Control Barrier Function based Quadratic Programs with Application to Adaptive Cruise Control

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Abstract— This paper develops a control methodology that unifies control barrier functions and control Lyapunov functions through quadratic programs. The result is demonstrated on adaptive cruise control, which presents both safety and performance considerations, as well as actuator bounds. We begin by presenting a novel notion of a barrier function associated with a set, formulated in the context of Lyapunov-like conditions; the existence of a barrier function satisfying these conditions implies forward invariance of the set. This formulation naturally yields a notion of control barrier function (CBF), yielding inequality constraints in the control input that, when satisfied, again imply forward invariance of the set. Through these constructions, CBFs can naturally be unified with control Lyapunov functions (CLFs) in the context of a quadratic program (QP); this allows for the simultaneous achievement of control objectives (represented by CLFs) subject to conditions on the admissible states of the system (represented by CBFs). These formulations are illustrated in the context of adaptive cruise control, where the control objective of achieving a desired speed is balanced by the minimum following conditions on a lead car and force-based constraints on acceleration and braking.

I. Introduction

Adaptive cruise control (ACC) [10] is being developed and deployed on passenger vehicles due to its promise to enhance driver convenience, safety, traffic flow, and fuel economy [13], [14], [23]. ACC is a multifaceted control problem because it involves asymptotic performance objectives (drive at a desired speed), subject to safety constraints (maintain a safe distance from the car in front of you), and constraints based on the physical characteristics of the car and road surface (maximum acceleration and deceleration). This control problem is made more challenging by the fact that the various objectives can often be in conflict, such as when the desired speed is faster than the speed of the leading car, and provably satisfying the safety-oriented constraints is of paramount importance. A variety of solutions have been proposed (see the survey paper [22]). The most relevant to the approach taken here is based on model predictive control [12], [16], which is natural in the ACC setting due to the existence of multiple control objectives. With a view toward providing Lyapunov-like conditions for the ACC problem with proofs of safety, this paper presents a novel approach to ACC through the use of barrier functions unified with control Lyapunov functions through quadratic programs.

Barrier functions—first utilized in optimization [6]—are now common throughout the control and verification literature due to their natural relationship with Lyapunov-like functions [21], [25], their ability to provably establish invariance of sets [5], [18], [26], and their relationship to multi-objective control [17]. This paper presents a novel form of barrier function, B, associated with a set, $\mathcal{C}$, i.e., $B(x) \to \infty$ as $x \to \partial\mathcal{C}$, and proves that if B satisfies Lyapunov-like conditions, then forward invariance of $\mathcal{C}$ is guaranteed. In contrast to existing formulations of barrier functions, which assume invariant level sets of B [21], i.e., $\dot{B} \leq 0$, we allow for B to grow when it is far way from the boundary of $\mathcal{C}$, i.e., we only require that

$$\dot{B} \leq \frac{\gamma}{B}$$

for $\gamma > 0$. This greatly increases the set of possible barrier functions and, more importantly, sets the stage for the introduction of control barrier functions (CBF). That is, given a control system, $\dot{B}$ becomes a function of the control input, and the existence of a control input that satisfies the barrier function condition implies the forward controlled-invariance of $\mathcal{C}$. Since B is allowed to grow in a preprescribed fashion, this greatly increases the available set of control inputs that satisfy the barrier function condition and, thereby, sets the stage for the unification of CBFs with control Lyapunov functions.

Control Lyapunov functions (CLFs), as pioneered in [19], [4], [20] and studied in depth in [7], utilize Lyapunov functions together with inequality constraints on their derivative to establish entire classes of controllers that stabilize a given system. Recently, these results were extended to achieve rapid exponential convergence in order to stabilize periodic orbits in hybrid systems, with experimental realization on bipedal robots [1], [2], [8]. These results motivated the observation that, since CLF conditions are affine in torque, they result in quadratic programs (QPs) [9]; this allows for the consideration of multiple control objectives (expressed via multiple CLFs) together with force- and torque-based constraints [3], [15]. This paper extends these ideas through the unification of CLFs and CBFs through QPs. In particular, given a control objective (expressed through a CLF) and an admissible set in the state space (expressed via a CBF), we formulate a QP that achieves the control objective subject to conditions that ensure the system stays in the safe set by inequalities from a CLF and a CBF. The safety critical nature of ACC will draw upon all of these elements.

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II. Control Barrier Functions

This section formulates a notion of *control Barrier functions (CBFs)*. Motivated by [21], we first present a novel reformulation of barrier functions, in a manner that is analogous (yet reciprocal) to how Lyapunov functions are used to establish exponential stability.

Consider a nonlinear system of the form:

$$\dot{x} = f(x) \tag{1}$$

for $x \in \mathbb{R}^n$ with f assumed to be locally Lipschitz. Given a set $\mathcal{C} \subset \mathbb{R}^n$, we determine conditions on functions $B : \mathcal{C} \to \mathbb{R}$ such that solutions of (1) are guaranteed to say in $\mathcal{C}$. These conditions will motivate the formulation of control barrier functions.

A. Motivation

Consider that solutions $x(t)$ of (1) that are forward complete and suppose we have a set $\mathcal{C}$ for which we wish to verify that $x(t) \in \mathcal{C}$ for all $t \geq 0$. For simplicity, further suppose that

$$
\begin{align}
\mathcal{C} &= \{x \in \mathbb{R}^n : h(x) \geq 0\}, \tag{2} \\
\partial\mathcal{C} &= \{x \in \mathbb{R}^n : h(x) = 0\}, \tag{3} \\
\text{Int}(\mathcal{C}) &= \{x \in \mathbb{R}^n : h(x) > 0\}, \tag{4}
\end{align}
$$

for a smooth (continuously differentiable) function $h : \mathbb{R}^n \to \mathbb{R}$. Motivated by methods in optimization [6], consider the barrier function candidate:

$$B(x) = -\log\left(\frac{h(x)}{1 + h(x)}\right). \tag{5}$$

Note that this function satisfies the important properties:

$$\inf_{x \in \text{Int}(\mathcal{C})} B(x) \geq 0, \qquad \lim_{x \to \partial\mathcal{C}} B(x) = \infty.$$

The question then becomes: *what conditions should be imposed on $\dot{B}$?* The conventional answer [21] has been to enforce $\dot{B} \leq 0$, but this may not be desirable since it will force invariance of sub-level sets of B; in particular, it will not allow solutions to leave a sub-level sets even if they are still contained in $\mathcal{C}$. In other words, this condition may be too strict. We therefore relax this condition to:

$$\dot{B} \leq \frac{\gamma}{B}, \tag{6}$$

where γ can be positive. This inequality allows for $\dot{B}$ to grow when solutions are far way from the boundary of $\mathcal{C}$. As solutions approach the boundary, the rate of growth slows.

For (6) to be an acceptable condition, we need to verify that its satisfaction guarantees that solutions to (1) stay in $\mathcal{C}$. To see this, we note that viewing B as a function of h yields

$$\dot{B}(h(x)) = \frac{\partial B}{\partial h}\dot{h} = -\frac{\dot{h}}{h + h^2}$$

Therefore, (6) implies that the rate of change in h is bounded by:

$$\dot{h} \geq \frac{\gamma(h + h^2)}{\log\left(\frac{h}{1+h}\right)}$$

Utilizing the Comparison Lemma [11] implies that:

$$h(x(t)) \geq \frac{1}{-1 + e^{\sqrt{2\gamma t + \log\left(\frac{h(x(0))+1}{h(x(0))}\right)^2}}}$$

Therefore, if $h(x(0)) > 0$, i.e., $x(0) \in C$, then $h(x(t)) > 0$ for all $t \geq 0$, i.e., $x(t) \in \mathcal{C}$ for all $t \geq 0$.

B. Barrier functions

Based on the motivation presented, we wish to formulate a general notion of barrier functions that give the same guarantees in a general context.

Definition 1: For the dynamical system (1), a function $B : \mathcal{C} \subset \mathbb{R}^n \to \mathbb{R}$ is a **barrier ruction (BF)** for the set $\mathcal{C}$ defined by (2)-(4) for a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ if there exist locally Lipschitz class $\mathcal{K}$ functions $\alpha_1, \alpha_2, \alpha_3$ such that, for all $x \in \text{Int}(\mathcal{C})$,

$$\frac{1}{\alpha_1(h(x))} \leq B(x) \leq \frac{1}{\alpha_2(h(x))} \tag{7}$$

$$\dot{B}(x) \leq \alpha_3(h(x)). \tag{8}$$

The motivation for the conditions on barrier functions is the following: Condition (7) implies that the barrier function B essentially behaves like the function $\frac{1}{\alpha(h(x))}$ for α a class $\mathcal{K}$ function which satisfies the essential conditions:

$$\inf_{x \in \text{Int}(\mathcal{C})} \frac{1}{\alpha(h(x))} \geq 0, \qquad \lim_{x \to \partial\mathcal{C}} \frac{1}{\alpha(h(x))} = \infty.$$

Moreover, the condition on $\dot{B}$ in (8) allows for B to grow quickly when solutions are far away from $\partial\mathcal{C}$, with this growth approaching zero as solutions approach $\partial\mathcal{C}$.

Main result. The notion of barrier functions introduced allows us to state the main result of this paper. First, we note that since (1) is assumed to be locally Lipschitz, for any initial condition $x_0 \in \mathbb{R}^n$ there exists a maximum time interval $I(x_0) = [0, \tau_{\max})$ such that $x(t)$ is the unique solution to (1) on $I(x_0)$; in the case when f is forward complete, $\tau_{\max} = \infty$. The set $\mathcal{C}$ is *forward invariant* if for every $x \in \mathcal{C}$, $x(t) \in \mathcal{C}$ for all $t \in I(x)$.

Theorem 1: *Given a set $\mathcal{C} \subset \mathbb{R}^n$ defined by (2)-(4), if there exists a barrier function $B : \mathcal{C} \to \mathbb{R}$, then $\mathcal{C}$ is forward invariant.*

Before proving Theorem 1, it is necessary to establish the following Lemma:

Lemma 1: *Consider the dynamical system:*

$$\dot{y} = \alpha\left(\frac{1}{y}\right), \qquad y(t_0) = y_0 \tag{9}$$

with α a locally Lipschitz class $\mathcal{K}$ function. For all $y_0 \in (0, \infty)$, the system has a unique solution defined for all $t \geq t_0$ and given by:

$$y(t) = \frac{1}{\sigma\left(\frac{1}{y_0}, t - t_0\right)} \tag{10}$$

where σ is a class $\mathcal{KL}$ function.

Proof: Consider the change of variables $z = \frac{1}{y}$. Under this change of variables, the dynamical system (9) implies that:

$$\dot{z} = -\frac{\dot{y}}{y^2} = -\frac{\alpha\left(\frac{1}{y}\right)}{y^2} = -\alpha(z)z^2 =: -\bar{\alpha}(z).$$

Since $\alpha(z)$ is a class $\mathcal{K}$ function, it follows that $\bar{\alpha}(z) = \alpha(z)z^2$ is a class $\mathcal{K}$ function. Therefore, by Lemma 4.4 of [11], it follows that

$$z(t) = \sigma(z_0, t - t_0)$$

with σ a class $\mathcal{KL}$ function. Converting back from z to y through $y = \frac{1}{z}$ yields the solution $y(t)$ given in (10). ∎

We now have the necessary framework in which to prove Theorem 1.

Proof: Utilizing (7) and (8), we have that

$$\dot{B} \leq \alpha_3 \circ \alpha_2^{-1}\left(\frac{1}{B}\right) =: \alpha\left(\frac{1}{B}\right).$$

Since the inverse of a class $\mathcal{K}$ function is a class $\mathcal{K}$ function, and the composition of class $\mathcal{K}$ functions is a class $\mathcal{K}$ function [11], $\alpha := \alpha_3 \circ \alpha_2^{-1}$ is a class $\mathcal{K}$ function. By Lemma 1 coupled with the Comparison Lemma [11], we have that

$$B(x(t)) \leq \frac{1}{\sigma\left(\frac{1}{B(x(t_0))}, t - t_0\right)}$$

for all $t \in I(x(t_0))$. This, coupled with the left inequality in (7), implies that

$$\alpha_1^{-1}\left(\sigma\left(\frac{1}{B(x(t_0))}, t - t_0\right)\right) \leq h(x(t)) \qquad (11)$$

for all $t \in I(x(t_0))$. By the properties of $\mathcal{K}$ and $\mathcal{KL}$ functions, if $x(t_0) \in \mathcal{C}$ and hence $B(x(t_0)) > 0$ it follows from (11) that $h(x(t)) > 0$ for all $t \in I(x(t_0))$. Therefore, $x(t) \in \mathcal{C}$ for all $t \in I(x(t_0))$ and thus $\mathcal{C}$ is forward invariant. ∎

Motivation revisited. We now have the framework to return to the motivating example considered in Section II-A. In particular, we note that the function considered in (5), subject to the conditions (6), is a Barrier Lyapunov function (by Definition 1), and therefore Theorem 1 establishes the invariance of the set $\mathcal{C}$ given in (2). This follows from the fact that

$$\alpha(r) = \frac{1}{-\log\left(\frac{r}{1+r}\right)}$$

is a class $\mathcal{K}$ function. Therefore, in Definition 1, we choose $\alpha_1(r) = \alpha_2(r) = \alpha(r)$ and $\alpha_3(r) = \gamma\alpha(r)$.

Note that the $\log$ barrier function was chosen for motivation due to its use in optimization; yet, Definition 1 suggests a simpler class of barrier functions. In particular, if

$$B(x) = \frac{1}{h(x)} \qquad (12)$$

satisfies (6) then B is a barrier function for $\mathcal{C}$ with $\alpha_1(r) = \alpha_2(r) = r$ and $\alpha_3(r) = \gamma r$.

C. Control Barrier Functions

Utilizing the formulation of barrier functions presented, it is natural to extend these concepts to the case of control systems through the use of control barrier functions (CBFs). It is important to note that control barrier functions have been considered in the context of existing notions of barrier certificates [25]. The construction presented here differ due to the novel formulation of barrier functions, i.e., the condition that the barrier function is allowed to grow in a preprescribed fashion increases the available control inputs that satisfy the control barrier function condition. Ultimately, the true usefulness of this will be seen when CBFs are unified with control Lyapunov functions through quadratic programs.

Suppose that we have an affine control system:

$$\dot{x} = f(x) + g(x)u \qquad (13)$$

with f and g locally Lipschitz, $x \in \mathbb{R}^m$ and $u \in U \subset \mathbb{R}^m$. In the case when the natural dynamics of the system, $\dot{x} = f(x)$, do not stay in a set $\mathcal{C}$, how can a controller be specified that will ensure containment in $\mathcal{C}$? This motivates the following:

Definition 2: Let $\mathcal{C} \subset \mathbb{R}^n$ be defined by (2)-(4) for a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$, then a function $B : \mathcal{C} \to \mathbb{R}$ is a **control barrier function (CBF)** if there exist class $\mathcal{K}$ functions α_1, α_2 and $0 < \gamma$ such that

$$\frac{1}{\alpha_1(\|x\|_{\partial \mathcal{C}})} \leq B(x) \leq \frac{1}{\alpha_2(\|x\|_{\partial \mathcal{C}})} \qquad (14)$$

$$\inf_{u \in U}\left[L_f B(x) + L_g B(x)u - \frac{\gamma}{B(x)}\right] \leq 0 \qquad (15)$$

for all $x \in \text{Int}(\mathcal{C})$.

Given a CBF, we can consider all control values that satisfy (15):

$$K_{\text{cbf}}(x) = \qquad (16)$$
$$\{u \in U : L_f B(x) + L_g B(x)u - \frac{\gamma}{B(x)} \leq 0\},$$

Considering control values in this set allows us to guarantee the forward invariance of $\mathcal{C}$ via the following straightfoward application of Theorem 1:

Corollary 1: Given a set $\mathcal{C} \subset \mathbb{R}^n$ defined by (2)-(4) with associated barrier function B, any Lipschitz continuous controller $u(x) \in K_{\text{cbf}}(x)$ for the system (13) renders the set $\mathcal{C}$ forward invariant.

III. CONTROL LYAPUNOV FUNCTIONS AND QPs

This section gives a brief overview of exponentially stabilizing control Lyapunov functions in the context of nonlinear systems. This formulation naturally leads to a Quadratic Program that allows for the unification of control Lyapunov and control barrier functions.

We now suppose that the dynamics of the system can be stated as a nonlinear affine control system of the form:

$$\dot{x} = f(x, z) + g(x, z)u \qquad (17)$$
$$\dot{z} = q(x, z),$$

where $x \in X$ are controlled (or output) states, $z \in Z$ are the uncontrolled states, and U is the set of admissible control values for u. In addition, we assume that $f(0, z) = 0$, i.e., that the zero dynamics surface Z defined by $x = 0$ with dynamics given by $\dot{z} = q(0, z)$ is invariant.

A. Control Lyapunov Functions

In this paper, we will focus on *exponentially stabilizing* control Lyapunov functions in order to motivate similar constructions in the case of barrier functions.

Definition 3: *A continuously differentiable function V : $X \to \mathbb{R}$ is an* **exponentially stabilizing control Lyapunov function (ES-CLF)** *[1], [2] if there exist positive constants $c_1, c_2, c_3 > 0$ such that*

$$c_1 \|x\|^2 \leq V(x) \leq c_2 \|x\|^2 \tag{18}$$

$$\inf_{u \in U} [L_f V(x, z) + L_g V(x, z)u + c_3 V(x)] \leq 0 \tag{19}$$

for all $(x, z) \in X \times Z$.

Min-Norm Controller. The existence of a ES-CLF yields a family of controllers that exponentially stabilize the system to the zero dynamics [7], [20]. In particular, consider the set of control values:

$$K_{\text{clf}}(x, z) = \tag{20}$$
$$\{u \in U : L_f V(x, z) + L_g V(x, z)u + c_3 V(x) \leq 0\}.$$

It follows that for locally Lipschitz controllers $u(x, z)$ satisfying:

$$u(x, z) \in K_{\text{clf}}(x, z) \qquad \Rightarrow \tag{21}$$
$$\|x(t)\| \leq \sqrt{\frac{c_2}{c_1}} e^{-\frac{c_3}{2}t} \|x(0)\|.$$

In addition, this yields specific feedback controllers, e.g., the min-norm controller:

$$
\begin{aligned}
m(x, z) &= \text{argmin}\{\|u\| : u \in K_{\text{clf}}(x, z)\} \tag{22} \\
&= \text{argmin}\{\|u\| : \psi_0(x) + \psi_1^T(x)u \leq 0\}
\end{aligned}
$$

where

$$
\begin{aligned}
\psi_0(x, z) &= L_f V(x, z) + c_3 V(x, z) \tag{23} \\
\psi_1(x, z) &= L_g V(x, z)^T
\end{aligned}
$$

While controller $m(x, z)$ that minimizes the control effort u can be stated in closed form as:

$$
m(x, z) = \begin{cases} -\frac{\psi_0(x,z)\psi_1(x,z)}{\psi_1(x,z)^T \psi_1(x,z)} & \text{if} \quad \psi_0(x, z) > 0 \\ 0 & \text{if} \quad \psi_0(x, z) \leq 0 \end{cases}
$$

it is important to note that this closed form solution is the solution to the quadratic program (QP):

$$m(x, z) = \underset{u \in U}{\text{argmin}} \quad u^T u \tag{24}$$
$$\text{s.t.} \quad \psi_0(x, z) + \psi_1^T(x, z)u \leq 0 \quad \text{(CLF)}$$

This formulation leads to a new class of controllers based upon CLF based QPs; these have been applied to locomotion and manipulation in bipedal robots [3], and have been utilized to experimentally achieve robotic walking [9], [8].

B. Combining CLFs and BLCFs via QPs

The advantage to the CLF and CBF formulations is that it allows for the unification of control objectives (represented by CLFs) that are regulated to give trajectories within desired sets (as dictated by CBFs).

Given a ES-CLF, V, and a CBF, B, they can be combined into a single controller through the use of a QP of the form:

$$
u^*(x, z) = \underset{\mathbf{u} = \begin{bmatrix} u \\ \delta \end{bmatrix} \in \mathbb{R}^{m+1}}{\text{argmin}} \quad \frac{1}{2}\mathbf{u}^T H(x, z)\mathbf{u} + F(x, z)^T \mathbf{u}
$$

$$\text{(CLF-CBF QP)}$$

$$\text{s.t.} \quad \psi_0(x, z) + \psi_1^T(x, z)u \leq \delta \tag{CLF}$$

$$L_f B(x, z) + L_g B(x, z)u \leq \frac{\gamma}{B(x, z)} \quad \text{(CBF)}$$

where here $H(x, z) \in \mathbb{R}^{m+1 \times m+1}$ and $F(x, z) \in \mathbb{R}^{m+1}$ are arbitrary cost functions, that can be chosen based upon desired (state based) weighting of the control inputs.

The form of the QP CLF-CBF QP together with Theorem 1, Corollary 1, and the results of [15] yields the following result:

Theorem 2: *Given a set $\mathcal{C} \subset \mathbb{R}^n$ defined by (2)-(4) with B an associated control barrier function, for any positive definite $H(x, z)$, the control law $u^*(x, z)$ obtained by solving the QP* (CLF-CBF QP) *is Lipschitz continuous and renders the set $\mathcal{C}$ forward invariant.*

Note that the QP (CLF-CBF QP) is guaranteed to have a solution based on the fact that we are relaxing the control objective, expressed via V, through δ. The consequence of this is that, while the set $\mathcal{C}$ will be rendered forward invariant, the control objective may not necessarily be archived. Practically speaking, if the control objective and the barrier function do not conflict, they will be simultaneously achievable. This will be justified further in the context of the adaptive cruise control problem.

IV. ACC PROBLEM SETUP

The remainder of this paper will be devoted to considering the adaptive cruise control (ACC) problem in the context of CBFs. We begin by setting up the dynamics of the automobile based upon [10] and [14]. We begin with the point-mass model of a vehicle moving in a straight line. We assume the dynamics of the vehicle are given by the model:

$$m\frac{dv}{dt} = F_w - F_r \tag{25}$$

where m is the mass of the vehicle (in kg), v is the velocity of the vehicle (in m/s), F_w is the wheel force in Newtons and F_r is the aerodynamic drag, also referred to as the rolling resistance, again in Newtons. In this case, we will assume the empirical form of the rolling resistance given by:

$$F_r = f_0 + f_1 v + f_2 v^2 \tag{26}$$

where f_0, f_1 and f_2 are determined empirically.

In the context of the simple formulation that will be presented here, we consider a second vehicle moving at a

constant speed, v_0. The distance between the vehicle that is being controlled at the second vehicle is given by:

$$\frac{d}{dt}D = v_0 - v \qquad (27)$$

where v is governed by (25).

The equations governing the system can be converted to an ODE. Let $x = (x_1, x_2)$ with x_1 the position of the vehicle and $x_2 = \dot{x}_1$ the velocity. Define $z = D$ to be the distance between the vehicle and the second vehicle traveling at a constant velocity. The dynamics of the system then become:

$$\dot{x} = \underbrace{\begin{bmatrix} x_2 \\ -\frac{1}{m}F_r \end{bmatrix}}_{f(x,z)} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{g(x,z)} u \qquad (28)$$

$$\dot{z} = \underbrace{v_0 - x_2}_{q(x,z)},$$

where $u = F_w$ is the control input and $F_r = f_0 + f_1 x_2 + f_2 x_2^2$.

Given the simple model introduced, we now present a series of control objectives and constraints that are necessary in the context of ACC. These will be divided into three classes of constraints: hard control objectives, soft control objectives, and force constraints.

Hard Constraints: These represent constraints that must not be violated under any condition. For ACC, this is simply the constraint: *"keep a safe distance from the car in front of you"*. More concretely, there are numerous formulations of this concept including Time Headway and Time to Collision [24]. In the context of this paper, to start with a simple formulation, we will use the general rule stated in [24]: the minimum distance between two cars is "half the speedometer." This translates into the hard constraint (with D in m and v in km/hr):

$$D \geq \frac{v}{2}. \qquad (HC1)$$

Soft Constraints: In the context of adaptive cruise control, in the case when adequate headway is assured, the goal is to achieve a desired speed, v_d. Or in other words:

$$\text{Drive} \qquad v - v_d \to 0. \qquad (SC1)$$

This translates into a soft constraint since this speed should only be achieved in the case when the hard constraints are satisfied.

Force Constraints: These constraints describe allowable wheel forces that are consistent with the driver convenience aspect of ACC; these are typically much less than the peak forces that can be generated by the car in emergency situations. Supposing that we do not want to accelerate or decelerate more than some fraction of g, we can write the constraints on acceleration and deceleration as an inequality:

$$-c_d g \leq \frac{F_w}{m} \leq c_a g \qquad (FC)$$

where c_d is the factor of g for deceleration and c_a is the factor of g for acceleration. For example, it may be desirable

to avoid accelerating and decelerating faster that $0.3g$, in which case $c_d = c_a = 0.3$.

V. FORMALLY ENCODING ACC CONSTRAINTS AS QPs

We now formulate hard constraints, soft constraints and force constraints as a Quadratic Program. This will be done through the conversion of hard and soft constraints into control Lyapunov functions (CLFs) and Control Barrier Functions (CBF). In addition, the force constraints will be utilized to construct an additional CBF that imply the satisfaction of force bounds. Finally, we will combine the constraints presented above to formulate a CLF-CBF based QP for ACC. Simulation results for the QP-based controller will be presented.

A. Soft Constraints as Control Lyapunov functions (CLFs)

We begin by formulating the soft constraint, speed regulation, as a CLF. The soft constraint (SC1) can be written as velocity based output:

$$\text{Drive} \qquad y(x, z) := x_2 - v_d \to 0. \qquad (SC1)$$

It is easy to verify that this is a relative degree 1 output since:

$$\dot{y} = \underbrace{-\frac{1}{m}F_r}_{L_f y} + \underbrace{\frac{1}{m}}_{L_g y} u$$

Picking the control input

$$u = \frac{1}{L_g y}(-L_f y + \mu) = F_r + m\mu$$

results in

$$\dot{y} = \mu.$$

Therefore, picking the Lyapunov function candidate $V(y) = y^2$, expressed in terms of the output dynamics, yields:

$$\dot{V}(y) = 2y\dot{y} = 2y\mu.$$

and therefore

$$\mu = -\frac{\varepsilon}{2}y \qquad \Rightarrow \qquad \dot{V}(y) = -\varepsilon V(y)$$

The end result is that V is a ES-CLF in the output dynamics, with $c_1 = c_2 = 1$ and $c_3 = \varepsilon$.

The trivial construction of V can also be used to motivate the conversion of the Lyapunov function back to the u dynamics of the system. In particular, we note that

$$\dot{V}(y) = \underbrace{-\frac{2y}{m}F_r}_{L_f V} + \underbrace{\frac{2y}{m}}_{L_g V} u$$

where F_r can be stated as a function of y via:

$$F_r(y) = f_0 + f_1(y + v_d) + f_2(y + v_d)^2$$

We need only verify that[1]

$$\inf_{u \in \mathbb{R}} [L_f V(y) + L_g V(y)u + \varepsilon V(y)] \leq 0$$

---

[1]Note that we will first search for any control input $u \in \mathbb{R}$; later, through the use of force constraints, u will be restricted to a subset of inputs.

but this holds since we can find a specific example of $u \in \mathbb{R}$ for which it is satisfied. Namely,

$$u = -\varepsilon \frac{m}{2} y + F_r(y).$$

These constructions indicate that we have a valid ES-CLF function V. Moreover, this function can be converted to constraints of the form (23) with

$$
\begin{align}
\psi_0(y) &= -\frac{2y}{m} F_r(y) + \varepsilon y^2 \tag{29}\\
\psi_1(y) &= \frac{2y}{m}
\end{align}
$$

and converting back to the (x, z) dynamics (28) yields:

$$
\begin{align}
\psi_0(x, z) &= -\frac{2(x_2 - v_d)}{m} F_r(x) + \varepsilon(x_2 - v_d)^2\\
\psi_1(x, z) &= \frac{2(x_2 - v_d)}{m} \tag{30}
\end{align}
$$

Summary. The end result of these constructions is the final formal form of the soft constraint (SC1), stated as a CLF constraint:

$$\psi_0(x, z) + \psi_1(x, z)u \le \delta_{sc} \qquad \text{(SC1-CLF)}$$

where δ_{sc} is a relaxation factor for the soft constraint. Note that it is this relaxation factor that makes the constraint a soft constraint, i.e, setting $\delta_{sc} = 0$ would make the constraint "hard" in that it would force exact exponential convergence at a rate of ε.

B. Hard Constraints as Control Barrier Functions (CBFs)

The goal is to construct an inequality constraint that enforces the hard constraint (HC1). For this, we utilize barrier functions as introduced in Sect. II. In particular, (HC1) results in a constraint of the form:

$$z \ge 1.8 x_2 \qquad \text{(HC1)}$$

for the (x, z) dynamics (28); here, the factor of 1.8 is a result of converting units to m and s. Correspondingly, we consider the function $h(x, z) = z - 1.8 x_2$, which yields the admissible set $\mathcal{C}$ as defined in (2)-(4). We choose the CBF candidate as B given in (5). To verify that this is a valid CBF, we must verify that

$$\inf_{u \in \mathbb{R}} \left[L_f B(x, z) + L_g B(x, z)u - \frac{\gamma}{B(x, z)} \right] \le 0 \qquad (31)$$

for $(x, z) \in \text{Int}(\mathcal{C})$ and $0 < \gamma$. To establish (31), note that

$$
\dot{B}(x, z, u) = \underbrace{-\frac{1.8 F_r(x) + m(v_0 - x_2)}{m(1 - 1.8 x_2 + z)(-1.8 x_2 + z)}}_{L_f B(x,z)}
$$
$$
+ \underbrace{\frac{1.8}{m(1 - 1.8 x_2 + z)(-1.8 x_2 + z)}}_{L_g B(x,z)} u
$$

For $(x, z) \in \text{Int}(\mathcal{C})$, it follows that $1.8 x_2 - z < 0$. Therefore, the feedback control law

$$u(x, z) = -\frac{1}{L_g B(x, z)} \left(L_f B(x, z) - \frac{\gamma}{B(x, z)} \right)$$

provides a specific example of a $u \in \mathbb{R}$ satisfying (31). As a result, B is a valid CBF.

Summary. The end result of these constructions is the final formal form of the hard constraint (HC1), stated as a CLF constraint:

$$L_f B(x, z) + L_g B(x, z)u - \frac{\gamma}{B(x, z)} \le 0 \qquad \text{(HC1-CBF)}$$

Since this is a hard constraint, no relaxation is used, i.e., this constraint must always be satisfied.

C. Force Based Constraints

The final set of constraints are the force constraints, i.e., (FC). These are easily encoded as inequality constraints via:

$$
\begin{align}
u &\le c_a m g \tag{FC1}\\
-u &\le c_d m g \tag{FC2}
\end{align}
$$

where $u = F_w$ is the wheel force, viewed as a control input.

Since it may be the case that these constraints will conflict with the torque values needed to satisfy the hard constraint (HC1-CBF), we introduce a force-based barrier function allowing the hard constraints and force constraints to be simultaneously satisfied. In particular, we seek a function h_F such that for all (x, z) satisfying $h_F(x, z) \ge 0$, there exists a trajectory of (28) satisfying (HC1-CBF) and the maximum braking limit in (FC2).

We consider the dynamics (25), and drop the aerodynamic and rolling resistance terms (26), yielding

$$m\dot{v} = u. \tag{32}$$

Under maximum braking,

$$v(t + \tau) = v(t) - \tau c_d g \tag{33}$$

and thus solving for T such that $v(t + T) = v_0$ (following at the velocity of the lead car) yields

$$T = \frac{v_0 - v(t)}{-c_d g}. \tag{34}$$

Substituting (33) and (34) into (27) gives

$$
\begin{align}
D(t + T) &= D(t) + \int_0^T [v_0 - v(t + \tau)] \, d\tau \tag{35}\\
&= D(t) + \int_0^T [v_0 - v(t) + \tau c_d g] \, d\tau \tag{36}\\
&= D(t) - \frac{1}{2} \frac{[v_0 - v(t)]^2}{c_d g} \tag{37}
\end{align}
$$

Hence, if the current headway is $D(t)$, the headway looking T seconds ahead, which is based on decelerating at the maximum rate to a following speed of v_0, is bounded by

$$D(t + T) = D(t) - \frac{1}{2} \frac{[v_0 - v(t)]^2}{c_d g}. \tag{38}$$

Therefore, within the set

$$\mathcal{C}_F = \left\{ (D, v) \mid D - \frac{1}{2} \frac{[v_0 - v]^2}{c_d g} \ge 1.8 v \right\}$$

the ACC-controlled car can always brake fast enough to maintain a half-speedometer headway using an allowed amount of deceleration.

Expressing the result in the coordinates (x, z), the end result is the function:

$$h_F(x, z) = -1.8x_2 - \frac{1}{2}\frac{(v_0 - x_2)^2}{c_d g} + z \qquad (39)$$

The superlevel set of this function defines the set $\mathcal{C}_F$, which yields a barrier function, B_F, utilizing (12).

Summary. The force based constraints are ultimately expressed via constraints (FC1) and (FC2) together with the control barrier function condition:

$$L_f B_F(x, z) + L_g B_F(x, z)u - \frac{1}{B_F(x, z)} \le 0 \qquad \text{(FC3)}$$

obtained from the force-based CBF B_F.

D. The ACC CLF-CBF based QP

Motivated by the constructions in [3], this will be achieved by combining the above constraints in a way consistent with the aforementioned ACC objectives. In particular, the end result will be a CLF-CBF QP of the form:

$$u^*(x, z) = \underset{\mathbf{u} = \begin{bmatrix} u \\ \delta_{sc} \end{bmatrix} \in \mathbb{R}^2}{\operatorname{argmin}} \quad \frac{1}{2}\mathbf{u}^T H_{\text{acc}}\mathbf{u} + F_{\text{acc}}^T \mathbf{u}$$

$$\text{(ACC QP)}$$

$$\text{s.t.} \quad A_{\text{clf}}\mathbf{u} \le b_{\text{clf}} \qquad \text{(CLF)}$$
$$A_{\text{cbf}}\mathbf{u} \le b_{\text{cbf}} \qquad \text{(CBF)}$$
$$A_{\text{fcbf}}\mathbf{u} \le b_{\text{fcbf}} \qquad \text{(FCBF)}$$
$$A_{\text{fc}}\mathbf{u} \le b_{\text{fc}} \qquad \text{(CC)}$$

The remainder of this section will be devoted to constructing the constraints and cost of this QP.

Inequality Constraints. The inequality constraints for (ACC QP) follow from the constraints constructed in the previous section. In particular, following from (SC1-CLF), we have:

$$A_{\text{clf}} = \begin{bmatrix} \psi_1(x, z) & -1 \end{bmatrix},$$
$$b_{\text{clf}} = -\psi_0(x, z). \qquad (40)$$

Following from (HC1-CBF), the hard constraints result in:

$$A_{\text{cbf}} = \begin{bmatrix} L_g B(x, z) & 0 \end{bmatrix},$$
$$b_{\text{cbf}} = -L_f B(x, z) + \frac{\gamma}{B(x, z)}. \qquad (41)$$

Finally, the comfort constraints in (FC1) and (FC2) yield the final set of inequality constraints:

$$A_{\text{cc}} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \qquad b_{\text{cc}} = \begin{bmatrix} c_a mg \\ c_d mg \end{bmatrix}, \qquad (42)$$

and (FC3) results in:

$$A_{\text{fcbf}} = \begin{bmatrix} L_g B_F(x, z) & 0 \end{bmatrix},$$
$$b_{\text{fcbf}} = -L_f B_F(x, z) + \frac{1}{B_F(x, z)}. \qquad (43)$$

Parameter	Value	Units
g	9.81	$\frac{m}{s^2}$
m	1650	kg
f_0	0.1	$Newton$
f_1	5	$\frac{(Newton)(s)}{m}$
f_2	0.25	$\frac{(Newton)(s^2)}{m}$
v_d	24	$\frac{m}{s}$
v_0	13.89	$\frac{m}{s}$
ε	10	Unitless
γ	1	Unitless
c_a	0.3	Unitless
c_d	0.3	Unitless
p_{sc}	$1e^{-5}$	Unitless

TABLE I

TABLE OF PARAMETERS USED IN THE SIMULATION RESULTS.

Cost. The cost will be presented with a view toward achieving the control objective encoded in the CLF, i.e., achieving the desired speed, subject to balancing the relaxation factors that ensure solvability and continuity of the CLF-CBF QP. In particular, we note that the CLF was constructed by first partially linearizing the system via the relative 1 degree output. This was done through the relationship:

$$u = F_r + m\mu$$

As a result, the cost relative to this control will be chosen as $\mu^T \mu$, which yields the following function in u:

$$\mu^T \mu = \frac{1}{m^2}\left(u^T u - 2u^T F_r + F_r^2\right).$$

This can then be converted into a cost of the form given in (ACC QP) via:

$$H_{\text{acc}} = 2\begin{bmatrix} \frac{1}{m^2} & 0 \\ 0 & p_{sc} \end{bmatrix}, \quad F_{\text{acc}} = -2\begin{bmatrix} \frac{F_r}{m^2} \\ 0 \end{bmatrix} \qquad (44)$$

Here p_{sc} is the penalty for the relaxation δ_{sc}.

E. Simulation Results

Simulation results obtained by applying the QP controller are shown in Fig. 1. For these simulation results, the parameters given in Table I were used, and the system (28) is started from the initial condition $(x_0, z_0) = (900, 20, 100)$. In particular, two cases are considered:

Case I: In this case, the QP (ACC QP) is solved using only constraints (CLF) and (CBF) in order to mirror the QP (CLF-CBF QP) utilized in Theorem 2. In this case, the CBF constraint (CBF) implies the forward invariance of the set defined by the hard constraint encoding the "half the speedometer" rule (right plot of Fig. 1). The control objective, encoded through the CLF constraint, is achieved when it does not conflict with the hard constraint. This is evidenced by the fact that the speed converges exponentially to the desired speed, v_d, until the distance to the lead car becomes sufficiently small (left plot of Fig. 1); at this point, due to the CBF, the speed of the following car converges to the speed of the lead car, v_0, in order to maintain a safe following distance. Note that in this case, the force
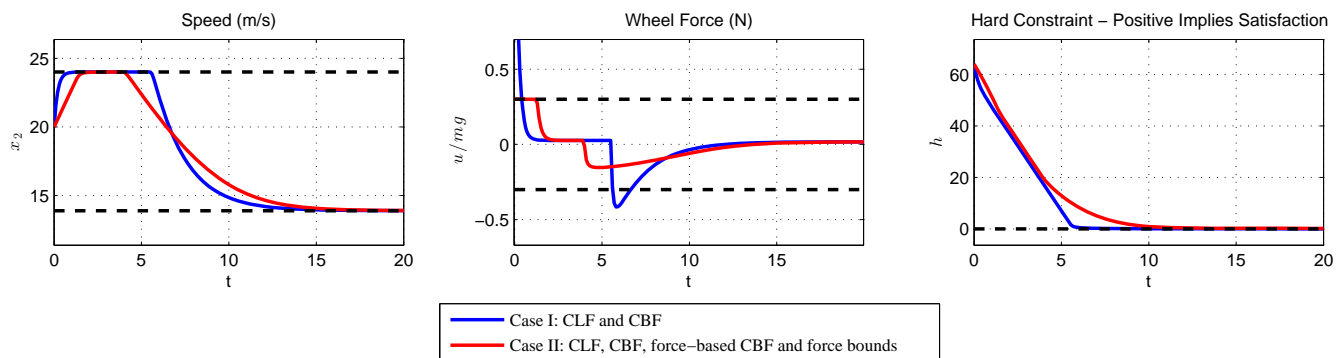
Fig. 1. Results of the QP based controller for Case I and Case II, including the speed (left) with the desired speed and speed of the lead car indicated by black lines, the wheel force (middle) with the upper and lower bounds indicated, and the hard constraint (right) where positive values indicate satisfaction.

constraints are violated (as indicated by the middle plot in Fig. 1), both when the car accelerates and breaks, since no force-based constraints are utilized.

Case II: In order to guarantee satisfaction of the force constraints, in this case all of the constraints in (ACC QP) are utilized. The end result is that, due to the force-based barrier function (FCBF) and force constraints (CC), the force constraints are satisfied for all time (as seen in the middle plot in Fig. 1). Note that, due to the required forces, the speed converges to v_d more slowly, and begins braking earlier as evidenced by the comparison between velocity profiles in Fig. 1. Finally, since the force-based barrier function is conservative, the car maintains a more conservative following distance (this case be seen in the behavior of the hard constraint in Fig. 1). Ultimately, the QP based controller (ACC QP) is able to satisfy all of the control objectives and constraints for the ACC problem outlined in Sect. IV through a single unified control methodology.

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