Mixed SDP/SOCP Moment Relaxations of the Optimal Power Flow Problem

Daniel K. Molzahn and Ian A. Hiskens
Department of Electrical Engineering and Computer Science
University of Michigan
Ann Arbor, MI USA 48109
{molzahn, hiskens}@umich.edu

***Abstract*—Recently, convex "moment" relaxations developed from the Lasserre hierarchy for polynomial optimization problems have been shown capable of globally solving many optimal power flow (OPF) problems. The moment relaxations, which take the form of semidefinite programs (SDP), generalize a previous SDP relaxation of the OPF problem. This paper presents an approach for improving the computational performance of the moment relaxations for many problems. This approach enforces second-order cone programming (SOCP) constraints that establish necessary (but not sufficient) conditions for satisfaction of the SDP constraints arising from the higher-order moment relaxations. The resulting "mixed SDP/SOCP" formulation implements the first-order relaxation using SDP constraints and the higher-order relaxations using SOCP constraints. Numerical results demonstrate that this mixed SDP/SOCP relaxation is capable of solving many problems for which the first-order moment relaxation fails to yield a global solution. For several examples, the mixed SDP/SOCP relaxation improves computational speed by factors from 1.13 to 18.7.**

***Index Terms*—Optimal power flow, convex relaxation, semidefinite programming, second-order cone programming**

I. Introduction

The optimal power flow (OPF) problem determines an optimal operating point for an electric power system in terms of a specified objective function (typically generation cost per unit time). The OPF problem is constrained by network equality constraints (the power flow equations, which model the relationship between voltages and power injections) and engineering limits (e.g., inequality constraints on voltage magnitudes, active and reactive power generations, and line flows).

The OPF problem is generally non-convex due to the nonlinear power flow equations and may have local solutions [1]. Many OPF solution techniques have been proposed, including successive quadratic programs, Lagrangian relaxation, interior point methods, and many types of heuristic optimization [2], [3]. Some of these techniques are quite mature and capable of finding at-least-locally optimal solutions to many large-scale OPF problems with reasonable computational burden. However, while typical local solution techniques often in fact find globally optimal solutions [4], they may fail to converge or converge to a local optimum [1], [5].

Recently, significant research attention has focused on convex relaxations of the OPF problem. Convex relaxations provide lower bounds on the optimal objective value and can certify infeasibility. Further, convex relaxations based on semidefinite programming (SDP) [6] and second-order cone programming (SOCP) [7] formulations have proven capable of globally solving many OPF problems.

The SDP relaxation of [6] has been generalized to a family of "moment relaxations" using the Lasserre hierarchy [8] for polynomial optimization problems [9]–[11]. The moment relaxations take the form of SDPs. The first-order relaxation in this hierarchy is equivalent to the SDP relaxation of [6]. Increasing the relaxation order in this hierarchy enables global solution of a broader class of OPF problems.

The ability to globally solve a broader class of OPF problems comes with the computational cost of larger SDPs. The moment relaxations quickly become computationally intractable with increasing order. Fortunately, second- and third-order moment relaxations globally solve many small problems for which the first-order relaxation fails to yield the globally optimal decision variables.

However, increasing system size results in computational challenges even for low-order moment relaxations. For instance, the second-order relaxation is computationally intractable for OPF problems with more than ten buses. In order to globally solve larger OPF problems, one must exploit network sparsity. This enables solution of the first-order relaxation for systems with thousands of buses [12], [13] and the second-order relaxation for systems with approximately forty buses [11], [14]. To solve larger problems, recent work [14] has both exploited network sparsity and only applied the computationally intensive higher-order moment relaxations to specific regions of the network. This approach has enabled solution of systems with up to 300 buses.

To further improve computational performance, we propose a "mixed SDP/SOCP" moment relaxation. The first-order relaxation is formulated using SDP constraints, while the SDP constraints of the higher-order relaxations are replaced by SOCP constraints that establish necessary (but not sufficient) conditions for satisfaction of the displaced SDP constraints. This mixed SDP/SOCP relaxation is thus a "middle ground" between the first- and higher-order moment relaxations implemented with SDP constraints.

Since the SOCP constraints are less computationally challenging than SDP constraints, the mixed SDP/SOCP moment relaxation generally solves more quickly than formulations of the moment relaxation that use only SDP constraints (hereafter referred to as SDP-based moment relaxations). Numerical results detailed in this paper illustrate typical speed up factors between 1.13 and 18.7 for many moderate-size systems.[1] We emphasize that the mixed SDP/SOCP moment relaxation finds the global solution to these problems. Hence, the additional computational effort required for the SDP-based moment relaxation is unnecessary for these problems.

However, there are some example problems for which the mixed SDP/SOCP formulation requires additional higher-order constraints, which may result in slower solution times for these problems. This was observed for one example in this paper. We also present two examples for which the SDP-based relaxation succeeds but the mixed SDP/SOCP formulation fails to find the global solution.

This paper is organized as follows. After introducing the OPF formulation in Section II and reviewing the moment relaxations in Section III, we describe the mixed SDP/SOCP moment relaxation in Section IV. Numerical results are then presented in Section V, and Section VI concludes the paper.

II. Optimal Power Flow Problem

We first present an OPF formulation in terms of rectangular voltage coordinates, active and reactive power injections, and apparent-power line-flow limits. Consider an n-bus power system, where $\mathcal{N} = \{1, 2, \ldots, n\}$ is the set of all buses, $\mathcal{G}$ is the set of generator buses, and $\mathcal{L}$ is the set of all lines. Let $P_{Dk} + jQ_{Dk}$ represent the active and reactive load demand and $V_k = V_{dk} + jV_{qk}$ the voltage phasors at each bus $k \in \mathcal{N}$. Superscripts "max" and "min" denote specified upper and lower limits. Buses without generators have maximum and minimum generation set to zero. Let $\mathbf{Y} = \mathbf{G} + j\mathbf{B}$ denote the network admittance matrix.

The power flow equations describe the network physics:

$$\begin{aligned} P_{Gk} =& f_{Pk}\left(V_d, V_q\right) = V_{dk} \sum_{i=1}^{n} \left(\mathbf{G}_{ik} V_{di} - \mathbf{B}_{ik} V_{qi}\right) \\ &+ V_{qk} \sum_{i=1}^{n} \left(\mathbf{B}_{ik} V_{di} + \mathbf{G}_{ik} V_{qi}\right) + P_{Dk} \end{aligned} \tag{1a}$$

$$\begin{aligned} Q_{Gk} =& f_{Qk}\left(V_d, V_q\right) = V_{dk} \sum_{i=1}^{n} \left(-\mathbf{B}_{ik} V_{di} - \mathbf{G}_{ik} V_{qi}\right) \\ &+ V_{qk} \sum_{i=1}^{n} \left(\mathbf{G}_{ik} V_{di} - \mathbf{B}_{ik} V_{qi}\right) + Q_{Dk} \end{aligned} \tag{1b}$$

Define a quadratic cost function for active power generation:

$$f_{Ck}\left(V_d, V_q\right) = c_{k2} \left(f_{Pk}\left(V_d, V_q\right)\right)^2 + c_{k1} f_{Pk}\left(V_d, V_q\right) + c_{k0} \tag{2}$$

Define a function for squared voltage magnitude:

$$\left(V_k\right)^2 = f_{Vk}\left(V_d, V_q\right) = V_{dk}^2 + V_{qk}^2 \tag{3}$$

[1]The higher-order forms of both the SDP-based moment relaxation and the mixed SDP/SOCP moment relaxation are currently limited to moderate-size systems due to numerical convergence issues with larger systems.

We use a line model with an ideal transformer that has a specified turns ratio $\tau_{lm} e^{j\theta_{lm}} : 1$ in series with a Π circuit with series impedance $R_{lm} + jX_{lm}$ (equivalent to an admittance of $g_{lm} + jb_{lm} = \frac{1}{R_{lm}+jX_{lm}}$) and shunt admittance $jb_{sh,lm}$.

$$\begin{aligned} P_{lm} &= f_{Plm}\left(V_d, V_q\right) = \left(V_{dl}^2 + V_{ql}^2\right) \frac{g_{lm}}{\tau_{lm}^2} \\ &+ \left(V_{dl} V_{dm} + V_{ql} V_{qm}\right)\left(b_{lm} \sin\left(\theta_{lm}\right) - g_{lm} \cos\left(\theta_{lm}\right)\right)/\tau_{lm} \\ &+ \left(V_{dl} V_{qm} - V_{ql} V_{dm}\right)\left(g_{lm} \sin\left(\theta_{lm}\right) + b_{lm} \cos\left(\theta_{lm}\right)\right)/\tau_{lm} \end{aligned} \tag{4a}$$

$$\begin{aligned} P_{ml} &= f_{Pml}\left(V_d, V_q\right) = \left(V_{dm}^2 + V_{qm}^2\right) g_{lm} \\ &- \left(V_{dl} V_{dm} + V_{ql} V_{qm}\right)\left(g_{lm} \cos\left(\theta_{lm}\right) + b_{lm} \sin\left(\theta_{lm}\right)\right)/\tau_{lm} \\ &+ \left(V_{dl} V_{qm} - V_{ql} V_{dm}\right)\left(g_{lm} \sin\left(\theta_{lm}\right) - b_{lm} \cos\left(\theta_{lm}\right)\right)/\tau_{lm} \end{aligned} \tag{4b}$$

$$\begin{aligned} Q_{lm} &= f_{Qlm}\left(V_d, V_q\right) = -\left(V_{dl}^2 + V_{ql}^2\right)\left(b_{lm} + \frac{b_{sh,lm}}{2}\right)/\tau_{lm}^2 \\ &+ \left(V_{dl} V_{dm} + V_{ql} V_{qm}\right)\left(b_{lm} \cos\left(\theta_{lm}\right) + g_{lm} \sin\left(\theta_{lm}\right)\right)/\tau_{lm} \\ &+ \left(V_{dl} V_{qm} - V_{ql} V_{dm}\right)\left(g_{lm} \cos\left(\theta_{lm}\right) - b_{lm} \sin\left(\theta_{lm}\right)\right)/\tau_{lm} \end{aligned} \tag{4c}$$

$$\begin{aligned} Q_{ml} &= f_{Qml}\left(V_d, V_q\right) = -\left(V_{dm}^2 + V_{qm}^2\right)\left(b_{lm} + \frac{b_{sh,lm}}{2}\right) \\ &+ \left(V_{dl} V_{dm} + V_{ql} V_{qm}\right)\left(b_{lm} \cos\left(\theta_{lm}\right) - g_{lm} \sin\left(\theta_{lm}\right)\right)/\tau_{lm} + \\ &+ \left(-V_{dl} V_{qm} + V_{ql} V_{dm}\right)\left(g_{lm} \cos\left(\theta_{lm}\right) + b_{lm} \sin\left(\theta_{lm}\right)\right)/\tau_{lm} \end{aligned} \tag{4d}$$

$$\left(S_{lm}\right)^2 = f_{Slm}\left(V_d, V_q\right) = \left(f_{Plm}\left(V_d, V_q\right)\right)^2 + \left(f_{Qlm}\left(V_d, V_q\right)\right)^2 \tag{4e}$$

$$\left(S_{ml}\right)^2 = f_{Sml}\left(V_d, V_q\right) = \left(f_{Pml}\left(V_d, V_q\right)\right)^2 + \left(f_{Qml}\left(V_d, V_q\right)\right)^2 \tag{4f}$$

The classical OPF problem is then

$$\min_{V_d, V_q} \sum_{k \in \mathcal{G}} f_{Ck}\left(V_d, V_q\right) \qquad \text{subject to} \tag{5a}$$
$$P_{Gk}^{\min} \leq f_{Pk}\left(V_d, V_q\right) \leq P_{Gk}^{\max} \qquad \forall k \in \mathcal{N} \tag{5b}$$
$$Q_{Gk}^{\min} \leq f_{Qk}\left(V_d, V_q\right) \leq Q_{Gk}^{\max} \qquad \forall k \in \mathcal{N} \tag{5c}$$
$$\left(V_k^{\min}\right)^2 \leq f_{Vk}\left(V_d, V_q\right) \leq \left(V_k^{\max}\right)^2 \qquad \forall k \in \mathcal{N} \tag{5d}$$
$$f_{Slm}\left(V_d, V_q\right) \leq \left(S_{lm}^{\max}\right)^2 \qquad \forall \left(l, m\right) \in \mathcal{L} \tag{5e}$$
$$f_{Sml}\left(V_d, V_q\right) \leq \left(S_{lm}^{\max}\right)^2 \qquad \forall \left(l, m\right) \in \mathcal{L} \tag{5f}$$
$$V_{q1} = 0 \tag{5g}$$

Constraint (5g) sets the reference bus angle to zero.

III. Moment Relaxations

All constraints and the objective function in the OPF problem (5) are polynomial functions of the voltage components V_d and V_q. This enables application of polynomial optimization tools from algebraic geometry. This section first reviews the "moment" relaxations from the Lasserre hierarchy [8] for polynomial optimization problems and then summarizes a method for exploiting network sparsity.

A. Review of Moment Relaxations

Polynomial optimization problems are a special case of "generalized moment problems" [8]. Global solutions to generalized moment problems can be approximated using moment relaxations that are formulated as SDPs. For polynomial optimization problems with bounded variables, such as OPF problems, the approximation approaches the global solution(s) as the relaxation order increases [8].

Formulating the moment relaxations requires several definitions. Define a vector containing all first-order monomials of the decision variables in (5): $\hat{x} = \begin{bmatrix} V_{d1} & V_{d2} & \dots & V_{qn} \end{bmatrix}^\intercal$. Given a vector $\alpha \in \mathbb{N}^{2n}$ representing monomial exponents, the expression $\hat{x}^\alpha = V_{d1}^{\alpha_1} V_{d2}^{\alpha_2} \cdots V_{qn}^{\alpha_{2n}}$ defines the monomial associated with $\hat{x}$ and α. A polynomial $g\left(\hat{x}\right)$ is

$$g\left(\hat{x}\right) \triangleq \sum_{\alpha \in \mathbb{N}^{2n}} g_\alpha \hat{x}^\alpha \tag{6}$$

where g_α is the scalar coefficient corresponding to $\hat{x}^\alpha$.

Define a linear functional $L_y\left\{g\right\}$ which replaces the monomials $\hat{x}^\alpha$ in a polynomial $g\left(\hat{x}\right)$ with scalar variables y_α:

$$L_y\left\{g\right\} \triangleq \sum_{\alpha \in \mathbb{N}^{2n}} g_\alpha y_\alpha \tag{7}$$

When $g\left(\hat{x}\right)$ is a matrix, the functional $L_y\left\{g\right\}$ is applied to each element of $g\left(\hat{x}\right)$.

Consider the vector $\hat{x} = \begin{bmatrix} V_{d1} & V_{d2} & V_{q2} \end{bmatrix}^\intercal$ corresponding to the voltage components of a two-bus system, where the angle reference (5g) is used to eliminate V_{q1}, and the polynomial $g\left(\hat{x}\right) = -\left(0.95\right)^2 + f_{V2}\left(V_d, V_q\right) = -\left(0.95\right)^2 + V_{d2}^2 + V_{q2}^2$. (The constraint $g\left(\hat{x}\right) \geq 0$ forces the voltage magnitude at bus 2 to be greater than or equal to 0.95 per unit.) Then $L_y\left\{g\right\} = -\left(0.95\right)^2 y_{000} + y_{020} + y_{002}$. Thus, $L_y\left\{g\right\}$ converts a polynomial $g\left(\hat{x}\right)$ to a linear function of y.

The order-γ relaxation forms a vector x_γ composed of all monomials of the voltage components up to order γ:

$$\begin{aligned} x_\gamma \triangleq \big[\; 1 \quad V_{d1} \quad \dots \quad V_{qn} \quad V_{d1}^2 \quad V_{d1}V_{d2} \quad \dots \\ \dots \quad V_{qn}^2 \quad V_{d1}^3 \quad V_{d1}^2 V_{d2} \quad \dots \quad V_{qn}^\gamma \;\big]^\intercal \end{aligned} \tag{8}$$

We now define *moment* and *localizing* matrices. The symmetric moment matrix $\mathbf{M}_\gamma\{y\}$ has entries y_α corresponding to all monomials $\hat{x}^\alpha$ up to order 2γ:

$$\mathbf{M}_\gamma\left\{y\right\} \triangleq L_y\left\{x_\gamma x_\gamma^\intercal\right\} \tag{9}$$

Symmetric localizing matrices are defined for each constraint of (5). The localizing matrices consist of linear combinations of the moment matrix entries y. Each polynomial constraint of the form $f\left(\hat{x}\right) - a \geq 0$ in (5) (e.g., $f_{V2}\left(\hat{x}\right) - V_2^{\min} \geq 0$) corresponds to the localizing matrix

$$\mathbf{M}_{\gamma-\beta}\left\{\left(f\left(\hat{x}\right) - a\right) y\right\} \triangleq L_y\left\{\left(f\left(\hat{x}\right) - a\right) x_{\gamma-\beta} x_{\gamma-\beta}^\intercal\right\} \tag{10}$$

where the polynomial f has degree 2β. Example moment and localizing matrices for the second-order relaxation of a two-bus system are presented in (13) and (14), respectively.

The order-γ moment relaxation of (5) is

$$\min_y L_y\left\{\sum_{k \in \mathcal{G}} f_{Ck}\right\} \qquad \text{subject to} \tag{11a}$$

$$\mathbf{M}_{\gamma-1}\left\{\left(f_{Pk} - P_k^{\min}\right) y\right\} \succeq 0 \qquad \forall k \in \mathcal{N} \tag{11b}$$

$$\mathbf{M}_{\gamma-1}\left\{\left(P_k^{\max} - f_{Pk}\right) y\right\} \succeq 0 \qquad \forall k \in \mathcal{N} \tag{11c}$$

$$\mathbf{M}_{\gamma-1}\left\{\left(f_{Qk} - Q_k^{\min}\right) y\right\} \succeq 0 \qquad \forall k \in \mathcal{N} \tag{11d}$$

$$\mathbf{M}_{\gamma-1}\left\{\left(Q_k^{\max} - f_{Qk}\right) y\right\} \succeq 0 \qquad \forall k \in \mathcal{N} \tag{11e}$$

$$\mathbf{M}_{\gamma-1}\left\{\left(f_{Vk} - V_k^{\min}\right) y\right\} \succeq 0 \qquad \forall k \in \mathcal{N} \tag{11f}$$

$$\mathbf{M}_{\gamma-1}\left\{\left(V_k^{\max} - f_{Vk}\right) y\right\} \succeq 0 \qquad \forall k \in \mathcal{N} \tag{11g}$$

$$\mathbf{M}_{\gamma-2}\left\{\left(S_{lm}^{\max} - f_{Slm}\right) y\right\} \succeq 0 \qquad \forall \left(l, m\right) \in \mathcal{L} \tag{11h}$$

$$\mathbf{M}_{\gamma-2}\left\{\left(S_{lm}^{\max} - f_{Sml}\right) y\right\} \succeq 0 \qquad \forall \left(l, m\right) \in \mathcal{L} \tag{11i}$$

$$\mathbf{M}_\gamma\{y\} \succeq 0 \tag{11j}$$

$$y_{00\dots0} = 1 \tag{11k}$$

$$y_{0\dots00\eta0\dots0} = 0 \qquad \eta = 1, \dots, 2\gamma \tag{11l}$$

where $\succeq 0$ indicates that the corresponding matrix is positive semidefinite. The moment relaxation is thus a SDP. The constraint (11k) enforces $x^0 = 1$. The constraint (11l) corresponds to the angle reference constraint (5g); the η in (11l) is in the $(n+1)$-th location, ensuring that $V_{q1}^\eta = 0$.[2]

The objective function and apparent-power line-flow constraints are quartic polynomials in the voltage components V_d and V_q. For $\gamma = 1$, these fourth-order polynomials can be rewritten as second-order using a Schur complement reformulation [14].

The order-γ moment relaxation yields a single global solution if $\operatorname{rank}\left(\mathbf{M}_\gamma\{y\}\right) = 1$. The global solution x^* to the OPF problem (5) is then determined by a spectral decomposition of the diagonal block of the moment matrix corresponding to the second-order terms. Specifically, let μ be a unit-length eigenvector associated with the non-zero eigenvalue λ from the diagonal block of the moment matrix corresponding to the second-order monomials (i.e., $\left[\mathbf{M}_\gamma\{y\}\right]_{2:k,2:k}$, where $k = 2n + 1$ and subscripts indicate entries in MATLAB notation). Then the vector $V^* = \sqrt{\lambda}\left(\mu_{1:n} + j\mu_{(n+1):2n}\right)$ is the global solution.

In the absence of multiple global optima to the OPF problem (5), a solution with $\operatorname{rank}\left(\mathbf{M}_\gamma\{y\}\right) > 1$ indicates that the order-γ moment relaxation only yields a lower bound on the objective value.[3] The order-$(\gamma + 1)$ moment relaxation will improve the lower bound and may give a global solution.

B. Summary of Sparsity-Exploiting Moment Relaxations

The matrices in the moment relaxations quickly grow with both the relaxation order and the system size. Specifically, for an n-bus system, the number of rows and columns in the order-γ relaxation's moment matrix is $\left(2n+\gamma\right)!/\left(\left(2n\right)!\gamma!\right)$. The formulation in (11) is computationally intractable for systems with more than ten buses. Solving the moment relaxation for larger OPF problems requires exploiting network

[2] The angle reference can alternatively be used to eliminate all terms corresponding to V_{q1} to reduce the size of the SDP.

[3] OPF problems generically have a single global optimum. The solution recovery procedure in [8] can be used to recover multiple global optima.

$$x_2 = \begin{bmatrix} 1 & V_{d1} & V_{d2} & V_{q2} & V_{d1}^2 & V_{d1}V_{d2} & V_{d1}V_{q2} & V_{d2}^2 & V_{d2}V_{q2} & V_{q2}^2 \end{bmatrix}^\intercal \tag{12}$$

$$\mathbf{M}_2\{y\} = L_y\{x_2 x_2^\intercal\} = \left[\begin{array}{c|ccc|cccccc}
y_{000} & y_{100} & y_{010} & y_{001} & y_{200} & y_{110} & y_{101} & y_{020} & y_{011} & y_{002} \\ \hline
y_{100} & y_{200} & y_{110} & y_{101} & y_{300} & y_{210} & y_{201} & y_{120} & y_{111} & y_{102} \\
y_{010} & y_{110} & y_{020} & y_{011} & y_{210} & y_{120} & y_{111} & y_{030} & y_{021} & y_{012} \\
y_{001} & y_{101} & y_{011} & y_{002} & y_{201} & y_{111} & y_{102} & y_{021} & y_{012} & y_{003} \\ \hline
y_{200} & y_{300} & y_{210} & y_{201} & y_{400} & y_{310} & y_{301} & y_{220} & y_{211} & y_{202} \\
y_{110} & y_{210} & y_{120} & y_{111} & y_{310} & y_{220} & y_{211} & y_{130} & y_{121} & y_{112} \\
y_{101} & y_{201} & y_{111} & y_{102} & y_{301} & y_{211} & y_{202} & y_{121} & y_{112} & y_{103} \\
y_{020} & y_{120} & y_{030} & y_{021} & y_{220} & y_{130} & y_{121} & y_{040} & y_{031} & y_{022} \\
y_{011} & y_{111} & y_{021} & y_{012} & y_{211} & y_{121} & y_{112} & y_{031} & y_{022} & y_{013} \\
y_{002} & y_{102} & y_{012} & y_{003} & y_{202} & y_{112} & y_{103} & y_{022} & y_{013} & y_{004}
\end{array}\right] \tag{13}$$

$$\mathbf{M}_1\left\{\left(f_{V2} - (0.95)^2\right) y\right\} = \\
\left[\begin{array}{c|ccc}
y_{020} + y_{002} - (0.95)^2 y_{000} & y_{120} + y_{102} - (0.95)^2 y_{100} & y_{030} + y_{012} - (0.95)^2 y_{010} & y_{021} + y_{003} - (0.95)^2 y_{001} \\ \hline
y_{120} + y_{102} - (0.95)^2 y_{100} & y_{220} + y_{202} - (0.95)^2 y_{200} & y_{130} + y_{112} - (0.95)^2 y_{110} & y_{121} + y_{103} - (0.95)^2 y_{101} \\
y_{030} + y_{012} - (0.95)^2 y_{010} & y_{130} + y_{112} - (0.95)^2 y_{110} & y_{040} + y_{022} - (0.95)^2 y_{020} & y_{031} + y_{013} - (0.95)^2 y_{011} \\
y_{021} + y_{003} - (0.95)^2 y_{001} & y_{121} + y_{103} - (0.95)^2 y_{101} & y_{031} + y_{013} - (0.95)^2 y_{011} & y_{022} + y_{004} - (0.95)^2 y_{002}
\end{array}\right] \tag{14}$$

sparsity [11], [14]. Full details are excluded from this summary for the sake of brevity; see [14] for a complete description.

Exploiting sparsity in the moment relaxations is accomplished using a matrix completion theorem [15] which draws on graph theory. The graph in question for the OPF problem is the power system network. Several graph theoretic definitions are necessary for understanding the matrix completion theorem. A *clique* is a subset of the graph nodes for which each node in the clique is connected to all other nodes in the clique. A *maximal clique* is a clique that is not a proper subset of another clique. A graph is *chordal* if each cycle of length four or more nodes has a chord, which is an edge connecting two nodes in the cycle that are not adjacent.

The maximal cliques of a chordal graph can be determined in linear time [16]. Since realistic power networks are generally not chordal, we use a chordal extension technique which adds fictitious edges to the network graph to obtain a chordal super-graph. A chordal extension can be determined using a Cholesky factorization of the network Laplacian matrix with an approximate minimum-degree permutation of the buses to maintain sparsity.

To state the matrix completion theorem, define a matrix $\mathbf{W}$ with partial information (i.e., not all entries of $\mathbf{W}$ have known values) with an associated undirected chordal graph. The matrix completion theorem states that $\mathbf{W}$ can be completed to a positive semidefinite matrix (i.e., the unknown entries of $\mathbf{W}$ can be chosen such that $\mathbf{W} \succeq 0$) if and only if the submatrices associated with each of the maximal cliques of the graph defined by $\mathbf{W}$ are all positive semidefinite. Thus, this theorem enables decomposition of a positive semidefinite constraint for a single large matrix to constraints on many smaller matrices. This effectively eliminates the need to consider many terms which do not appear in the constraint equations of the OPF problem (5). See [12]–[14], [17] for details.

Exploiting network sparsity enables solution of the second-order relaxation for problems up to approximately 40 buses. Using the observation that the first-order relaxation gives a valid solution for large regions of typical OPF problems, larger problems can be solved by selectively applying the higher-order relaxation constraints to specific regions of the network [14]. The choice of where to apply higher-order relaxation constraints is achieved through a heuristic that is based on observations of power mismatches. This heuristic compares the power injections from the solution to the moment relaxation with the power injections implied by the rank-one matrix that is "closest" to the moment matrix. That is, let $\mathbf{W}$ represent the moment matrix given by the solution of the moment relaxation (i.e., $\mathbf{W} = \mathbf{M}_\gamma\{y\}$). The block of the moment matrix corresponding to the first-order relaxation terms is $\mathbf{W}_{2:k,2:k}$, where $k = 2n+1$. Define $\mathbf{W}^{(1)}_{2:k,2:k}$ as the projection of $\mathbf{W}_{2:k,2:k}$ onto the space of rank-one matrices.[4] The power injection mismatch is defined as the absolute value of the difference between the power injections implied by $\mathbf{W}_{2:k,2:k}$ and $\mathbf{W}^{(1)}_{2:k,2:k}$.

For first-order relaxations of typical OPF problems, the power injection mismatch is very small at the majority of buses, with large mismatches at only a few buses. The buses with large mismatch provide an indication of where to add the higher-order moment constraints. Each iteration of the algorithm in [14] applies the higher-order relaxation constraints to the two buses with greatest power injection mismatch. Thus, each iteration of the algorithm progressively tightens the relaxation until a rank-one (global) solution is obtained.

IV. Mixed SDP/SOCP Moment Relaxations

By exploiting sparsity and selectively applying the higher-order moment constraints, the SDP-based moment relaxation is capable of solving OPF problems of up to 300 buses. The feasible space for the moment relaxation is defined

[4]Projection on the space of rank-one matrices is calculated by only keeping the terms in an eigen-decomposition that correspond to the largest eigenvalue.

with positive semidefinite matrix constraints, resulting in a computationally intensive solution process.

In order to improve the computational performance of the moment relaxations, this section describes a "mixed SDP/SOCP" relaxation that implements the higher-order relaxation constraints with a SOCP formulation that is less computationally intensive than the SDP formulation.

We begin with necessary conditions for an arbitrary symmetric matrix $\mathbf{A}$ to be positive semidefinite. If $\mathbf{A} \succeq 0$, then all off-diagonal terms $\mathbf{A}_{ik}$ satisfy the SOCP constraints

$$\mathbf{A}_{ii}\mathbf{A}_{kk} \geq |\mathbf{A}_{ik}|^2 \qquad \forall \{(i,k) \mid k > i\} \tag{15}$$

Only off-diagonal terms in the upper triangular part of $\mathbf{A}$ need to be considered in (15) due to matrix symmetry. Since the SOCP constraints are necessary but not sufficient for ensuring that the corresponding matrix is positive semidefinite, (15) forms a relaxation of $\mathbf{A} \succeq 0$.

In [7], the SOCP constraints (15) are applied to a complex form of the first-order moment relaxation. While this significantly reduces the computational burden compared to using SDP constraints, the SOCP relaxation in [7] typically only yields the global solution to a limited set of OPF problems.[5]

Conversely, the relaxation proposed in this paper formulates the first-order moment relaxation with SDP constraints. This alone is sufficient to globally solve many OPF problems [6], [13]. For OPF problems where the first-order relaxation does not yield the global solution, this paper applies a SOCP formulation for the higher-order constraints rather than use the computationally intensive SDP formulation. Thus, the proposed "mixed SDP/SOCP" relaxation forms a middle ground between the first- and higher-order moment relaxations.

Specifically, consider the moment matrix constraint in (11j). For notational convenience, define $\mathbf{W}^k$ as the submatrix of the moment matrix corresponding to the k^{th} maximal clique. (i.e., $\mathbf{W}^k$ contains all terms in the moment matrix that are only associated with buses in the k^{th} maximal clique; see Section III-B and [14] for further details.) The mixed SDP/SOCP relaxation applies a SDP constraint to the submatrix corresponding to the first-order relaxation:

$$\mathbf{W}^k_{2:2n_k+1,2:2n_k+1} \succeq 0 \qquad \forall \text{ maximal cliques } k \tag{16}$$

where n_k is the number of buses in the k^{th} maximal clique. For all maximal cliques k determined to require higher-order constraints by the algorithm summarized in Section III-B, enforce the SOCP necessary constraints defined in (15) for

$$\mathbf{W}^k - \begin{bmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}^k_{2:2n_k+1,2:2n_k+1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \forall \text{ maximal cliques } k \tag{17}$$

where the $\mathbf{0}$ entries are appropriately sized blocks of zeros. That is, enforce (15) for the terms in the moment matrix corresponding to the higher-order constraints. Similarly, apply SOCP constraints using (15) to the localizing matrices in (11).[6]

This relaxation is tighter than the first-order SDP relaxation, and therefore solves many problems where the first-order relaxation fails. However, it is generally not as tight as the SDP-based higher-order moment relaxation (i.e., the relaxation that enforces $\mathbf{W}^k \succeq 0$); there are problems for which the mixed SDP/SOCP relaxation fails but the second-order SDP-based moment relaxation succeeds in finding the global solution.

V. Results

The computational efficiency of the mixed SDP/SOCP moment relaxation will be illustrated through comparison with the SDP-based relaxation. Salient test cases require that the first-order relaxation fail to yield the global solution. Accordingly, this paper uses test cases from [1] and [14].

This comparison used the iterative algorithm from [14], which is summarized in Section III-B, with a termination criterion that all power injection mismatches are less than 1 MVA. (The solutions typically had significantly smaller mismatch.) The implementation used MATLAB 2013a, YALMIP 2014.12.18 [18], and Mosek 7.1.0.12, and the results were generated using a computer with a quad-core 2.70 GHz processor and 16 GB of RAM.

A second-order relaxation with the mixed SDP/SOCP formulation globally solves a variety of small test problems for which the first-order relaxation does not yield the global solution, including the two- and five-bus systems in [1]. However, these small problems do not demonstrate the computational advantages of the mixed SDP/SOCP relaxations.

Fig. 1 shows the solver times for several moderate-size test cases from [14]. The first and second columns in each group show the solver times for the SDP-based and mixed SDP/SOCP moment relaxations, respectively. The stacked bars show the solver time required for each iteration of the algorithm summarized in Section III-B.

These results demonstrate significant computational speed improvements from the mixed SDP/SOCP relaxation. Both the SDP-based and mixed SDP/SOCP relaxations globally solve all test cases in Fig. 1. For case14Q, case57L, case118Q, case118L, and case300 from [14], the mixed SDP/SOCP relaxation requires the same number of iterations, but the solver times are between a factor of 1.13 and 18.70 faster than the SDP-based relaxation.

Solver times are most improved when the iterative algorithm increases the relaxation order for a bus which corresponds to a maximal clique containing many buses. In such cases, the moment and localizing matrices are large, so the higher-order relaxation constraints make a major contribution to the solver times (e.g., case118Q, which has higher-order constraints corresponding to maximal cliques with up to nine buses). Thus, replacing the SDP constraints with SOCP constraints

[5]The SOCP relaxation in [7] is guaranteed to globally solve OPF problems with radial networks that satisfy certain non-trivial technical conditions, but generally fails for mesh network topologies.

[6]Terms corresponding to odd-order monomials in the off-diagonal blocks of the moment and localizing matrices do not appear in the OPF constraints. Therefore, terms corresponding to the odd-order monomials are set to zero and do not require SOCP constraints.

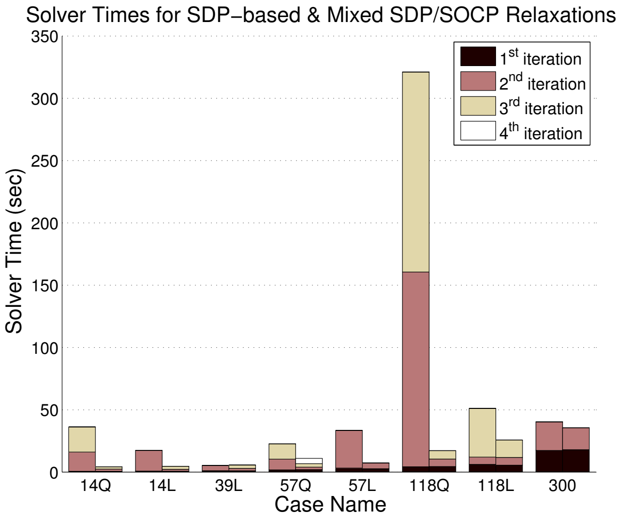

Fig. 1. Solver times for test cases from [14]. The first column in each group corresponds to the SDP-based moment relaxation. The second column in each group corresponds to the mixed SDP/SOCP relaxation. The stacked bars represent the time required for each iteration of the algorithm summarized in Section III-B and described fully in [14].

has a large impact. If the solver times are dominated by the first-order relaxation (e.g., case300, which has higher-order constraints corresponding to maximal cliques with only up to five buses), enforcing the higher-order relaxation with SOCP constraints does not yield a large improvement.

For case14L, case39L, and case57Q, the algorithm requires one additional iteration for the mixed SDP/SOCP relaxation compared to the SDP-based relaxation (i.e., the algorithm applies higher-order relaxation constraints to additional buses for the mixed SDP/SOCP relaxation). Despite the additional higher-order constraints required by case14L and case57Q, the mixed SDP/SOCP relaxation was still faster by factors of 3.72 and 2.07, respectively, than the SDP-based relaxation. As demonstrated by the 5% speed decrease for case39L, though, it is possible for the mixed SDP/SOCP relaxation to have worse performance than the SDP-based relaxation. However, the results in Fig. 1 suggest that this is atypical.

There are also cases where the SDP-based relaxation succeeds but the mixed SDP/SOCP relaxation fails to yield a global solution. For instance, a second-order SDP-based relaxation globally solves case9mod from [1], but even third- and fourth-order mixed SDP/SOCP relaxations fail to yield the global optimum. Similarly, a mix of second- and third-order relaxations results in the global solution to case39mod from [1] using the SDP-based formulation, but even applying third-order constraints to all buses in this system fails to yield the global solution with the mixed SDP/SOCP relaxation.

VI. Conclusion

A hierarchy of convex "moment" relaxations formulated as SDPs are capable of globally solving a broader class of OPF problems than other relaxations. However, even low-order relaxations in this hierarchy can be computationally prohibitive. This paper has proposed a middle ground between the first- and higher-order relaxations that implements the first-order relaxation with SDP constraints and higher-order relaxations with SOCP constraints. This mixed SDP/SOCP relaxation is capable of solving many OPF problems for which the first-order relaxation fails to yield a global solution. Several numerical examples demonstrate that the mixed SDP/SOCP relaxation typically has a significant computational speed advantage over the SDP-based formulation.

Acknowledgment

We acknowledge the support of the Dow Postdoctoral Fellowship in Sustainability, ARPA-E grant DE-AR0000232, and Los Alamos National Laboratory subcontract 270958.

References

[1] W. Bukhsh, A. Grothey, K. McKinnon, and P. Trodden, "Local Solutions of the Optimal Power Flow Problem," *IEEE Trans. Power Syst.*, vol. 28, no. 4, pp. 4780–4788, 2013.

[2] J. Momoh, R. Adapa, and M. El-Hawary, "A Review of Selected Optimal Power Flow Literature to 1993. Parts I and II," *IEEE Trans. Power Syst.*, vol. 14, no. 1, pp. 96–111, Feb. 1999.

[3] A. Castillo and R. O'Neill, "Survey of Approaches to Solving the ACOPF (OPF Paper 4)," US Federal Energy Regulatory Commission, Tech. Rep., Mar. 2013.

[4] D. Molzahn, B. Lesieutre, and C. DeMarco, "A Sufficient Condition for Global Optimality of Solutions to the Optimal Power Flow Problem," *IEEE Trans. Power Syst.*, vol. 29, no. 2, pp. 978–979, Mar. 2014.

[5] A. Castillo and R. O'Neill, "Computational Performance of Solution Techniques Applied to the ACOPF (OPF Paper 5)," US Federal Energy Regulatory Commission, Tech. Rep., Jan. 2013.

[6] J. Lavaei and S. Low, "Zero Duality Gap in Optimal Power Flow Problem," *IEEE Trans. Power Syst.*, vol. 27, no. 1, pp. 92–107, Feb. 2012.

[7] S. H. Low, "Convex Relaxation of Optimal Power Flow–Part I: Formulations and Equivalence," *IEEE Trans. Control Network Syst.*, vol. 1, no. 1, pp. 15–27, Mar. 2014.

[8] J.-B. Lasserre, *Moments, Positive Polynomials and Their Applications*. Imperial College Press, 2010, vol. 1.

[9] D. K. Molzahn and I. A. Hiskens, "Moment-Based Relaxation of the Optimal Power Flow Problem," *18th Power Syst. Comput. Conf. (PSCC)*, 18-22 Aug. 2014.

[10] C. Josz, J. Maeght, P. Panciatici, and J. C. Gilbert, "Application of the Moment-SOS Approach to Global Optimization of the OPF Problem," To appear in *IEEE Trans. Power Syst.*

[11] B. Ghaddar, J. Marecek, and M. Mevissen, "Optimal Power Flow as a Polynomial Optimization Problem," To appear in *IEEE Trans. Power Syst.*

[12] R. Jabr, "Exploiting Sparsity in SDP Relaxations of the OPF Problem," *IEEE Trans. Power Syst.*, vol. 27, no. 2, pp. 1138–1139, May 2012.

[13] D. Molzahn, J. Holzer, B. Lesieutre, and C. DeMarco, "Implementation of a Large-Scale Optimal Power Flow Solver Based on Semidefinite Programming," *IEEE Trans. Power Syst.*, vol. 28, no. 4, pp. 3987–3998, 2013.

[14] D. Molzahn and I. Hiskens, "Sparsity-Exploiting Moment-Based Relaxations of the Optimal Power Flow Problem," To appear in *IEEE Trans. Power Syst.*

[15] R. Gron, C. Johnson, E. Sá, and H. Wolkowicz, "Positive Definite Completions of Partial Hermitian Matrices," *Linear Algebra Appl.*, vol. 58, pp. 109–124, 1984.

[16] R. Tarjan and M. Yannakakis, "Simple Linear-Time Algorithms to Test Chordality of Graphs, Test Acyclicity of Hypergraphs, and Selectively Reduce Acyclic Hypergraphs," *SIAM J. Comput.*, vol. 13, p. 566, 1984.

[17] H. Waki, S. Kim, M. Kojima, and M. Muramatsu, "Sums of Squares and Semidefinite Program Relaxations for Polynomial Optimization Problems with Structured Sparsity," *SIAM J. Optimiz.*, vol. 17, no. 1, pp. 218–242, 2006.

[18] J. Löfberg, "YALMIP: A Toolbox for Modeling and Optimization in MATLAB," in *IEEE Int. Symp. Compu. Aided Control Syst. Des.*, 2004, pp. 284–289.