial^2 \ln(LF)}{\partial^2 \beta} = -\frac{n}{\beta^2} - a\alpha\sum_{i=1}^{n} \ln^2(x_i) x_i^{-\beta} + (b-1)\left[\sum_{i=1}^{n} \frac{c\alpha \ln^2(x_i) x_i^{-\beta} \exp(-c\alpha x_i^{-\beta})}{1-\exp(-c\alpha x_i^{-\beta})} - \sum_{i=1}^{n} \frac{c^2\alpha^2 \ln^2(x_i) x_i^{-2\beta} \exp(-c\alpha x_i^{-\beta})}{\left\{1-\exp(-c\alpha x_i^{-\beta})\right\}^2}\right]$$

Appedix B:

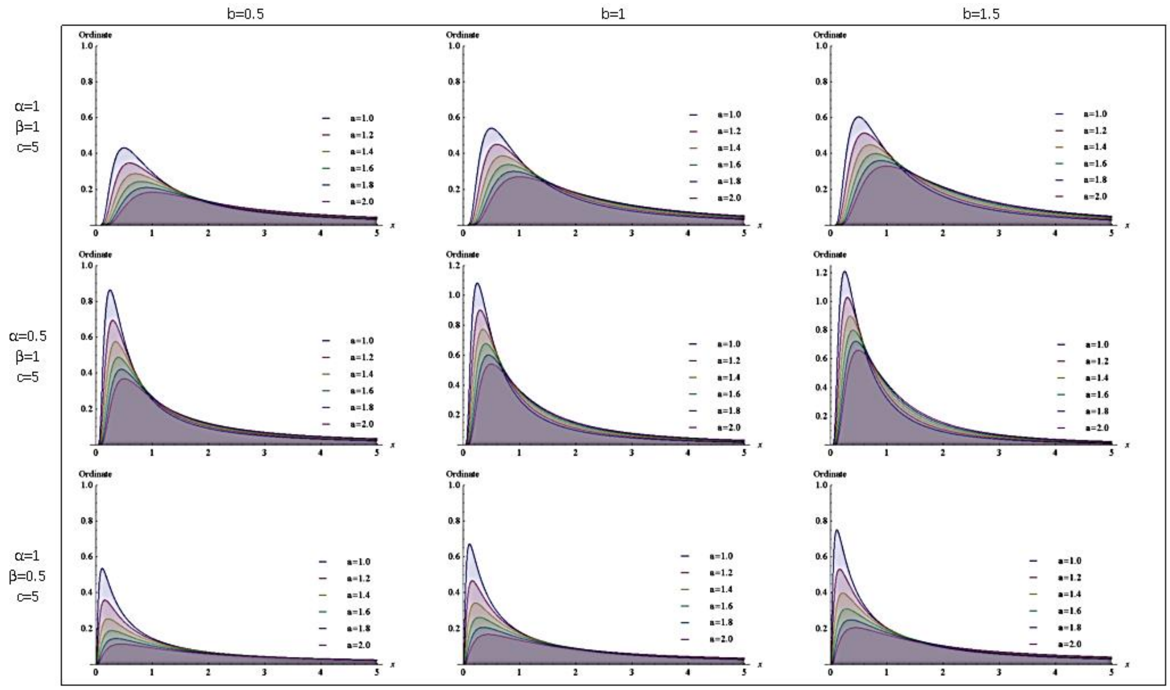

Figure 1: Plots of density function (pdf) of McDonald's Inverse Weibull distribution for various values of a, b, c, α and β

Appendix C:

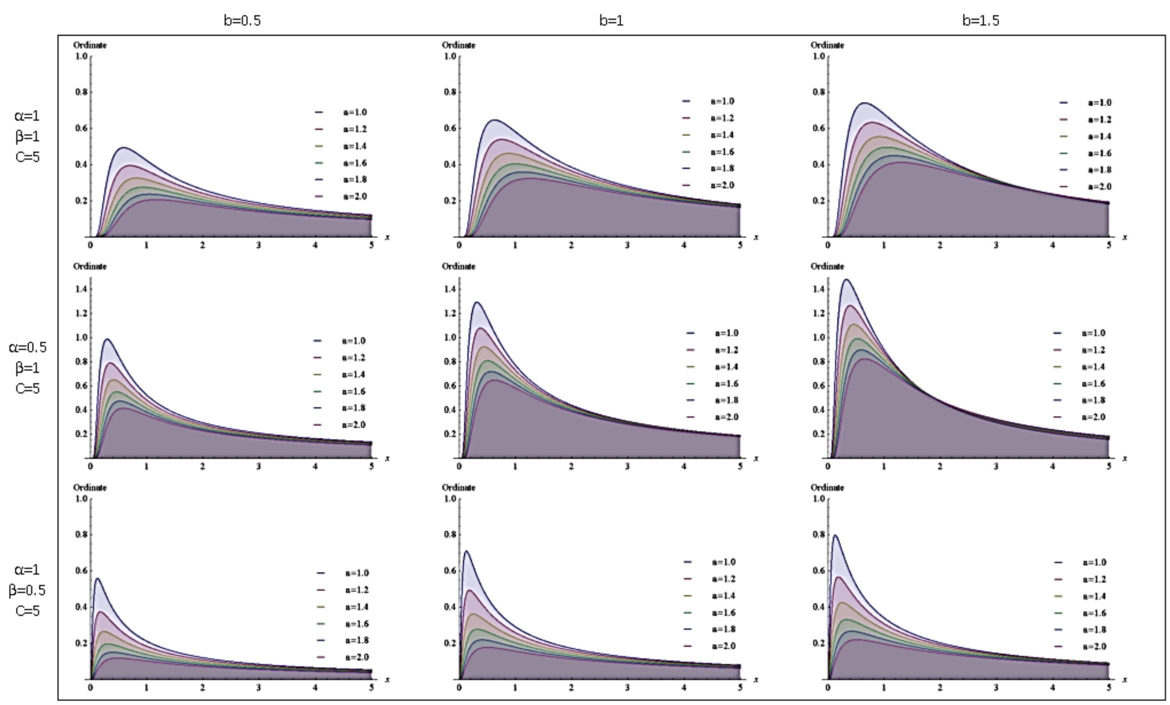

Figure 2: Plots of hazard function of McDonald's Inverse Weibull distribution for various values of a, b, c, α and β