

# Investigating the Maximum Number of Real Solutions to the Power Flow Equations: Analysis of Lossless Four-Bus Systems

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**Abstract**—The power flow equations model the steady-state relationship between the power injections and voltage phasors in an electric power system. By separating the real and imaginary components of the voltage phasors, the power flow equations can be formulated as a system of quadratic polynomials. Only the real solutions to these polynomial equations are physically meaningful. This paper focuses on the maximum number of real solutions to the power flow equations. An upper bound on the number of real power flow solutions commonly used in the literature is the maximum number of complex solutions. There exist two- and three-bus systems for which all complex solutions are real. It is an open question whether this is also the case for larger systems. This paper investigates four-bus systems using techniques from numerical algebraic geometry and conjectures a negative answer to this question. In particular, this paper studies lossless, four-bus systems composed of PV buses connected by lines with arbitrary susceptances. Computing the Galois group, which is degenerate, enables conversion of the problem of counting the number of real solutions to the power flow equations into counting the number of positive roots of a univariate sextic polynomial. From this analysis, it is conjectured that the system has at most 16 real solutions, which is strictly less than the maximum number of complex solutions, namely 20. We also provide explicit parameter values where this system has 16 real solutions so that the conjectured upper bound is achievable.

## I. INTRODUCTION

The power flow equations are at the heart of many electric power system computations. These equations model the steady-state relationship between the power injections and voltage phasors in a power system. Power flow solutions correspond to the equilibria of the differential-algebraic equations describing the power system dynamics. The power flow equations also form the key constraints in many power system optimization problems, such as optimal power flow [1], unit commitment [2], and state estimation [3].

Despite their importance to power system analyses, there are open theoretical questions regarding power flow solution characteristics. This paper focuses on an open question regarding the maximum number of power flow solutions.

For typical operating conditions, there is a single “high-voltage” power flow solution which corresponds to a stable

equilibrium of reasonable dynamical models. There exists a voluminous literature of mature algorithms for calculating this solution for large-scale systems (e.g., Newton-Raphson [4] and Gauss-Seidel [5]). The performance of Newton-based algorithms is sensitive to the initialization. While reasonable initializations (e.g., the solution to a related problem or a “flat start” of  $1\angle 0^\circ$  per unit voltages) typically converge to the high-voltage solution, Newton-based methods have fractal basins of attraction [6]. Initializing Newton-based methods is thus challenging when parameters move outside typical operating ranges. Accordingly, significant research has been focused on modifications and alternatives to Newton-based methods. See [7] for a discussion of some recent developments.

It is well known that the power flow equations may have multiple solutions [8]. While the high-voltage solution is often of primary interest, other solutions are relevant for certain stability assessments [9]–[13]. There may also exist multiple stable solutions [14], [15]. Further, non-convexities in the power flow feasible space related to multiple solutions are associated with non-zero relaxation gaps for convex relaxations of optimal power flow problems [16], [17].

The literature details a variety of methods for calculating multiple power flow solutions. These include using a range of initializations for a Newton-based method [18], [19], a semidefinite relaxation [20], [21], an auxiliary gradient approach [22], monotone operator theory [23], and a holomorphic embedding of the power flow equations [24]. While these approaches can often find multiple power flow solutions, they are not guaranteed to find *all* solutions.

The numerical continuation method of [25] scales with the *actual* number of power flow solutions rather than an upper bound on the *potential* number of solutions meaning that it is computationally tractable for large test cases. Although it is claimed in [25] that this method will find all power flow solutions for all systems, [26] shows that the proof of this claim is flawed with an explicit counterexample presented in [27]. A recent modification of this method [28] formulates the power flow equations as intersecting ellipsoids which results in all continuation traces being bounded. The method in [28] finds all solutions to a variety of small systems, including the counterexample in [27]. However, there is currently no known proof showing that this method always succeeds in finding all power flow solutions.

There are several methods which are guaranteed to find all power flow solutions. These methods formulate the power flow equations as polynomials whose variables are the real and imaginary parts of the voltage phasors. This enables the

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application of algorithms which find all *complex* solutions to these polynomials. Relevant algorithms include interval analysis [29], Gröbner bases [30], an eigenvalue technique [31], “moment/sum-of-squares” relaxations [32], and numerical polynomial homotopy continuation (NPHC) [33]–[37].

While these methods find all complex solutions, only the real solutions are physically meaningful. Since the number of complex solutions is typically much greater than the number of real solutions, these methods are computationally limited to small systems. Of the numerical methods known to find all power flow solutions, NPHC is currently the most computationally tractable. NPHC has been applied to power system test cases with up to 14 buses [35], and up to the equivalent of 18 buses for the related Kuramoto model [38].

NPHC computes solutions to the power flow equations by tracking so-called solution paths. The number of such paths is based on an upper bound on the number of complex solutions to the power flow equations. Thus, this method can be improved by deriving tighter upper bounds. Moreover, such tighter upper bounds improve the characterization of power flow feasible spaces and are therefore both interesting in their own right and relevant to, e.g., analyzing convex relaxations [16], [17]. For an  $n$ -bus system, one upper bound on the number of complex solutions to the power flow equations is based on Bézout’s theorem, namely  $2^{2n-2}$  complex solutions [39]. By exploiting structure in the equations of lossless systems of PV buses, this upper bound can be reduced to  $\binom{2n-2}{n-1} = \frac{(2n-2)!}{((n-1)!)^2}$  [39]. This bound was extended to (potentially lossy) systems of PQ buses in [40] and to general power systems in [41]. (See [42] for an alternative proof of this bound.)

Whereas the bounds in [39]–[42] are network agnostic, the monomial structure of the power flow equations is determined by the network topology. Topology-dependent upper bounds have the potential to be tighter than previous bounds for specific problems. Recent work [43] uses extensive numerical experiments to verify the topology-dependent bound in [34] and conjecture a new bound for a different class of topologies. Related work [44] proposes topology-dependent bounds for a broader class of network topologies. For generic parameter values, the approach in [44] tightly bounds the number of complex solutions.

All known existing approaches have studied the maximum number of complex solutions in order to bound the number of real solutions. This paper addresses the question of whether systems exist for which the number of real solutions is equal to the number of complex solutions.<sup>1</sup> Otherwise, the equations may have additional structure that could be exploited to produce tighter bounds on the maximum number of real solutions. Two-bus systems can have two real solutions and there exist three-bus systems with six real solutions [45]–[47], both of which match the upper bound provided by the number of complex solutions. Thus, this question is answered in the affirmative for these systems. For a lossless, four-bus system of PV buses with unity voltage magnitudes and a

restricted set of line susceptances, the maximum number of real solutions is 14 [39], which is strictly less than the bound of 20 provided by the number of complex solutions.

We consider lossless, four-bus systems of PV buses connected by lines with arbitrary susceptances. The first contribution of this paper is a choice of susceptance parameters that has 16 real solutions, thus demonstrating that open regions in the parameter space exist with more than 14 real solutions. This choice, and many others, arose by solving 100,000 random instances of the susceptance parameters for systems with zero power injections. The second contribution of this paper is an analysis of the Galois group which allows for one to count the number of real solutions of the power flow equations in this particular case, namely lossless, four-bus systems of PV buses with unity voltage magnitudes and zero power injections, by simply counting the number of positive roots of a univariate sextic polynomial. Although we are unable to explicitly compute this polynomial where the coefficients are functions of the parameters, we can compute it explicitly when the parameters are specified. This analysis and computations lead to the conjecture that no choice of parameters will produce more than 16 isolated real solutions. If this conjecture holds, this would provide a negative answer to the question of whether the number of real solutions can be equal to the bound provided by the number of complex solutions. In particular, this suggests that there is the potential for developing tighter bounds on the maximum number of real solutions to the power flow equations beyond the bounds provided by the number of complex solutions.

The rest of this paper is organized as follows. Section II overviews the power flow equations and presents the four-bus test case studied in this paper along with a choice of parameters, resulting from a computational experiment, that yields 16 real solutions. Section III reviews computing Galois groups and provides the reduction down to counting the number of positive roots of a univariate polynomial. Section IV concludes the paper and discusses future work.

## II. THE POWER FLOW EQUATIONS

This section provides an overview of the power flow equations and then presents the lossless, four-bus system of PV buses which is the main focus of this paper. We conclude this section with a summary of a computational experiment and a choice of parameters for which the power flow equations have 16 real solutions.

### A. Overview of the Power Flow Equations

Consider an  $n$ -bus electric power system with buses labeled  $1, \dots, n$ . The network topology and electrical parameters are described via the admittance matrix  $\mathbf{Y} = \mathbf{G} + \mathbf{jB}$ , where  $\mathbf{j} = \sqrt{-1}$ . (See, e.g., [48] for details on the admittance matrix construction.) Lossless systems have  $\mathbf{G} = 0$ .

Each bus has two associated complex quantities: the active and reactive power injections  $P + \mathbf{j}Q$  where  $P, Q \in \mathbb{R}^n$  and the voltage phasors  $V_d + \mathbf{j}V_q$  denoted using rectangular coordinates with  $V_d, V_q \in \mathbb{R}^n$ . Let  $|V_i|$  denote the voltage

<sup>1</sup>This was first posed as Question 5.1 in [39].

magnitude at bus  $i$ . The power flow equations are quadratic polynomials in the voltage components:

$$P_i = V_{di} \sum_{k=1}^n (\mathbf{G}_{ik} V_{dk} - \mathbf{B}_{ik} V_{qk}) + V_{qi} \sum_{k=1}^n (\mathbf{B}_{ik} V_{dk} + \mathbf{G}_{ik} V_{qk}) \quad (1a)$$

$$Q_i = V_{di} \sum_{k=1}^n (-\mathbf{B}_{ik} V_{dk} - \mathbf{G}_{ik} V_{qk}) + V_{qi} \sum_{k=1}^n (\mathbf{G}_{ik} V_{dk} - \mathbf{B}_{ik} V_{qk}) \quad (1b)$$

$$|V_i|^2 = V_{di}^2 + V_{qi}^2. \quad (1c)$$

Each bus is classified as PQ, PV, or slack. PQ buses, which typically correspond to loads, enforce the active power (1a) and reactive power (1b) equations with specified values for  $P_i$  and  $Q_i$ . PV buses, which typically model generators, enforce (1a) and (1c) with specified  $P_i$  and  $|V_i|$ . The reactive power  $Q_i$  may be computed as an “output quantity” via (1b).<sup>2</sup> A single slack bus is selected with specified voltage magnitude  $|V_i|$ . The voltage phasor at the slack bus is usually chosen to provide a  $0^\circ$  reference angle, i.e.,  $V_{di} = |V_i|$  and  $V_{qi} = 0$ . The active power  $P_i$  and reactive power  $Q_i$  at the slack bus are determined from (1a) and (1b), thus ensuring conservation of power.

With the complex voltage phasors separated into real and imaginary parts  $V_d$  and  $V_q$ , a solution to (1) is only physically meaningful if all variables  $V_d$  and  $V_q$  are *real valued*.

### B. Four-Bus Test Case

The case of interest in this paper is a lossless, four-bus system of PV buses connected by lines with arbitrary susceptances.<sup>3</sup> Fig. 1 shows the one-line diagram for this system, where bus 1 is chosen as the slack bus. The parameter  $b_{ik}$  is the susceptance of the line connecting buses  $i$  and  $k$ .

The maximum number of real solutions for a system of PV buses is non-increasing with increasing resistances [46]. Thus, allowing for losses is not necessary for the purposes of this paper: the maximum number of real solutions for lossy systems is bounded by the maximum number of real solutions for lossless systems of the same size.

With the potential for load and generation at each bus, the active power injection parameters  $P_2$ ,  $P_3$ , and  $P_4$  may be positive, negative, or zero. The voltage magnitude parameters  $|V_i|$ ,  $i = 1, \dots, 4$ , are positive. The line susceptance parameters may be positive (inductive), negative (capacitive) or zero.<sup>4</sup> Note that  $b_{ik} = b_{ki}$ .

A susceptance parameter  $b_{ik} = 0$  indicates that there is no connection between buses  $i$  and  $k$ . Thus, the formulation we study allows for the possibility that the system is not completely connected.<sup>5</sup>

<sup>2</sup>This paper does not consider reactive power limited generators.

<sup>3</sup>We follow a tradition of initially studying power flow related results for this restricted class of power systems (e.g., [39] uses a related, more specific class of systems to obtain initial results that were later generalized [40]–[44]). Future work includes generalizing the results in this paper to four-bus systems that may have PQ buses as well as larger systems.

<sup>4</sup>Shunt susceptance parameters are not necessary since all buses are PV.

<sup>5</sup>The network connectivity is reflected in the monomial structure of the power flow equations. As discussed in [34], [43], [44], the upper bound on the number of real power flow solutions provided by the maximum number of complex solutions is non-increasing with increasing network sparsity.

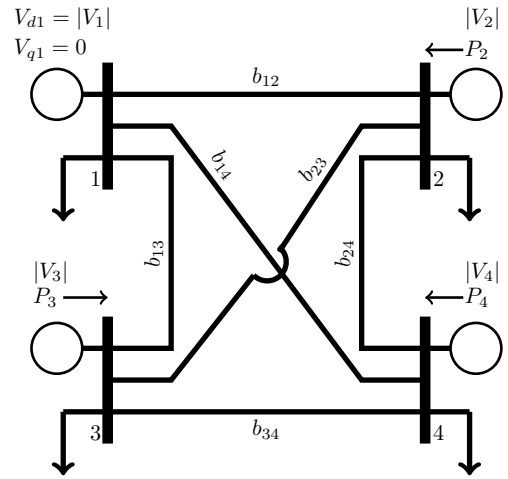


Fig. 1. Four-Bus System

The power flow equations for the system in Fig. 1 are

$$\sum_{\substack{k=2,3,4 \\ k \neq i}} b_{ik} (V_{dk} V_{qi} - V_{di} V_{qk}) + |V_1| V_{qi} b_{1i} = P_i \quad i = 2, 3, 4 \quad (2a)$$

$$V_{di}^2 + V_{qi}^2 = |V_i|^2 \quad i = 2, 3, 4 \quad (2b)$$

While there are 12 parameters in (2), the three voltage magnitude parameters  $|V_i|$  can be set to 1 per unit without loss of generality for this system. This is observed by reformulating the power injection equations in polar coordinates:

$$\sum_{\substack{k=1,\dots,4 \\ k \neq i}} |V_i| |V_k| b_{ik} \sin(\theta_i - \theta_k) = P_i \quad i = 2, 3, 4 \quad (3)$$

where  $\theta_i = \arctan\left(\frac{V_{qi}}{V_{di}}\right)$  is the voltage angle at bus  $i$ . Any valid change to the voltage magnitudes can be compensated by modifying the corresponding susceptance parameters such that the products of voltage magnitudes and electrical parameters are unchanged. In other words, for any choice of  $|V_i|$  and  $b_{ik}$ , we can construct an equivalent system with susceptances  $\hat{b}_{ik} = |V_i| |V_k| b_{ik}$  and 1 per unit voltage magnitudes such that  $|V_i| |V_k| b_{ik} = 1 \cdot 1 \cdot \hat{b}_{ik}$ .

In the particular case of lossless, four-bus systems with zero power injections ( $P_i = 0$ ), unity voltage magnitudes ( $|V_i| = 1$ ), and a restricted set of susceptance parameters  $b_{ik}$  (see [39, Prop. 2.14]), it is shown in [39] that the maximum number of real power flow solutions is 14. In a computational experiment, we also take zero power injections and unity voltage magnitudes but allow the electrical parameters to be generic. Using the NPHC method implemented in Bertini [49], we find all complex solutions for 100,000 random instances of the 6 line susceptance parameters  $b_{ik}$  which were selected from a normal distribution with mean 0 and standard deviation 8. In this experiment, 750 of the 100,000 instance had 16 real solutions. Table I presents line susceptances for a test case that has 16 real solutions, which are listed in Table II. (The characteristics of these solutions are discussed in Section III.) From this example, we can

apply the implicit function theorem to guarantee that there is an open set of parameters for which there are 16 real solutions to the power flow equations for each value of the parameters in this set. This statement also naturally extends to the case with non-zero power injections as well as non-unity and non-equal voltage magnitudes.

TABLE I

SUSCEPTANCES (PER UNIT) FOR A CASE WITH 16 REAL SOLUTIONS.

$b_{12}$	$b_{13}$	$b_{14}$	$b_{23}$	$b_{24}$	$b_{34}$
1.612	-4.649	-5.472	-7.504	10.05	-13.571

TABLE II

THE 16 REAL SOLUTIONS FOR THE SYSTEM DESCRIBED IN TABLE I

Sol. #	$V_{d2}$	$V_{q2}$	$V_{d3}$	$V_{q3}$	$V_{d4}$	$V_{q4}$
1	1	0	1	0	-1	0
2	-1	0	1	0	-1	0
3	1	0	1	0	1	0
4	1	0	-1	0	-1	0
5	-1	0	1	0	1	0
6	-1	0	-1	0	-1	0
7	-1	0	-1	0	1	0
8	1	0	1	0	1	0
9	.30976	-.95082	-.82212	-.56932	-.97906	.20359
10	-.88313	.46912	.97310	.23039	.99834	-.05754
11	.57067	-.82118	-.61912	.78530	.41658	-.90910
12	.89239	-.45127	.84624	-.53281	-.94751	.31973
13	-.88313	-.46912	.97310	-.23039	.99834	.05754
14	.57067	.82118	-.61912	-.78530	.41658	.90910
15	.30975	.95082	-.82212	.56932	-.97906	-.20359
16	.89239	.45127	.84624	.53281	-.94751	-.31973

### III. GALOIS GROUPS AND STRUCTURE

The test case in Section II-B shows that there exist four-bus systems with at least 16 real solutions. We next use Galois groups to provide evidence for an upper bound of 16 real solutions for four-bus systems of PV buses.

One way to possibly understand the maximum number of real solutions to a parameterized system of polynomial equations is to consider the Galois group, which corresponds geometrically to the monodromy group, e.g., see [51], [52]. For a general set of the parameters, the implicit function theorem guarantees that the solutions are analytic functions of the parameters locally. As one extends the domains of these analytic functions, the Galois group encodes the relationships between these analytic extensions. For example, consider the equation  $x^2 - t = 0$  with solutions  $x_{\pm}(t) = \pm\sqrt{t}$ . Although these functions are not analytic at the origin, one can extend them to analytic functions on, say, the complex unit circle  $t(\theta) = e^{j\theta}$ . Hence,  $x_{\pm}(t(\theta)) = \pm e^{j\theta/2}$ . These analytic extensions are the same since  $x_{\pm}(t(\theta)) = x_{\mp}(t(\theta + 2\pi))$ . In this case, the Galois group is the full symmetric group on 2 elements,  $S_2$ , i.e., the plus and minus branches of the square root function are equivalent.

The size of the Galois group is inversely related to the structure in the solutions. That is, the smaller the Galois group, the more structure there is. The structure identified by the Galois group can then be exploited, for example, to help solve the system more efficiently or to aid in the derivation of upper bounds on the number of real solutions.

We start with the lossless, four-bus systems of PV buses with unity voltage magnitudes ( $|V_i| = 1$ ). With this, the system defined by (2) has 9 free parameters (six line susceptances  $b_{ik}$  and three active power injections  $P_i$ ) and 6 variables ( $V_{di}, V_{qi}, i = 2, 3, 4$ ). The values in Table I with  $P_i = 0$  corresponds to an instance of the parameters  $p^* \in \mathbb{C}^9$ . The bound from [39] indicates that there exist a maximum of  $\binom{2n-2}{n-1} = \binom{6}{3} = 20$  complex solutions for this system which is sharp for parameters  $p^*$ . Upon fixing an ordering of the 20 complex solutions, the relationship between the Galois group and the monodromy group describe all the ways that the 20 solutions can be permuted, i.e., reordered, as the parameters are moved along a loop starting and ending at  $p^*$ . This computation can be performed using numerical algebraic geometry, e.g., see [53], [54]. In this case, the Galois group is as large as possible, namely the symmetric group on 20 elements,  $S_{20}$ . That is, for any permutation, i.e., reordering, of the 20 solutions, there is a loop starting and ending at  $p^*$  which realizes this permutation. As in the case with the square root function, each of the 20 solution branches are equivalent meaning that there is no structure that we can ascertain from the Galois group in this case.

We now restrict to the case of zero active power injections, i.e.,  $P_i = 0$  for  $i = 2, 3, 4$ . While systems with zero active power injections are a special case, the number of real power flow solutions decreases as the power injections approach the network's maximum loadability limit, beyond which no real solutions exist [18]. We therefore expect that the maximum number of real solutions is achieved with small or zero active power injections. Future work includes generalizing the following results to cases of active power injections that are not near zero.

After specifying zero active power injections, the remaining parameter space is 6 dimensional. The approach in [55] yields that the Galois group is a subgroup of  $S_{20}$  of order 46,080. Since  $S_{20}$  has order  $20! \approx 2.43 \cdot 10^{18}$ , this Galois group is degenerate which means that there is a significant amount of structure in the solutions as a function of the six parameters that we now aim to determine and exploit.

There are two reasons for this degeneracy, both of which can be observed in the real solutions in Table II. First, eight of the solutions are always  $V_{qi} = 0$  and  $V_{di} = \pm 1$  for  $i = 2, 3, 4$ . These solutions are associated with all choices of voltage angle differences of  $0^\circ$  or  $180^\circ$  between each pair of buses. This results in the sine of the angle differences in (3) being equal to zero, and thus both zero active power injections and zero active power flows in the network.

Second, the other 12 solutions are in a two-way symmetry, i.e., if  $(V_d, V_q)$  is a solution, so is  $(V_d, -V_q)$ . This arises from the odd symmetry of the sine function in (3). With angle differences that are not equal to  $0^\circ$  or  $180^\circ$ , active power flows in loops around the network in these solutions, but no active power is *injected* at any of the buses. Related phenomena have been observed in practical power systems [50].

These two observations impose restrictions on the Galois group. The first shows that only 12 of the 20 complex solutions are nonconstant with respect to the parameters. The

second shows that these 12 nonconstant solutions arise in six pairs. The largest order that the Galois group could be with these two restrictions is  $2^6 \cdot 6! = 46,080$ . Since the Galois group indeed has this order, these two observations completely describe the degeneracy.

We ignore the 8 solutions that are constant with respect to parameters and focus on the other 12 (potentially complex) solutions. Since every coordinate  $V_{qi}$  for  $i = 2, 3, 4$  is generically distinct for these 12 solutions, we select the coordinate  $V_{q4}$ . With this setup, there is a univariate sextic polynomial  $r_6(x)$  whose coefficients are polynomials of the parameters  $b_{ik}$  such that  $r_6(V_{q4}^2) = 0$  exactly on the 12 solutions. That is, the  $V_{q4}$  coordinates of the 12 solutions arise by taking the positive and negative square roots of the solutions of  $r_6(x) = 0$ . Therefore, the real solutions of the original system can be counted and computed from the positive roots of  $r_6(x)$ . For example, with the setup from Table I, the sextic polynomial  $r_6$  is

$$x^6 + 13.4913x^5 + 136.2685x^4 - 144.4123x^3 + 18.9004x^2 - 0.5871x + 0.0017.$$

By Descartes' rule of signs,  $r_6$  has at most 4 positive roots, which is sharp in this case, yielding that the original system has exactly  $8 + 2 \cdot 4 = 16$  real solutions as listed in Table II.

Suppose now that we additionally set one of the parameters  $b_{ik}$  to zero, i.e., no line directly connects buses  $i$  and  $k$ . Then, the system generically has only 16 complex solutions.<sup>6</sup> For example, consider taking  $b_{12} = 0$  and the other parameters as in Table I. With this setup, the 8 nontrivial solutions correspond with the square roots of the roots of a quartic polynomial  $r_4$ , namely

$$x^4 - 1.438x^3 + 0.611x^2 - 0.070x + 0.002.$$

Since all four roots are positive, the corresponding system has  $8 + 2 \cdot 4 = 16$  real solutions showing that all 16 complex solutions can all be real.

There are 20 complex solutions when the parameters  $b_{ik}$  are generic, but only 16 complex solutions when one of them is zero and the others are generic. Therefore, for concreteness, we consider the case as we take the parameter  $b_{12}$  towards 0. Since there are only 16 solutions with  $b_{12} = 0$ , the other four solutions must diverge to infinity as  $b_{12} \rightarrow 0$ . The solutions at infinity are not identically zero and satisfy the system of homogeneous equations resulting from the highest degree terms in each of the polynomials. For example, the finite solutions satisfy  $V_{di}^2 + V_{qi}^2 = 1$  for  $i = 2, 3, 4$ , while the infinite solutions must satisfy the corresponding homogeneous equations  $V_{di}^2 + V_{qi}^2 = 0$ .<sup>7</sup> Since the only real solution satisfying  $V_{di}^2 + V_{qi}^2 = 0$  for  $i = 2, 3, 4$  is the origin, the 4 solutions at infinity arising from the four divergent paths must be nonreal. In particular, these four solutions must

<sup>6</sup>A maximum of 16 complex solutions for this case was first shown by a similar argument in [39, Prop. 3.2]. We show that all 16 solutions can be real for this case.

<sup>7</sup>For  $f(x, y) = x^2 + y^2 - 1$ , the homogenization of  $f$  is the polynomial  $f^h(x, y, z) = z^2 f(x/z, y/z) = x^2 + y^2 - z^2$ . Solutions at infinity have  $z = 0$ , i.e., satisfy  $f^h(x, y, 0) = x^2 + y^2 = 0$ .

be nonreal for all values of  $b_{12}$  near zero. The same argument applies to any parameter  $b_{ik}$  near zero.

In summary, in the lossless, four-bus systems of PV buses with unity voltage magnitudes and zero power injections, we have shown that the maximum number of real solutions is 16 when any of the parameters  $b_{ik}$  is at or near zero (i.e., a pair of buses  $i$  and  $k$  have a "weak" direct connection). The implicit function theorem extends this result to more general cases. For example, in the lossless, four-bus systems of PV buses with unity voltage magnitudes, the maximum number of real solutions is 16 when the power injections are at or near zero and any one of the parameters  $b_{ik}$  is at or near zero.

Due to this result together with facts that no real solutions exist for sufficiently large injections and that the number of real solutions do not increase with increasing resistances [46], we conjecture that the upper bound of 16 real solutions extends to general four-bus systems of PV buses. This conjecture is further supported by the numerical experiment described in Section II-B, which did not restrict any of the line susceptances to be near zero.

#### IV. CONCLUSIONS AND FUTURE WORK

The power flow equations are key to many power system computations, and the characteristics of their solutions are relevant to problems in dynamics and optimization. The two main contributions of this paper are related to the maximum number of real power flow solutions. First, a test case with 16 real solutions surpasses the previously known maximum of 14 real solutions for a four-bus system. Second, Galois group theory is used to analyze the system and leads to the conjecture that the power flow equations for a lossless, four-bus system of PV buses has no more than 16 real solutions. If true, this conjecture suggests that the number of real solutions for this class of problems is strictly less than the upper bound of 20 provided by the maximum number of complex solutions. This result provides evidence for an affirmative answer regarding the open question of whether there exists a gap between the maximum number of real solutions and the upper bound provided by the number of complex solutions for systems with more than three buses.

Future work includes completing the proof of the conjecture that there are at most 16 real solutions for lossless, four-bus systems of PV buses by considering cases that do not restrict any of the parameters to be near zero. Other future work includes considering systems with PQ buses as well as developing tighter upper bounds on the number of real power flow solutions for larger test cases and other network topologies.

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