Another way out of the Preface Paradox?

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1 Introduction

The so called Preface Paradox runs as follows [11]. Suppose you write a book, in which you advance a great number of claims $b_1, b_2, \ldots, b_m$. Since you can adequately defend each one of them, it seems rational for you to accept their conjunction, call it b. Even so, you admit in the preface that your book will contain at least a few errors. This apparently amounts to say that at least one of the claims in your book is false, i.e., that you accept the disjunctive statement $\neg b_1 \vee \neg b_2 \vee \cdots \vee \neg b_m$. But this statement is logically equivalent to $\neg b$; thus, it seems that you are entitled to rationally accept both b and its negation. "Rationality, plus modesty, thus forces [you] to a contradiction" [16, p. 162].

In this note, I explore a possible way out of the Preface Paradox based on the notion, to be introduced below, of "approximate" belief: i.e., on the idea that, in some circumstances, you may assert b while believing, in fact, a different statement h which is "close" to b (in a suitably defined sense). This idea is inspired by a solution to the Preface Paradox recently put forward by Hannes Leitgeb ("A way out of the preface paradox?", *Analysis*, 2014) which is presented in section 2. Another relevant suggestion comes from a paper by Sven Ove Hansson [9], who highlights an interesting link between the Preface Paradox and the logic of belief change. I discuss this suggestion in section 3, where an account of approximate belief is proposed. I conclude, in section 4, by briefly discussing the main differences between the present account and Leitgeb's solution.

2 What does the author really believe?

According to a well-known definition [16, p. 1], a paradox is a "an apparently unacceptable conclusion derived by apparently acceptable reasoning from apparently acceptable premises". Thus, solving or dissolving a paradox amounts to showing that "either the conclusion is not really unacceptable, or else the starting point, or the reasoning, has some non-obvious flaw" (*ibidem*). In our case, the line of reasoning leading to the Preface Paradox is quite clear. First, some general, background assumptions are more or less explicitly stated. They are labeled A0–A2 below:

A0. (Rationality) The author of the book is (ideally) rational.

A1. (Conjunctive closure) The beliefs of a rational author are closed under conjunction; i.e., if the author accepts $b_1, b_2, \ldots, b_m$ then he accepts b.

A2. (Consistency) The beliefs of a rational author are (logically) consistent.

Secondly, the premises of the paradox are presented:

P1. The author accepts $b_1, b_2, \ldots, b_m$.

P2. The author accepts $\neg b_1 \vee \neg b_2 \vee \cdots \vee \neg b_m$.

Then the paradox is easily derived. On the one hand, given A0, it follows from A1 and P1 that the author accepts b. On the other hand, P2 implies, by logic alone, that the author accepts $\neg b$. It follows, against A2, that the author accepts both b and $\neg b$ (and hence their conjunction, again by A0 and A1).

In the attempt to find a way out of the Preface Paradox, most commentators have questioned either A1 or A2 as the most problematic assumptions [7, 5]. In his analysis, Leitgeb [10] focuses instead on P1, and challenges the assumption that the author of the book actually believes b. His idea is that, by publishing the book, the author doesn't really accept all the claims $b_1, \ldots, b_m$ in the book. Thus, he doesn't believe their conjunction b, but a strictly weaker claim: namely, that "the vast majority" of these claims are true. This provides a straightforward way out of the paradox, since this weaker claim is logically compatible with $\neg b$, i.e., with what the author states in the preface. More generally, Leitgeb argues that when someone makes a great number m of assertions, as opposed to one or few claims, what he really believe is just that most of them are true.

More formally, let k be a natural number not greater than m, but "sufficiently close" to m.[1] According to Leitgeb [10, p. 12], what the author accepts by publishing the book is not b, but its "statistical weakening" $S_k(b)$, defined as the disjunction of all the conjunctions of k different sentences among $b_1, \ldots, b_m$.

EXAMPLE 1. In the following, I'll repeatedly make use of the toy (and, as such unrealistic) example where $m = 3$ and $k = 2$ [10, p. 12]. In this case, the statistical weakening of $b = b_1 \,\&\, b_2 \,\&\, b_3$ is

$$S_2(b_1 \,\&\, b_2 \,\&\, b_3) = (b_1 \,\&\, b_2) \vee (b_1 \,\&\, b_3) \vee (b_2 \,\&\, b_3).$$

Note that the precise value of k is highly context-dependent and does not need to be explicitly stated, not even by the author of the book [10, pp. 12, 14]. In any case, as far as k is smaller than m, $S_k(b)$ is strictly weaker than b in the sense that b entails $S_k(b)$, but not vice versa. Hence, $S_k(b)$ is compatible with $\neg b$, so that the author could accept both of them and still maintain the consistency of what he believes.

Leitgeb's solution is interesting also because it naturally suggests a different, more general account of the Preface Paradox. From a purely logical point of view, it is clear that any statement h which, like $S_k(b)$, is compatible with $\neg b$ (and hence doesn't entail b) provides a way out of the paradox, if h is taken to represent the "real" content of the author's beliefs. In this connection, a recent paper by Hansson

[1] Leitgeb [10, p. 12] assumes $1 \leq k \leq m$ but, given the intended interpretation, it seems safe to say that k should be not smaller than $\frac{m}{2}$.

[9] provides a potentially fruitful suggestion. Hansson notes, in passing, that the author in the Preface Paradox apparently faces a problem of "belief contraction" as studied in the AGM theory of belief revision [9, pp. 1024–1025].[2] This means that our author initially accepts b but has reasons to believe that $\neg b$ is the case; accordingly, he should give up his belief in b or, in the AGM jargon, he should perform a contraction of b by b itself, denoted $(b - b)$. This would lead him to accept a new statement $h = (b - b)$ that is strictly weaker than b and hence compatible with $\neg b$. As in the case of Leitgeb's solution, this would provide a way out of the paradox.

Hansson's suggestion, however, adds an important idea to Leitgeb's strategy of weakening b in order to solve the paradox. In belief revision theory, in fact, a relevant *caveat* applies: belief contraction, and belief change in general, has to be "conservative" [8, sec. 3.5 and pp. 91 ff.]. This means that, after the change, the beliefs of the author should be as close as possible to his previous beliefs; in other words, belief change should be "minimal", in that it preserves as much as possible of the content of the original belief state.

This idea of minimal change leads us to the following proposal, inspired by both Leitgeb's solution and Hansson's suggestion. Let say that someone *approximately believes* b—or has an *approximate belief* in b—when, while asserting that b is the case, he actually accept some other statement h which is "close" to b in some adequately defined sense (to be clarified in the next section). If h is compatible with $\neg b$, but still close to b, this offers a solution to the Preface Paradox in line with Leitgeb's strategy. Both Leitgeb's and Hansson's proposals can then be recovered as the special cases where h is, respectively, the statistical weakening $S_k(b)$ of b or the contraction $(b - b)$. In the former case, "approximation" to b is construed as k being close to m, i.e., the "vast majority" of the claims $b_1, \ldots, b_m$ being true. In my proposal, what matters is not the number k of purportedly true claims, but the overall closeness or similarity of h to b. The following section shows how this notion of approximate belief can be made precise.[3]

3 Approximate belief

In this section, we will consider a couple of different ways of formally reconstructing the notion of approximate belief in the context of the Preface Paradox.

Preliminaries To keep things simple, let's consider a propositional language $\mathcal{L}_n$ with a finite number n of atomic sentences $a_1, \ldots, a_n$.[4] The constituents of $\mathcal{L}_n$ are

[2] The AGM account of belief revision [8] has been developed in the eighties by Carlos Alchourrón, Peter Gärdenfors, and David Makinson, and is named after them. Note that I'm not suggesting that Hansson would underwrite the proposal advanced below. Hansson is not proposing a solution to the Preface Paradox; he just highlights that what is paradoxical in this situation is exactly that the author "has reasons to contract by [b] but refrains from doing so since such a contraction would be cognitively unmanageable", and hence retains his belief in b.

[3] Philosophers of science are familiar with various notions of approximation in different contexts [12]; the need for such notions is increasingly acknowledged also in traditional and formal epistemology (see, respectively, [1, pp. 327 ff.] and [6]).

[4] All definitions in this section can be easily generalized to more complex languages, including monadic and "nomic" languages [12, 13, 4].

the $q = 2^n$ maximally informative conjunctions $c_1, \ldots, c_q$ of $\mathcal{L}_n$. Each constituent has the form $\pm a_1 \,\&\, \ldots \,\&\, \pm a_n$, where $\pm$ can be $\neg$ or nothing, and can be thought of as the most complete description of a possible world given the expressive resources of $\mathcal{L}_n$.

It is well-known that any statement x of $\mathcal{L}_n$ can be expressed, in normal form, as the disjunction of all constituents in its "range", which is the set of constituents entailing x: $x \equiv \bigvee_{c_i \vDash x} c_i$. (For this reason, we abuse notation by letting x denote both a statement and its range.) Equivalently, one may think of x as the set of possible worlds in which x is true. It is often instructive to consider what may be called the "conjunctive statements" of $\mathcal{L}_n$ [3]. These are finite, consistent conjunctions of "basic" statements $\pm a_i$, i.e., of atomic sentences or their negations. Constituents are a special case of conjunctive statements, containing exactly n conjuncts. I will often refer to conjunctive statements in the examples below.

To make sense of the notion of approximation, one needs to introduce a distance measure $\Delta(c_i, c_j)$ defined on any pair of constituents c_i and c_j of $\mathcal{L}_n$, expressing the similarity or closeness between the two corresponding possible worlds. In the following, I will assume that $\Delta(c_i, c_j)$ is the normalized Hamming (also known as Dalal) distance between c_i and c_j, i.e., the plain number of atomic sentences on which c_i and c_j disagree, divided by n.[5] There are various ways to define, on the basis of Δ, the distance between a statement x and a constituent c_i. For instance, the minimum distance $\Delta_{min}(x, c_i)$ between x and c_i is defined as $\min_{c_j \in x} \Delta(c_j, c_i)$, i.e., as the distance from c_i of the closest constituents of x.

Approximation by minimal belief contraction For our purposes, the following notion will prove useful [13, p. 171]. Given two statements x and y in normal forms, the set $D_x(y)$ of the y-worlds closest to x is defined as follows:

$$D_x(y) = \{\, c_i \in y : \Delta_{min}(c_i, x) \leq \Delta_{min}(c_j, x) \text{ for all } c_j \in y \,\}.$$

In words, $D_x(y)$ contains all constituents in (the range of) y at minimum distance from x. In the context of belief revision theory, this immediately provides a definition of the contraction $(x - y)$ of x by y, as follows (*ibidem*):

$$(x - y) = \bigvee D_x(\neg y) \vee x$$

The contraction of x by y thus enlarges the set of possibilities admitted by x with the set of the $\neg y$-worlds closest to x (see below for examples).

Now, following Hansson, suppose that the author in the Preface Paradox initially believes b but decides to give up his belief in b. In this case, the contraction $(b - b)$ will contain all the possibilities within b, along with all the closest possibilities "around" b (see figure 1). To see the above definition at work, it is instructive to consider the special case where the claims $b_1, \ldots, b_m$ in the book are basic statements in the sense defined before, and hence b is a conjunctive statement. In such case, one can check that, by giving up b, the author comes to believe that at least $m - 1$ of his m

[5] This assumption will significantly simplify the definitions and the examples below, but it is not essential. See [12, 13] for a more general treatment.

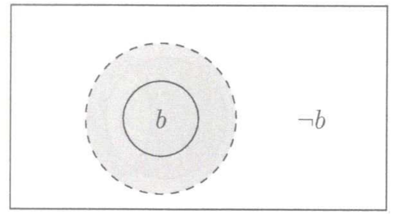

Figure 1. Each point of the rectangular surface represents a constituent or possible world. The solid circle represents the range of statement b. The dashed circle includes the worlds at minimum distance from b; the shadowed area is the contraction $(b - b)$.

beliefs are true:[6]

$$(b - b) = S_{m-1}(b) \tag{1}$$

Thus, when b is a conjunctive statement, belief contraction leads to a special case of Leitgeb's solution, where the author accepts the strongest possible statistical weakening of b (with $k = m - 1$).

EXAMPLE 2. Suppose that b is the conjunctive statement $b_1 \,\&\, b_2 \,\&\, b_3$. Then:

$$(b - b) = (b_1 \,\&\, b_2) \vee (b_1 \,\&\, b_3) \vee (b_2 \,\&\, b_3) = S_2(b)$$

(compare example 1 in section 2).

All other cases admitted by Leitgeb's solution, and corresponding to values of k smaller than $m - 1$, are excluded here since they would result in non-conservative contractions of b, i.e., statements too distant from b. (With reference to Figure 1, such statements would be represented, for decreasing k, by increasingly larger circles around b.)

Approximation by distance minimization Up to this point, I followed Hansson's suggestion of reconstructing the Preface Paradox as a problem of belief change. According to this idea, and in agreement with a special case of Leitgeb's solution, the author escapes the paradox by accepting a statement h which is close to b in the sense that h coincides with a conservative contraction of b. The idea of approximate belief introduced in section 2 is however more general than this, since h can be close to b without being a contraction of b. To make sense of this notion in full generality, one needs to define a measure for the distance between two arbitrary statements x

[6] Proof. Suppose that b is a conjunctive statement. An arbitrary constituent c belongs to $D_b(\neg b)$ iff c disagrees with b (otherwise it would be in the range of b) exactly on one of the conjuncts of b (otherwise $\Delta_{min}(c, b)$ wouldn't be minimal). It follows that $b \cup D_b(\neg b)$ contains all constituents which disagree with b at most on one claim of b. This is the range of the contraction $(b - b)$, which then says that at least $m - 1$ claims of b are true.

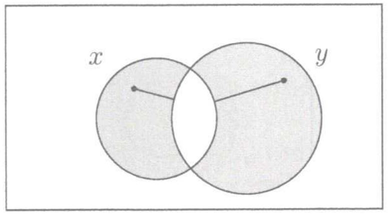

Figure 2. The symmetric difference (shadowed) between (the ranges of) x and y. The minimum distance of an arbitrary constituent of each theory from the other theory is displayed.

and y of $\mathcal{L}_n$.[7] Niiniluoto [12, p. 248] proposes the following normalized measure:

$$\delta(x,y) = \frac{\alpha}{q} \sum_{c_i \in y \setminus x} \Delta_{min}(c_i, x) + \frac{\alpha'}{q} \sum_{c_j \in x \setminus y} \Delta_{min}(c_j, y) \qquad (2)$$

where $0 < \alpha, \alpha' \leq 1$. The distance between x and y is thus based on the symmetric difference $(x \setminus y) \cup (y \setminus x)$ between (the ranges of) x and y (see section 2). If, e.g., y is construed as the "target" which x has to approximate, then the worlds in the symmetric difference between x and y reflect two kind of "errors" of x. The members of $y \setminus x$ can be construed as the mistaken exclusions of x, i.e., possibilities admitted by y and wrongly excluded by x; while the elements of $x \setminus y$ are the mistaken inclusions of x, i.e., possibilities excluded by y and wrongly admitted by x (see also [4]). The minimum distances of all errors are then summed up, with weights α and α' reflecting the relative seriousness of the two kinds of errors. Note that $\delta(x,y)$ takes is minimal value just in case x and y are the same statement.

The above distance measure can be employed to define a notion of approximate belief as applied to the case of the Preface Paradox, as follows. Let us say that the author approximately believes b when he accepts a statement h such that:

$$h \nvDash b \text{ and } \delta(h,b) \text{ is minimal}$$

This guarantees that the author's beliefs are both close to b and compatible with $\neg b$. Thus, any statement h meeting the condition above provides a possible way out of the paradox.

Note that, in order to be a good approximation of b, h has to include possibilities which are close to b and exclude possibilities which are far from b. This is guaranteed when h is chosen as a subset of $b \cup D_b(\neg b)$, since $D_b(\neg b)$ contains the closest possibilities to b among those excluded by b itself (see again figure 1). Thus, belief contraction turns out to be the special case where h is chosen as $b \cup D_b(\neg b)$ itself. In

[7] Different measures of this kind have been studied in the philosophy of science literature concerning verisimilitude or truthlikeness [12, 14]. In fact, note that when x is an arbitrary statement and y is the true constituent of $\mathcal{L}_n$ (describing the actual world, i.e., "the whole truth" about the domain), the verisimilitude of x can be defined as a decreasing function of the distance between x and y.

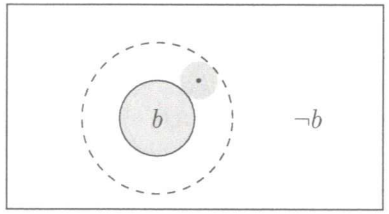

Figure 3. A statement h (shadowed) at minimum distance from b.

general, however, $(b - b)$ will be too weak a statement to be a good approximation of b; accordingly, h will typically be stronger than $(b - b)$. More precisely, one can check that $\delta(h,b)$ is minimized when:[8]

$$h = b \cup \{c_i\} \text{ where } c_i \in D_b(\neg b),$$

i.e., when h includes all worlds in b and exactly one of the worlds at minimum distance from b (see figure 3).

EXAMPLE 3. Let b be the conjunctive statement $b_1 \& b_2 \& b_3$ in $\mathcal{L}_4$. Then the following statements are at minimum distance from b:

$$\begin{gathered} b_1 \& b_2 \& (b_3 \vee b_4), \\ b_1 \& b_2 \& (b_3 \vee \neg b_4), \\ b_1 \& b_3 \& (b_2 \vee b_4), \\ b_1 \& b_3 \& (b_2 \vee \neg b_4), \\ b_2 \& b_3 \& (b_1 \vee b_4), \\ b_2 \& b_3 \& (b_1 \vee \neg b_4). \end{gathered}$$

In any case, the author will keep believing two of his original claims and will suspend the judgment on the remaining one. By taking the disjunction of all the statements above, one finds again the contraction $(b - b)$ of example 2.

The example above shows that there are in general many different statements h at minimum distance from b. In specific cases, one may think that pragmatic factors will guide the choice in favor of one or the other of these different approximations of b. In this connection, contracting b by b can be construed as the safe strategy of choosing *all* the best candidate approximations to b. This avoids the problem posed by their multiplicity and guarantees an unique result, $(b - b)$, which, however, is not maximally close to b. While sub-optimal in this sense, such a strategy may be rational, if one recalls that $(b - b)$ is after all a good approximation to b if compared to other solutions, like $S_k(b)$ for low values of k (cf. figure 1).

[8] Proof. Distance $\delta(h,b)$ is minimal when both addenda in equation 2 are minimal. For fixed values of α and α' (and a given choice of $\mathcal{L}_n$), this is guaranteed when both $\sum_{c_i \in b \setminus h} \Delta_{min}(c_i, h)$ and $\sum_{c_j \in h \setminus b} \Delta_{min}(c_j, b)$ are minimized. The former sum is minimized, and equals 0, if h is chosen such that $b \vDash h$, since in that case $b \setminus h = \emptyset$. On the other hand, if, as required, $h \nvDash b$ then the latter sum cannot be zero, since $h \setminus b$ has to include at least a constituent "outside" b. Thus, $\delta(h,b)$ is minimized if h is chosen to include just one of the closest constituents to b, besides those of b itself.

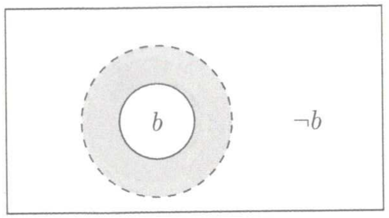

Figure 4. The revision $(b * \neg b)$ of b by $\neg b$ (shadowed).

Approximation by minimal belief revision Before concluding, it may be instructive to consider still another strategy of determining an unique approximation h of b. The two solutions considered above share a common feature: both the statistical weakening $S_k(b)$ and the contraction $(b - b)$ are entailed by b. Indeed, it is easy to check that, in order to minimize the distance from b, h has to be a consequence of b (since in this case the second addendum in equation 2 is 0). However, also statements not entailed by b can be quite (although not maximally) close to b. In particular, it may be the case that h is logically incompatible with b while being close to b. In this connection, an interesting special case is when h is the revision of b by $\neg b$, i.e., the result of accepting $\neg b$ when one believes b. This is also a way of reconstructing the situation of an author who, having published b in the book, asserts in the preface that $\neg b$ is actually the case (in line with premise P2 of the Preface Paradox).

The revision of x by y is defined in general as follows [13, p. 171]:

$$(x * y) = \bigvee D_x(\neg y),$$

i.e., as the set of possibilities admitted by $\neg y$ which are closest to x. In the present case, the revision of b by $\neg b$ reduces to the worlds "around" b, i.e., at minimum distance from b (see figure 4).

EXAMPLE 4. If b is the conjunctive statement $b_1 \,\&\, b_2 \,\&\, b_3$ then:

$$(b * \neg b) = (b_1 \,\&\, b_2 \,\&\, \neg b_3) \vee (b_1 \,\&\, \neg b_2 \,\&\, b_3) \vee (\neg b_1 \,\&\, b_2 \,\&\, b_3)$$

In short, the author believes that exactly one of the claims in the book is false, the others being true.

As in the case of contraction, revision turns out to be a special case of distance minimization, where h is chosen as $D_b(\neg b)$. In general, however, h will differ from both the contraction and the revision of b (as the foregoing examples show). Still, contraction and revision provide two instructive illustrations of approximate belief, especially as far as the Preface Paradox is concerned. These correspond to two alternative ways of of approximating b through minimal belief changes, which lead in turn to two alternative ways out of the paradox. The first, contraction-based solution amounts to choose h such that h entails neither b nor $\neg b$; this amounts to questioning both premises P1 and P2 of the paradox, since in this case the author believes neither the conjunction of the claims in the book nor its negation.The second, revision-based solution is to say that h entails $\neg b$, so that the author indeed accepts $\neg b$ and rejects b; in turn, this amounts to rejecting P1 while fully endorsing P2. In this connection, both Leitgeb's proposal from section 2 and the one based on distance minimization favor the former, contraction-based solution over the latter.

4 Concluding remarks

In this paper, I followed Leitgeb's idea that when someone makes a great number of different claims $b_1, \ldots, b_m$, he doesn't actually accept their conjunction b but some weaker statement h. I also argued that h should be construed as a good approximation of b, or that h should be close to b. These notions of approximation and closeness can be made precise once a distance measure among the possibilities in the logical space is defined. In the case of the Preface Paradox, the author of the book approximately believes b in the sense that he accepts a statement h which doesn't entail b but still is close as possible to b.

As shown in section 3, h doesn't coincide, in general, with Leitgeb's statistical weakening $S_k(b)$ of b, or with the minimal changes of b obtained through contraction and revision. In fact, h will be closer to b than each of the three statements $S_k(b)$, $(b - b)$, and $(b * \neg b)$, which are all too weak to be good approximations of b. Still, these weaker statements, and especially the latter two, may be plausible approximations of b in some contexts, since they sometimes uniquely determine the actual beliefs of the author (as in the simple examples from section 3). On the contrary, as already observed, there are in general many statements h which are maximally close to b; in this sense, the notion of approximate belief as defined here may be "cognitively unmanageable" [9, p. 1024] and hence less psychologically plausible than those alternatives. In other words, the author may be unable to specify the statement h which he really believes; and this may be the reason why, in the book, he actually asserts b [10, p. 14].

In any case, the main conceptual difference between the account proposed here and Leitgeb's solution has to do with the notion of belief itself. As Leitgeb [10, p. 14] notes, his solution of the paradox has the advantage of allowing the author to accept $S_k(b)$ with high confidence, in the sense that the probability of $S_k(b)$ can be high even if the probability of b is very low. This depends on the fact that $S_k(b)$ is a much weaker statement than b, and probability is inversely related to logical strength (in the sense that if x entails y then x cannot be more probable than y). On the contrary, approximation as defined here is positively correlated with logical strength at least in the following sense. In order to be a good approximation of b, h has to hold in roughly the same set of possible worlds (cf. figure 2); this means that h will entail most of the consequences of b (recall the principle of conservatism of the AGM theory). Accordingly, as compared to $S_k(b)$, h is much stronger, and hence a less probable statement.

As a consequence, if belief requires high probability, h (as well as b itself) cannot be really believed by the author of the book. On the other hand, it is well-known that the "high probability" view of belief is problematic, and the Preface Paradox is often used exactly to show that it is untenable [7]. For this reason, it is useful to consider other notions of belief or acceptance, compatible with the possibility of believing also propositions which are not highly probable. One such notion is adopted within the fallibilist tradition in epistemology and philosophy of science, according to which even our best beliefs (e.g., scientific hypotheses) are typically false or highly improbable [15, 12]. The idea of approximate belief defended here apparently provides an account of the Preface Paradox in line with this tradition (for a related but different treatment see [2]). While publishing $b_1, \ldots, b_m$ in his book, the author is conscious of his own fallibility, and, accordingly, doesn't believe that b is actually the case. Still, he remains committed to the claim that b is, so to speak, roughly the case. This situation may be understood by saying that what the author really believes is a statement h, which, while not highly probable, is a good approximation of what the author in fact asserts (i.e., b). In this way, this notion of approximate belief provides another possible solution to the Preface Paradox, alternative to Leitgeb's one.

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Characterizing Logical Consequence in Paraconsistent Weak Kleene

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ABSTRACT. In this paper we present Parconsistent Weak Kleene (PWK), a logic that first appeared in the works of Sören Halldén and Arthur Prior, and we establish a characterization result for PWK-consequence, thus providing necessary and sufficient conditions for B to be a consequence of Γ in PWK.

1 Introduction

In [7] and [15], Sören Halldén and Arthur Prior independently discuss a logic based on the following three tenets: (a) there are cases where classical truth value assignment is not possible, (b) in such cases, the presence of a third, non-classical, truth value propagates from one sentence to any compound sentence including it, and finally, (c) valid inferences go from non-false premises to non-false conclusions. The so-called Weak Kleene Logic (or Bochvar Logic) is built in accordance with tenets (a)–(b), but it assumes that classical truth is the only value to be preserved by valid inference.[1] If we endorse (c) and include the non-classical value among the designated values, we get a *paraconsistent counterpart* of Bochvar Logic, that we call *Paraconsistent Weak Kleene* or PWK for short.[2]

In this paper, we give a characterization result of the relation of logical consequence in propositional PWK, that is, we provide necessary and sufficient conditions for a formula B to be the logical consequence of a set Γ of formulas.

There are two main rationales for this result. First, our result has a general mathematical interest in the areas of three-valued logics. Indeed, few results have been provided on PWK, but an exploration of the formalism reveals interesting connections with Relevant Logic. Second, our result generalizes a result by Paoli [12], that considers syntactical restrictions that obtain by imposing the First-Degree-Entailment (FDE for short) requirements to PWK. It is thus of interest in relation

[1] For Bochvar Logic, see [4], [10] and [16].

[2] In this paper, we are using the label 'paraconsistent Kleene logic' as short for 'paraconsistent counterpart of a Kleene Logic'. This use is suggested by the fact that paraconsistent logics as Priest's Logic of Paradox LP and the present PWK obtain by keeping the 'strong matrices' introduced by [8] and the 'weak matrices' by [4] and [8], respectively, and extending the set of designated elements as to include the non-classical value. Our choice does not presuppose anything more, since paraconsistency does not belong to the range of applications for which Kleene Logics have been designed (which included phenomena of *underdetermination*, by contrast). We use the label PWK accordingly.