Semidefinite Relaxations of Equivalent Optimal Power Flow Problems: An Illustrative Example

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***Abstract*—Recently, there has been significant interest in convex relaxations of the optimal power flow (OPF) problem. A semidefinite relaxation globally solves many OPF problems. However, there exist practical problems for which the semidefinite relaxation fails to yield the global solution. Conditions for the success or failure of the semidefinite relaxation are valuable for determining whether the relaxation is appropriate for a given OPF problem. To move beyond existing conditions, which only apply to a limited class of problems, a typical conjecture is that failure of the semidefinite relaxation can be related to physical characteristics of the system. By presenting an example OPF problem with two equivalent formulations, this paper demonstrates that physically based conditions cannot universally explain algorithm behavior. The semidefinite relaxation fails for one formulation but succeeds in finding the global solution to the other formulation. Since these formulations represent the same system, success (or otherwise) of the semidefinite relaxation must involve factors beyond just the network physics.**

I. Introduction

The optimal power flow (OPF) problem determines a minimum cost operating point for an electric power system subject to both network equality constraints (the power flow equations, which model the relationship between the voltage phasors and the power injections) and engineering limits. Typical objectives are minimization of losses or generation costs. The OPF problem is generally non-convex due to the non-linear power flow equations [1] and may have local solutions [2]. Non-convexity of the OPF problem has made solution techniques an ongoing research topic. Many OPF solution techniques have been proposed, including successive quadratic programs, Lagrangian relaxation, and interior point methods [3].

Recently, there has been significant interest in a semidefinite relaxation of the OPF problem [4]. Using a rank relaxation, the OPF problem is reformulated as a convex semidefinite program. If the relaxed problem satisfies a rank condition (i.e., the relaxation is said to be "exact"), the global solution to the original OPF problem can be determined in polynomial time. Prior OPF solution methods do not guarantee finding a global solution in polynomial time; semidefinite programming approaches thus have a substantial advantage over traditional solution techniques. However, the rank condition is not satisfied for all practical OPF problems [2], [5].

There is substantial interest in developing sufficient conditions for exactness of the semidefinite relaxation. Existing sufficient conditions include requirements on power injection, voltage magnitude, and line-flow limits and either radial networks (typical of distribution system models), appropriate placement of controllable phase shifting transformers, or a limited subset of mesh network topologies [6], [7].

The semidefinite relaxation globally solves many OPF problems which do not satisfy any known sufficient conditions [6], [7]. In other words, the set of problems guaranteed to be exact by known sufficient conditions is much smaller than the set of problems for which the relaxation is exact. This suggests the potential for developing broader conditions. A natural speculation is that some physical characteristics of an OPF problem may inform such conditions. With solutions that are close to voltage collapse, several problems for which the semidefinite relaxation fails to be exact support this speculation [5].

This paper tends to dampen enthusiasm for this avenue of research by considering a small problem with two equivalent formulations. The semidefinite relaxation succeeds in globally solving one formulation but fails to solve the other. Since both formulations represent the same system, strictly physically based conditions for the success of the relaxation cannot differentiate between these formulations.[1] It will be shown that the feasible spaces illustrate why the semidefinite relaxation succeeds for one formulation but fails for the other.

The small example presented in this paper is relatively simple. In fact, this example "OPF" problem reduces to finding the minimum loss solution to power flow constraint equations for a specified set of power injections and voltage magnitudes. Thus, this example further demonstrates that the semidefinite relaxation may fail even for simple OPF problems.

This paper is organized as follows. Section II introduces the OPF problem. Section III gives the semidefinite relaxation. Section IV presents the example OPF problem that is the main contribution of this paper. Section V concludes the paper.

II. OPF Problem Formulation

We first present a formulation of the OPF problem in terms of rectangular voltage coordinates and active and reactive power generation. Consider an n-bus power system, where

[1] See also [7] for an example where different line-flow limit formulations determine success or failure of the semidefinite relaxation.

$\mathcal{N} = \{1, 2, \ldots, n\}$ is the set of all buses and $\mathcal{G}$ is the set of generator buses. Let $P_{Dk} + jQ_{Dk}$ represent the active and reactive load demand at each bus $k \in \mathcal{N}$. Let $V_k = V_{dk} + jV_{qk}$ represent the voltage phasors in rectangular coordinates at each bus $k \in \mathcal{N}$. Superscripts "max" and "min" denote specified upper and lower limits. Buses without generators have maximum and minimum generation set to zero (i.e., $P_{Gk}^{\max} = P_{Gk}^{\min} = Q_{Gk}^{\max} = Q_{Gk}^{\min} = 0, \quad \forall k \in \mathcal{N} \setminus \mathcal{G}$). Let $\mathbf{Y} = \mathbf{G} + j\mathbf{B}$ denote the network admittance matrix.

The power flow equations describe the network physics:

$$P_{Gk} = V_{dk} \sum_{i=1}^{n} \left(\mathbf{G}_{ik} V_{di} - \mathbf{B}_{ik} V_{qi}\right) + V_{qk} \sum_{i=1}^{n} \left(\mathbf{B}_{ik} V_{di} + \mathbf{G}_{ik} V_{qi}\right) + P_{Dk} \tag{1a}$$

$$Q_{Gk} = V_{qk} \sum_{i=1}^{n} \left(\mathbf{G}_{ik} V_{di} - \mathbf{B}_{ik} V_{qi}\right) - V_{dk} \sum_{i=1}^{n} \left(\mathbf{B}_{ik} V_{di} + \mathbf{G}_{ik} V_{qi}\right) + Q_{Dk} \tag{1b}$$

The OPF problem considered in this paper is

$$\min_{V_d, V_q} \sum_{k \in \mathcal{G}} P_{Gk} \qquad \text{subject to} \tag{2a}$$

$$P_{Gk}^{\min} \leq P_{Gk} \leq P_{Gk}^{\max} \qquad \forall k \in \mathcal{N} \tag{2b}$$

$$Q_{Gk}^{\min} \leq Q_{Gk} \leq Q_{Gk}^{\max} \qquad \forall k \in \mathcal{N} \tag{2c}$$

$$\left(V_k^{\min}\right)^2 \leq V_{dk}^2 + V_{qk}^2 \leq \left(V_k^{\max}\right)^2 \qquad \forall k \in \mathcal{N} \tag{2d}$$

$$V_{q1} = 0 \tag{2e}$$

Note that constraint (2e) sets the reference bus angle to zero.

This formulation minimizes the active power losses, but other convex objective functions, such as piecewise-linear and quadratic functions of active power generation, may be modeled. See [8] for the semidefinite relaxation of a more general OPF formulation.

III. Semidefinite Relaxation of the OPF Problem

This section describes a semidefinite relaxation of the OPF problem, which was first presented in [4]. Let e_k denote the k^{th} standard basis vector in $\mathbb{R}^n$. Define $Y_k = e_k e_k^\intercal \mathbf{Y}$.

Matrices employed in the bus power injection, voltage magnitude, and angle reference constraints are

$$\mathbf{Y}_k = \frac{1}{2} \begin{bmatrix} \mathrm{Re}\left(Y_k + Y_k^\intercal\right) & \mathrm{Im}\left(Y_k^\intercal - Y_k\right) \\ \mathrm{Im}\left(Y_k - Y_k^\intercal\right) & \mathrm{Re}\left(Y_k + Y_k^\intercal\right) \end{bmatrix} \tag{3a}$$

$$\bar{\mathbf{Y}}_k = -\frac{1}{2} \begin{bmatrix} \mathrm{Im}\left(Y_k + Y_k^\intercal\right) & \mathrm{Re}\left(Y_k - Y_k^\intercal\right) \\ \mathrm{Re}\left(Y_k^\intercal - Y_k\right) & \mathrm{Im}\left(Y_k + Y_k^\intercal\right) \end{bmatrix} \tag{3b}$$

$$\mathbf{M}_k = \begin{bmatrix} e_k e_k^\intercal & \mathbf{0} \\ \mathbf{0} & e_k e_k^\intercal \end{bmatrix} \tag{3c}$$

$$\mathbf{N}_k = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & e_k e_k^\intercal \end{bmatrix} \tag{3d}$$

Define the vector of voltage components

$$x = \begin{bmatrix} V_{d1} & V_{d2} & \ldots & V_{dn} & V_{q1} & V_{q2} & \ldots & V_{qn} \end{bmatrix}^\intercal \tag{4}$$

Then define the rank-one matrix

$$\mathbf{W} = x x^\intercal \tag{5}$$

The active and reactive power injections at bus k are $\mathrm{tr}\left(\mathbf{Y}_k \mathbf{W}\right)$ and $\mathrm{tr}\left(\bar{\mathbf{Y}}_k \mathbf{W}\right)$, respectively, where tr indicates the matrix trace operator. The square of the voltage magnitude at bus k is $\mathrm{tr}\left(\mathbf{M}_k \mathbf{W}\right)$. The constraint $\mathrm{tr}\left(\mathbf{N}_1 \mathbf{W}\right) = 0$ sets the reference angle.

Replacing the rank-one requirement from (5) by the less stringent constraint $\mathbf{W} \succeq 0$, where $\succeq 0$ indicates positive semidefiniteness, yields the semidefinite relaxation of (2):

$$\min_{\mathbf{W}} \sum_{k \in \mathcal{G}} \mathrm{tr}\left(\mathbf{Y}_k \mathbf{W}\right) + P_{Dk} \qquad \text{subject to} \tag{6a}$$

$$P_{Gk}^{\min} \leq \mathrm{tr}\left(\mathbf{Y}_k \mathbf{W}\right) + P_{Dk} \leq P_{Gk}^{\max} \qquad \forall k \in \mathcal{N} \tag{6b}$$

$$Q_{Gk}^{\min} \leq \mathrm{tr}\left(\bar{\mathbf{Y}}_k \mathbf{W}\right) + Q_{Dk} \leq Q_{Gk}^{\max} \qquad \forall k \in \mathcal{N} \tag{6c}$$

$$\left(V_k^{\min}\right)^2 \leq \mathrm{tr}\left(\mathbf{M}_k \mathbf{W}\right) \leq \left(V_k^{\max}\right)^2 \qquad \forall k \in \mathcal{N} \tag{6d}$$

$$\mathrm{tr}\left(\mathbf{N}_1 \mathbf{W}\right) = 0 \tag{6e}$$

$$\mathbf{W} \succeq 0 \tag{6f}$$

If the condition $\mathrm{rank}\left(\mathbf{W}\right) = 1$ is satisfied, the relaxation is "exact" and the global solution to (2) is recovered using an eigen decomposition. Let λ be the non-zero eigenvalue of a rank-one solution $\mathbf{W}$ to (6) with associated unit-length eigenvector η. The globally optimal voltage phasor is

$$V^* = \sqrt{\lambda} \left(\eta_{1:n} + j\eta_{(n+1):2n}\right) \tag{7}$$

where subscripts denote vector entries in MATLAB notation.

IV. Equivalent Formulations of a Small Example Problem

Since the semidefinite relaxation globally solves many OPF problems which do not satisfy any known sufficient conditions [6], [7], there is potential for development of broader sufficient conditions. One speculation is that some physical characteristic of the OPF problem predicts the relaxation's success or failure.

The following example shows that strictly physically based sufficient conditions are unable to definitively predict success or failure of the semidefinite relaxation for all OPF problems. The example problem has equivalent two- and three-bus formulations. The relaxation globally solves the two-bus system but only gives a strict lower bound on the objective value (i.e., $P_{G1} + P_{G2}$) rather than the solution to the three-bus system.

A. Example Problem

Consider the two- and three-bus systems in Figs. 1 and 2. The upper and lower limits on active and reactive power injections and voltage magnitudes are equal (e.g., $V_1^{\max} = V_1^{\min} = 1$ per unit), which results in equality constraints on the corresponding quantities. For both systems, the voltage magnitudes at buses 1 and 2 are fixed to 1.0 and 1.3 per unit, respectively, the active power injection at bus 2 is fixed to zero, and there are no limits on the reactive power injections at buses 1 and 2. For bus 3 in the three-bus system, the active and reactive power injections are constrained to zero and there is no voltage magnitude constraint.

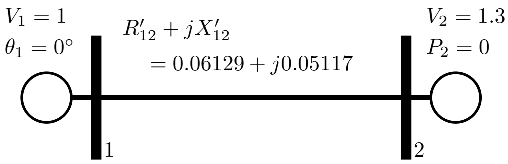


Fig. 1. Two-Bus System

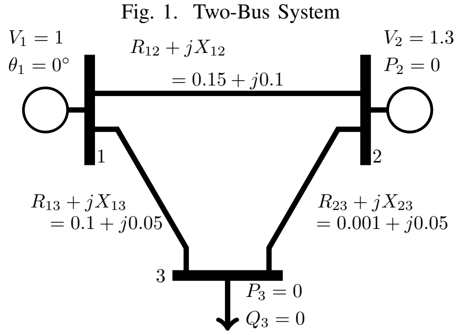


Fig. 2. Three-Bus System

With two quantities specified at each bus k along with two degrees of freedom (V_{dk} and V_{qk}), the feasible space for the OPF problem (2) for this example consists of a set of isolated points that are the solutions of the power flow equations. The OPF finds the solution point that has the lowest active power losses. Here, this solution corresponds to the “high voltage” power flow solution, which is commonly calculated using a Newton-Raphson iteration initialized from a flat start (i.e., unity voltage magnitudes and zero voltage angles). In this paper, however, we use this problem to explore the properties of the semidefinite relaxation.

Since bus 3 in the three-bus system has zero power injections, it can be eliminated by adding $R_{13} + jX_{13}$ and $R_{23} + jX_{23}$ to yield an equivalent two-bus system with two parallel lines. The parallel combination of these lines gives the line impedance $R'_{12} + jX'_{12}$ shown in the two-bus system of Fig. 1. Thus, the OPF problems for the two- and three-bus systems are *equivalent*. The voltage at bus 3 in the three-bus system can be directly computed from the solution to the two-bus system. The global solutions are given in Table I.

The semidefinite relaxation globally solves the two-bus system. However, for the three-bus system, the relaxation only provides a lower bound that is 22% less than the true global optimum (i.e., there exists a large relaxation gap).

B. Feasible Space Exploration

Although the OPF problems for the two- and three-bus systems share the same feasible spaces, this is not the case for their semidefinite relaxations. This section explores the feasible spaces of these relaxations to illustrate why the relaxation globally solves the two-bus system but fails for the equivalent three-bus system.

TABLE I
SOLUTIONS TO TWO- AND THREE-BUS SYSTEMS (PER UNIT)

	Two-Bus System	Three-Bus System
$V_{d1} + jV_{q1}$	$1.000 + j0.000$	$1.000 + j0.000$
$V_{d2} + jV_{q2}$	$1.049 - j0.767$	$1.049 - j0.767$
$V_{d3} + jV_{q3}$	N/A	$0.849 - j0.586$
$P_1 + jQ_1$	$5.68 - j7.77$	$5.68 - j7.77$
$P_2 + jQ_2$	$0.0 + j12.52$	$0.0 + j12.52$
$P_3 + jQ_3$	N/A	$0.0 + j0.0$

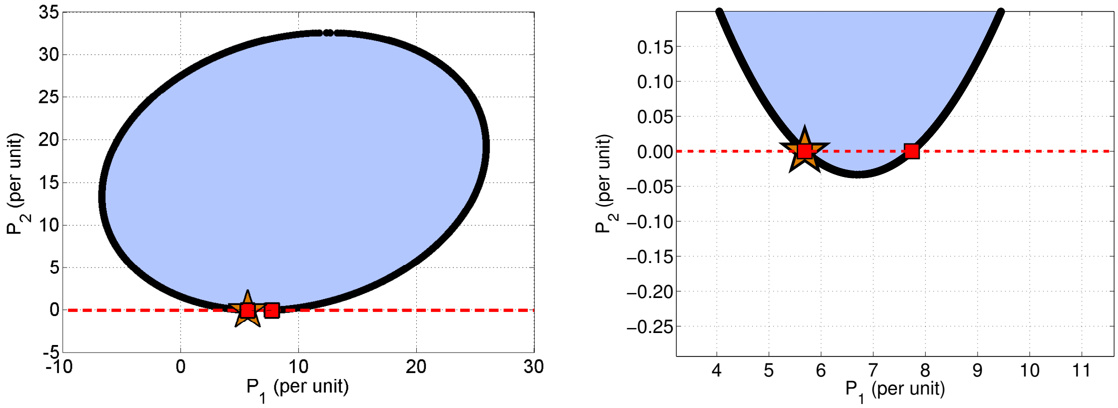
Projections of the feasible spaces of the two- and three-bus systems, in terms of the active power injections at each bus, are shown in Figs. 3 and 4, respectively. The boundary of the oval, shown by the black line in Fig. 3, is the feasible space of the OPF problem (2) for varying values of P_2. The shaded region comprising the interior of the oval in Fig. 3 is the feasible space of the semidefinite relaxation. For the specified value of $P_2 = 0$, shown by the red dashed line, the OPF problem has a feasible space consisting of the two red squares at the intersection of the red dashed line and the black oval. The semidefinite relaxation finds the global optimum of (2) (i.e., the leftmost red square) at the orange star.

In Fig. 4a, the black dots outline the feasible space of the OPF problem (2) for varying values of P_2 and P_3, as determined by repeated homotopy calculations [9]. This feasible space has an ellipsoidal shape with a hole in the interior. The red dashed line corresponds to zero active power injections at buses 2 and 3. The OPF solutions are shown by the red squares at the intersection of the exterior of the ellipsoidal shape with the red dashed line, and are near the hole in the feasible space. The feasible space of the semidefinite relaxation, shown by the shaded region in Fig. 4, “stretches over” this hole in the OPF’s feasible space. As seen in Fig. 4b, which shows a zoomed view of a cut through $P_3 = 0$, the exterior of the relaxation’s feasible space does not match the feasible space of the OPF problem near this hole. Thus, the solution to the semidefinite relaxation (6) at the orange star does not match the true global solution to the OPF problem (2) at the leftmost red square. The semdefinite relaxation fails to be exact for the three-bus problem.

The hole in the OPF’s feasible space is a non-convexity introduced by “nearby” problems (i.e., OPF problems with different values of P_3) in the three-bus system. Without the additional degrees of freedom associated with bus 3, there is no “nearby” non-convexity for the two-bus system. Thus, even though the OPF problems for the two- and three-bus systems share the same feasible space (i.e., the red squares in Figs. 3 and 4), the semidefinite relaxation succeeds in globally solving the two-bus system but not the three-bus system.

Modeling decisions among physically equivalent systems may determine the success or failure of the semidefinite relaxation. This example suggests that simplifying power system models may improve the performance of the semidefinite relaxation; however, further work is necessary to determine whether this holds more generally.

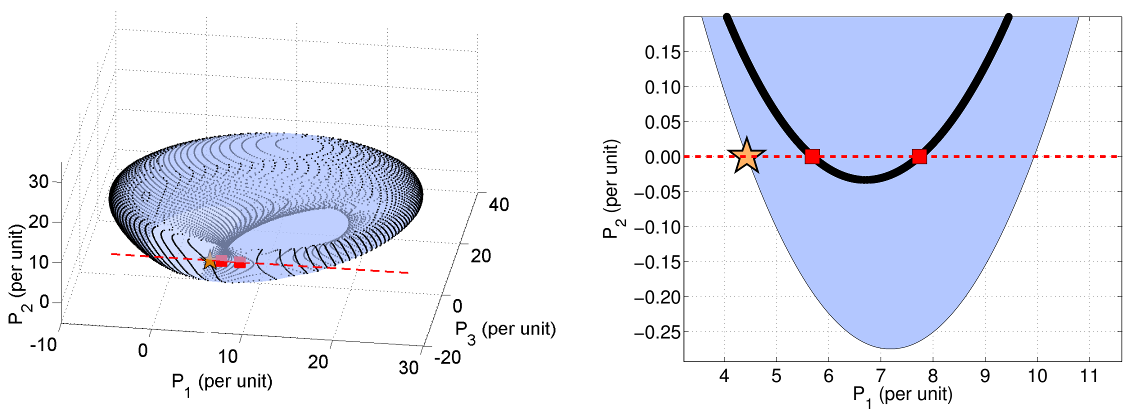
Finally, note that the second-order “moment” relaxation [10], which generalizes the semidefinite relaxation, globally solves both systems, but the penalization heuristic in [7] fails to globally solve the three-bus system.



(a) Projection of the Two-Bus System's Feasible Space

(b) Zoomed View of Fig. 3a

Fig. 3. Projection of the Two-Bus System's Feasible Space. The red squares at the intersection of the black oval and red dashed line are the feasible space for the OPF problem (2). The orange star is the solution to the semidefinite relaxation (6).



(a) Projection of the Three-Bus System's Feasible Space

(b) Projection of the Three-Bus System's Feasible Space with P_3=0

Fig. 4. Projection of the Three-Bus System's Feasible Space. The feasible space for the OPF problem (2) is denoted by the red squares at the intersection of the red dashed line and the region formed by the black dots. The orange star is the solution to the semidefinite relaxation (6), which does not match the global solution at the leftmost red square.

V. Conclusion

This paper has presented a small example OPF problem with two equivalent formulations. The semidefinite relaxation globally solves only one of the two formulations. This suggests that strictly physically based sufficient conditions for exactness of the semidefinite relaxation of the OPF problem cannot predict the relaxation's success or failure for all OPF problems.

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