Physical Simulation of Infiltration Equations

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Abstract. A method of using laminar (capillary) flow models for synthesizing the infiltration process on physical hydrologic models is described. The equations defining laminar flow in capillary tubes and between parallel plates were derived for several different geometric configurations representing horizontal, vertically downward, vertically upward, and radial flow. The theoretical equations for the different configurations are shown to be analogous in functional form to the theoretical and empirical infiltration equations used to define the time variation in the infiltration rate of a soil. The results from several tests conducted on capillary tubes were in close agreement with those obtained from the theoretical equations. Thus it was concluded that capillary flow models can be used to synthesize the infiltration process on physical models.

During the past few years several investigations have been conducted on the use of physical models to study hydrologic processes such as watershed runoff and overland flow. Most of these studies have used geometrically similar, scaled 'solid' models of the prototype system constructed from plexiglass, fiberglass, or other impervious materials [*Mamisao*, 1952; *Chery*, 1965; *Grace and Eagleson*, 1967]. Such studies have shown that physical modeling of the rainfall–runoff process is feasible when losses are not of major importance. When losses are significant, however, the application of physical models to the study of hydrologic systems has been inhibited because of the difficulty of synthesizing or simulating the 'loss' process. *Eagleson* [1969] indicates that since losses cannot be modeled, their indirect effect on flow dynamics will be absent in the model. In certain cases, however, their effects can be accounted for by making approximate corrections to the model-forecast output.

In most natural hydrologic systems the infiltration process plays a predominant role. Water that infiltrates the ground may be considered a loss in terms of surface runoff or a gain in terms of soil moisture and groundwater supply. Several methods have been tried to synthesize this process on physical models: (1) drilling small holes in the material so that the fluid (usually water) may be drained off; (2) scaling the precipitation rate to account for both the spatial and the time variability of the soil infiltration rate (a technique that usually causes droplets to form on the model at a low precipitation rate (high infiltration) because of the effects of surface tension); or (3) covering the model with some absorbent or porous material. To the author's knowledge, none of these methods of synthesizing the process have proved completely successful. Thus any method that would allow the synthesizing infiltration on a physical model would be extremely valuable in studies of hydrologic systems.

INFILTRATION

Infiltration of water into a soil, like many other flow processes in porous mediums, is governed by the *Richards* [1931] soil moisture equation,

$$\partial\theta/\partial t = \nabla \cdot k \nabla \Phi \tag{1}$$

in which θ is the volumetric moisture content, t is the time, k is the capillary conductivity, and Φ is the total potential. Equation 1 is the continuity equation for flow when the flux V at any point is defined by the Darcy equation,

$$V = -k\nabla\Phi \tag{2}$$

According to equation 2 the flux at any point in a soil system, including the soil surface, is proportional to the hydraulic or capillary conductivity k and the total potential gradient $\nabla\Phi$.

Many investigators have provided mathematical expressions that define the time variation in infiltration:

Horizontal infiltration

Kirkham and Feng [1949]

$$M = At^{1/2} \tag{3}$$

Vertical downward infiltration

Kostiakov [1932]; *Lewis* [1937]

$$M = At^{n} \tag{4}$$

Horton [1940]

$$M = At + B(1 - e^{-ct}) \tag{5}$$

Philip [1954]

$$M = S^{*}t^{1/2} + At \tag{6}$$

Green and Ampt [1911]

$$t = A[M - B\ln(1 + M/B)] \tag{7}$$

Vertical upward infiltration

Green and Ampt [1911]

$$t = A[-M - B\ln(1 - M/B)] \tag{8}$$

in which M is the mass infiltration occurring in time t, e is the base of the natural logarithms, and n, A, B, S^*, c are constants whose magnitudes must be evaluated experimentally.

THEORETICAL ANALYSIS AND DERIVATION OF EQUATIONS

In the derivation of equations, the following assumptions have been made:

1. The flow of the fluid is laminar.
2. The energy loss of the fluid caused by disturbances in the flow pattern at the entrance or at the front of the model is negligible.
3. The inertial force produced by deceleration of the flow is small and therefore negligible.
4. During the advance of the viscous fluid in a capillary tube or between parallel plates the capillary head at the liquid front caused by surface tension between the liquid meniscus and the adjoining wall is constant and equal in magnitude to that under an equilibrium flow condition at a fixed temperature and pressure.

Velocity of steady laminar flow in a capillary tube. *Rouse* [1961] shows that the basic equations of Navier-Stokes may be simplified for steady, laminar flow in a circular tube of constant diameter to:

$$\frac{\partial}{\partial x}(p + \rho gh) = \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{9}$$

$$(\partial/\partial y)(p + \rho gh) = 0 \tag{10}$$

$$(\partial/\partial z)(p + \rho gh) = 0 \tag{11}$$

where

x, y, z, rectangular Cartesian coordinates in which the x axis is coincident with the center line of the tube and positive in the direction of flow;
p, pressure intensity at a point in the tube;
ρ, density of the fluid;
g, acceleration due to gravity;
h, elevation to a point in the tube as measured from a horizontal datum;
μ, dynamic viscosity;
u, velocity component in the x direction.

The pressure intensity will vary in the axial direction as the result of viscous shear and change in elevation. However, it must be hydrostatically distributed over every normal cross section of flow.

The solution of equation 9 expressed as the mean velocity of flow in the x direction is:

$$V = -\frac{d^2}{32\mu}\frac{\partial}{\partial x}(p + \rho gh) \tag{12}$$

where V is the mean velocity of flow and d is the diameter of the tube. Since the mean velocity V is constant, the potential gradient is likewise constant and may be expressed as (Figure 1):

$$\frac{\partial}{\partial x}(p + \rho gh) = \rho g\left(\frac{H_2 + h_2 - H_1 - h_1}{L}\right) \tag{13}$$

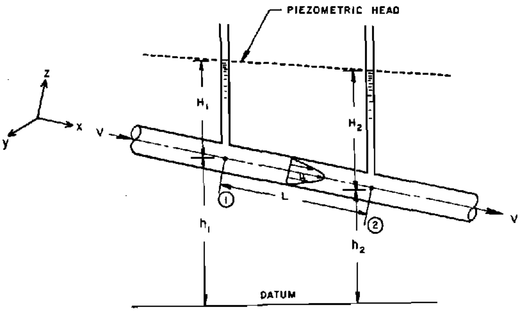


Fig. 1. Steady laminar flow in a circular tube.

where H_1, H_2 are the piezometric heads at cross sections 1 and 2, respectively, measured from the center line of the tube; h_1, h_2 are the elevations at cross sections 1 and 2, respectively, measured from a horizontal datum to the center line of the tube; and L is the distance between the two cross sections along the center line of the tube. Substituting equation 13 into equation 12, one obtains

$$V = \frac{\rho g d^2}{32\mu}\left(\frac{H_1 + h_1 - H_2 - h_2}{L}\right) \tag{14}$$

which is a generalized form of the well-known Hagen-Poiseuille formula.

Volumetric flow into a horizontal capillary tube. Consider the case in which a capillary tube of constant diameter is placed horizontally and connected to a container in which the level of fluid is maintained at a constant elevation (Figure 2). Flow into the tube is caused by the force of the constant liquid pressure at the entrance (point A) and by a constant capillary force at the moving front (point B). During an infinitesimal interval of time, flow in the tube can be considered steady, and thus equation 14 can be assumed to apply. Furthermore, since the tube is horizontal, $h_1 = h_2$; and if H_1 (the pressure head at point A) is taken as H and if

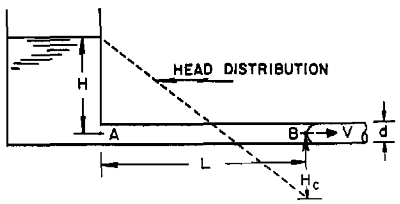


Fig. 2. Flow in horizontal capillary tube.

H_2 (the negative capillary head at point B) is taken as $(-H_c)$, then the average velocity of flow can be expressed as

$$V = \frac{\rho g d^2}{32\mu}\left(\frac{H + H_c}{L}\right) \tag{15}$$

If the capillary head is assumed equal to the equilibrium height of rise of a fluid in a vertical capillary tube, then

$$H_c = 4\sigma \cos \theta/\rho g d \tag{16}$$

where σ is the surface tension of the fluid and θ is the angle of contact between the fluid and the tube wall.

Combining equations 15 and 16, one obtains

$$V = \frac{\rho g d^2}{32\mu}\left[\frac{H + (4\sigma \cos \theta/\rho g d)}{L}\right] \tag{17}$$

However, since the flow in the horizontal tube is unsteady, the mean velocity V at any section changes with time. From the law of continuity, the mean velocity of the flow at any section at any time can be shown to be equal to the time rate of change of the length L of the liquid column AB; that is,

$$V = \frac{dL}{dt} = \frac{\rho g d^2}{32\mu}\left[\frac{H + (4\sigma \cos \theta/\rho d g)}{L}\right] \tag{18}$$

If one separates the variables in equation 18 and integrates in the limits, $L = 0$ when $t = 0$, $L = L$ when $t = t$, and the distance of advance is:

$$L = \left[\frac{\rho g d^2}{16\mu}\left(H + \frac{4\sigma \cos \theta}{\rho g d}\right)\right]^{1/2} t^{1/2} \tag{19}$$

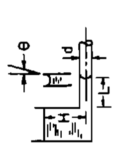
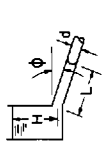
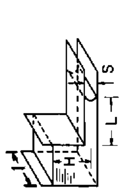
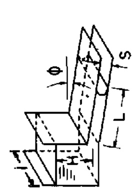
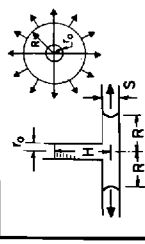
The cumulative flow of fluid into the tube M is therefore:

$$M = \frac{\pi d^2}{4}\left[\frac{\rho g d^2}{16\mu}\left(H + \frac{4\sigma \cos \theta}{\rho g d}\right)\right]^{1/2} t^{1/2} \tag{20}$$

The theoretical equations of laminar, capillary flow for several different geometric configurations are summarized in Table 1. The procedure used in developing these equations was similar to that outlined in deriving the equation of flow into a horizontal capillary tube. For additional details of these theoretical derivations, the reader is referred to the work of *Lin* [1967]. In Table 1 note that the generalized equations defining flow into inclined capillary tubes or flow per unit width between parallel plates (within $90° \geq \phi \geq -90°$) are applicable:

1. When the tube or parallel plates are inclined downward from the horizontal, the angle of inclination ϕ is taken as positive; when they are inclined upward, ϕ is negative.
2. When the tube or the parallel plates are horizontal ($\phi = 0$), by expanding the

TABLE 1. Flow Equations of Different Geometric Configurations and Their Analogous Infiltration Equations

GEOMETRIC FORMS	FLOW EQUATIONS	EQUATION NUMBER	ANALOGOUS INFILTRATION EQUATIONS
	$L = \sqrt{\frac{\rho g d^2}{16\mu}\left[H + \frac{4\sigma \cos\theta}{\rho g d}\right]}\, t^{\frac{1}{2}}$ $M = \frac{\pi d^2}{4}\sqrt{\frac{\rho g d^2}{16\mu}\left[H + \frac{4\sigma \cos\theta}{\rho g d}\right]}\, t^{\frac{1}{2}}$	19 20	$M = At^{\frac{1}{2}}$
	$t = \frac{32\mu}{\sin^2\phi\,\rho g d^2}\left\{L \sin\phi - \left(H + \frac{4\sigma\cos\theta}{\rho g d}\right)\ln\left[1 + \frac{L\sin\phi}{\left(H + \frac{4\sigma\cos\theta}{\rho g d}\right)}\right]\right\}$ $t = \frac{128\mu}{\sin^2\phi\,\rho g \pi d^4}\left\{M\sin\phi - \frac{\pi d^2}{4}\left(H + \frac{4\sigma\cos\theta}{\rho g d}\right)\ln\left[1 + \frac{M\sin\phi}{\frac{\pi d^2}{4}\left(H + \frac{4\sigma\cos\theta}{\rho g d}\right)}\right]\right\}$ $-90° \leq \phi \leq 90°$	21 22	$t = A\left[\pm M - B\ln(1 \pm M/B)\right]$
	$L = \sqrt{\frac{\rho g S^2}{6\mu}\left(H + \frac{2\sigma\cos\theta}{\rho g S}\right)}\, t^{\frac{1}{2}}$ $M = S\sqrt{\frac{\rho g S^2}{6\mu}\left(H + \frac{2\sigma\cos\theta}{\rho g S}\right)}\, t^{\frac{1}{2}}$	23 24	$M = At^{\frac{1}{2}}$
	$t = \frac{12\mu}{\sin^2\phi\,\rho g S}\left\{L\sin\phi - \left(H + \frac{2\sigma\cos\theta}{\rho g S}\right)\ln\left[1 + \frac{L\sin\phi}{\left(H + \frac{2\sigma\cos\theta}{\rho g S}\right)}\right]\right\}$ $t = \frac{12\mu}{\sin^2\phi\,\rho g S^2}\left\{M\sin\phi - S\left(H + \frac{2\sigma\cos\theta}{\rho g S}\right)\ln\left[1 + \frac{M\sin\phi}{S\left(H + \frac{2\sigma\cos\theta}{\rho g S}\right)}\right]\right\}$ $-90° \leq \phi \leq 90°$	25 26	$t = A\left[\pm M - B\ln(1 \pm M/B)\right]$
	$t = \frac{3\mu}{(\sigma\cos\theta)^2}\left\{\frac{\sigma\cos\theta}{S}(R^2 - r_0^2) - (R - r_0)\left(r_0\rho g H + \frac{2r_0\sigma\cos\theta}{S}\right)\right.$ $\left. + r_0\rho g H\left(1 + \frac{r_0\rho g H S}{2\sigma\cos\theta}\right)\ln\left[\frac{r_0\rho g H + \frac{2R\sigma\cos\theta}{S}}{r_0\rho g H + \frac{2r_0\sigma\cos\theta}{S}}\right]\right\}$	27	

logarithmic term or using the l'Hôspital rule, one can reduce the equations to those describing horizontal flow (e.g., equations 19 and 20).

It is important that the versatility of laminar flow models be recognized. For example, it has been shown that flow into horizontally placed capillary tubes may be expressed by an equation of the form $M = kt^{1/2}$, which is analogous to an infiltration equation of a similar form. In addition, the value of k is a function of factors such as the tube diameter, the liquid head at the entrance, and the density, viscosity, surface tension, and contact angle of the fluid. Because of these factors a wide range of flow rates may be obtained in a given model by changing the properties of the fluid without having to change the physical dimensions of the model.

Similarly flow through anisotropic soils may be synthesized with the models by changing the size of the tubes, the spacing of the plates, or the angle of inclination of either.

SIMILITUDE CRITERIA

The relationships given by equations 20, 22, 24, 26, and 27 (Table 1) and their analogies with currently used infiltration equations provide the basis for establishing similitude criteria. Thus the conditions for similitude between the prototype and the model for several cases are:

Horizontal flow. The infiltration equation for horizontal flow can be synthesized by using either capillary tubes or parallel plates placed horizontally. When capillary tubes are used, the mass infiltration to the model M_m in time t_m can be written as

$$M_m = A_m t_m^{1/2} \tag{28}$$

where

$$A_m = \frac{\pi d^2}{4}\left[\frac{\rho g d^2}{16\mu}\left(H + \frac{4\sigma \cos \theta}{\rho g d}\right)\right]^{1/2}$$

Thus the requirement for similitude determined by combining (3) and (28) is:

$$\frac{M}{M_m} = \frac{A}{A_m}\left(\frac{t}{t_m}\right)^{1/2} \tag{29}$$

Downward flow. The infiltration equation for downward flow can be synthesized by using either capillary tubes or parallel plates placed vertically. When capillary tubes are used, the model equation can be written as:

$$t_m = A_m[M_m - B_m \ln (1 + M_m/B_m)] \tag{30}$$

where $A_m = 128\mu/\rho g \pi d^4 \sin\phi$ and

$$B_m = \frac{\pi d^2}{4 \sin \phi}\left(H + \frac{4\sigma \cos \theta}{\rho g d}\right)$$

Equations 7 and 30 can be reduced to the following dimensionless forms:

$$\frac{t}{AM} = \left[1 - \frac{B}{M} \ln\left(1 + \frac{M}{B}\right)\right] \tag{31}$$

$$\frac{t_m}{A_m M_m} = \left[1 - \frac{B_m}{M_m} \ln\left(1 + \frac{M_m}{B_m}\right)\right] \tag{32}$$

From these dimensionless equations the similitude requirements can be established as $t/AM = t_m/A_m M_m$ and $B/M = B_m/M_m$ or

$$tA_m/t_m A = B/B_m = M/M_m \tag{33}$$

Upward flow. The infiltration equation for upward flow can be synthesized by using the same method described for downward flow. Similarly it can be shown that the conditions of similitude are also those given by equation 33.

Horizontal radial flow can be simulated by using two horizontally placed parallel plates to

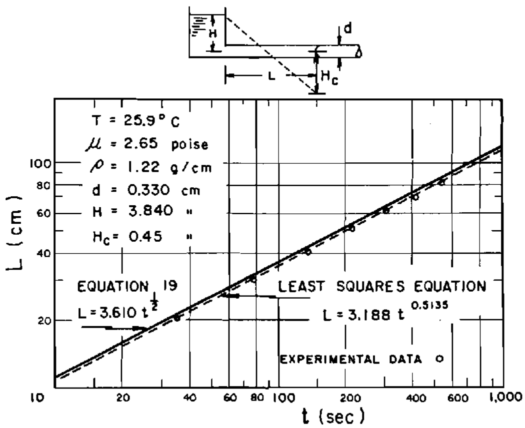


Fig. 3. Length of flow with time into a horizontal capillary tube. Solid line indicates the theoretical relationship calculated from equation 19. Broken line indicates the 'best-fit' line determined by a least squares analysis of the experimental data.

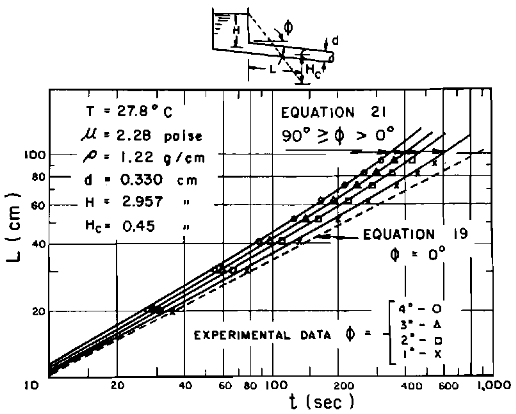


Fig. 4. Length of advance of flow with time into a downward-inclined capillary tube. Solid lines indicate the theoretical curves calculated by equation 21. Broken line indicates the theoretical curve of flow into a horizontal tube.

which fluid is supplied at a constant head at the center (Table 1).

EXPERIMENTAL RESULTS AND DISCUSSION

Several experiments were conducted in the laboratory to test the theoretical equations describing flow in horizontal, downward-inclined, and upward-inclined capillary tubes. In each trial the length of advance of the fluid in the tube was measured with time.

Prior to conducting the tests the average diameter of each capillary tube was determined by filling it with mercury and measuring the length and weight of the column in the tube. The average diameter was calculated by using the specific weight of mercury, the tube being assumed circular. All tests were conducted on dry tubes by using glycerine (about 93% concentration) as the fluid. The viscosity was measured at the beginning and end of each test. The term $4\sigma\cos\theta/\rho g d$, used in (19) and (20), was evaluated by measuring the height of glycerine rise in a vertical tube at equilibrium.

Horizontal flow. The results of a test conducted on a capillary tube placed horizontally are plotted in Figure 3. Although these results were obtained from a single test, they exemplify a pattern found in numerous other tests conducted on different tubes at different heads [*Lin*, 1967]. As shown in Figure 3, the agreement between theoretical and experimental curves is good. However, the fact that the experimental points fall below the theoretical line indicates that the measured length of advance in given time was less than that calculated. This discrepancy was attributed to possible errors in the basic assumptions underlying the theoretical equation:

1. Although the energy loss in the fluid on entry into the tube and the effect of the inertial force of the mass of fluid in the tube during deceleration were assumed to be negligible, they were of sufficient magnitude to affect the flow regime.
2. The head produced by the surface tension force may not be taken as equal to that at the equilibrium condition in a vertical tube (particularly at times shortly after entry of the fluid into the tube).

The slope of the fitted line was found to be statistically different (at the 1% level) from the theoretical value of ½. Differences were also found in the slope values obtained from repeated tests on the same tube under constant head. In part, the statistical differences may be attributed to the small variance in the points from the fitted line.

Downward flow. The results of tests conducted on capillary tubes inclined downward are plotted in Figure 4. In general the experimental and theoretical data are in good agreement. In addition note that the curves tend to shift to the

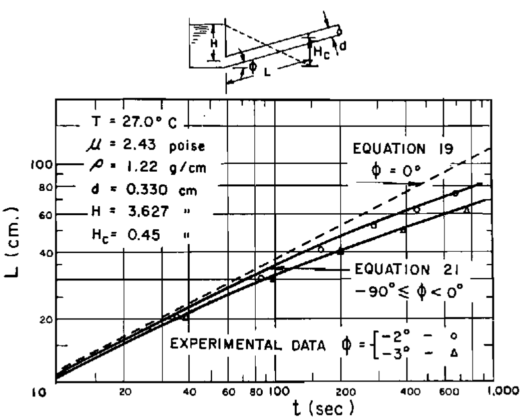


Fig. 5. Length of advance of flow with time into an upward-inclined capillary tube. Solid lines indicate the theoretical curves calculated by equation 21. Broken line indicates the theoretical curve of flow into a horizontal tube.

left and become more concave as the angle of inclination of the tube increases; that is, the length of advance of the fluid increases exponentially with the angle of inclination.

Upward flow. The results of tests conducted on capillary tubes inclined upward are plotted in Figure 5. The experimental points are in close agreement with the theoretical equation. In addition as the angle of inclination of the tube increases, the curves tend to exhibit a more convex shape. This relationship suggests that the length of advance of the front in a given time decreases exponentially with the angle of inclination.

SUMMARY AND CONCLUSIONS

The equations defining laminar flow in capillary tubes and between parallel plates were developed for several different geometric configurations: horizontal, downward-inclined, upward-inclined, and radial flow. These configurations were selected because of their similarity to systems in the movement of water through soils.

Comparing the theoretical equations with experimental data obtained from laboratory tests revealed that:

1. In general the experimental data were in close agreement with the theoretical equations.
2. Various geometric configurations may be selected by using capillary tubes and parallel plates that provide functional relationships analogous to the basic infiltration equations. Therefore these laminar flow systems may be used to synthesize infiltration on physical models.

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