# Harmony and the Context of Deducibility

Ole Thomassen Hjortland[†]

## 1 Introduction

The philosophical discussion about logical constants has only recently moved into the substructural era. While philosophers have spent a lot of time discussing the meaning of logical constants in the context of classical versus intuitionistic logic, very little has been said about the introduction of substructural connectives. Linear logic, affine logic and other substructural logics offer a more fine-grained perspective on basic connectives such as conjunction and disjunction, a perspective which I believe will also shed light on debates in the philosophy of logic. In what follows I will look at one particularly interesting instance of this: The development of the position known as *logical inferentialism* in view of substructural connectives. I claim that sensitivity to structural properties is an interesting challenge to logical inferentialism, and that it ultimately requires revision of core notions in the inferentialist literature. Specifically, I want to argue that current definitions of proof theoretic harmony give rise to problematic nonconservativeness as a result of their insensitivity to substructurality. These nonconservativeness results are undesirable because they make it impossible to consistently add logical constants that are of independent philosophical interest.

## 2 Background

When Prior (1961) introduced the mock-connective `tonk`, he hoped to show that merely defining inference rules for a new logical constant is not sufficient to determine its meaning. In short, that *logical inferentialism* is false.[1] Simply because `tonk` is equipped with a pair of natural deduction introduction and elimination rules it does not follow that we have succesfully stipulated the semantic content of the operator. Prior's tacit assumption is that if a constant—like `tonk`—trivializes the system, then it cannot be meaningful. Its stipulation semantically misfires.

Prior's `tonk` is not a decisive objection against inferentialism, but it does present its proponents with some difficult choices. On the one hand, one can opt for an all-inclusive approach on which any set of inference rules, no matter how blatantly inconsistent, can succesfully fix the meaning of the logical constant in question. This is the attitude that Dummett (1991) styled Wittgensteinan,

[†]Munich Center for Mathematical Philosophy (MCMP), LMU Munich, Germany.
Email: `ole.hjortland@lmu.de`

[1]Logical inferentialism is a thesis limited to the semantics of logical constants, in contrast to the universal inferentialism of, say, Brandom (1994).

if only losely connected to Wittgenstein. On the other hand, we can accept Prior's objection but attempt to identify criteria for successful stipulation. On the former approach `tonk` is perfectly meaningful, albeit still inconsistent; on the latter, `tonk` is the very litmus test for an adequate set of constraints on inference rules. Whatever the constraints, `tonk` ought to be excluded.

Belnap (1962) was an early attempt at devising a set of constraints on which ill-behaved connectives such as `tonk` are ruled out. Belnap suggests that what is wrong with `tonk` is that it is not a *conservative extension* of any (transitive) consistent system. Prawitz (1971) and Dummett (1991) followed up Belnap's criteria by offering more fine-grained analyses of the conditions under which a connective and its associated introduction and elimination rules can be conservatively added to an antecedent system.

In the subsequent literature there is a great deal of discussion of how conservativeness follows from normalization theorems (and the subformula and separability corollaries). Following Prawitz's insight that the intuitionistic connectives satisfy the *inversion principle*, a number of authors have subscribed to the view that an adequate constraint on meaningful logical constants is a *local constraint* on the associated inference rules, rather than a global constraint on the entire system. As such, conservativeness and normalisation are mere symptoms of an underlying property of the inference rules.

In the last couple of decades, highly successful work has been done on generalizing Prawitz's inversion principle. Schroeder-Heister (1984), von Plato (2001), Tennant (2002) have all discussed a generalization of natural deduction rules which facilitates a language-independent account of inversion. Schroeder-Heister (2004, 2007), Read (2000, 2010), and Francez and Dyckhoff (2012) have applied these very generalizations to give a constraint on inference rules—a constraint which after Dummett is known as *proof theoretic harmony*.[2] As a result we have a better grasp of the connections between the inversion principle, normalization, and—ultimately—consistency; there is now a better understanding of the connections between natural deduction and sequent calculus (see especially von Plato, 2001); and harmonious inference rules have been given for a range of new connectives, including modalities (*e.g.* Read, 2012). Even more importantly, Dummett and Prawitz's revisionary project of giving a justification of intuitionistic logic has run into serious problems (see Weir, 1986; Milne, 1994; Read, 2000; Rumfitt, 2000).

## 3 The Context of Deducibility

One motivation for moving from a global constraint to local constraints on inference rules is that the former is highly dependent on what Belnap calls 'the antecedent context of deducibility':

> It seems to me that the key to a solution lies in observing that even on the synthetic view, we are not defining our connectives *ab initio*, but rather in terms of an *antecedently given context of deducibility*, concerning which we have some definite notions. By that I mean

[2]For Dummett on harmony, see Dummett (1991, ch. 9). Schroeder-Heister prefers the term 'definitional reflection' but his constraints are nevertheless closely related.

> that before arriving at the problem of characterising connectives, we have already made some assumptions about the nature of deducibility. That this is so can be seen immediately observing Prior's use of transitivity of deducibility in order to secure his ingenious result. But if we note that we already *have* some assumptions about the context of deducibility within which we are operating, it becomes apparent that by a too careless use of definitions, it is possible to create a situation in which we are forced to say things inconsistent with those assumptions (Belnap, 1962).

Cook (2005) has already explored a non-transitive consequence relation for which `tonk` is a conservative extension. Similarly, Restall (2007) has discussed the contextual sensitivity of Belnap's other criterion: *uniqueness*. In both cases the moral is the same. The global properties engendered by introducing new logical constants largely depend on prior choices about the *structural properties* of the deducibility relation, for example: Is it transitive? Does it satisfy weakening (is it monotonic)? Is it single- or multiple-conclusion? My contention is that not only are they right about this; the context of deducibility is equally important for the proper formulation of local constraints.

In fact, Dummett (1991, pp. 205–6) contains an early discussion of such structural properties. He considers the difference between a classical disjunction and a quantum logic disjunction. Dummett observes that whereas classical $\vee$-elimination allows (possibly distinct) auxiliary formulae in each subderivation (the minor premises), its quantum counterpart is restricted to subderivations where the conclusion follows from the discharged assumptions alone (*i.e.* the disjuncts). More precisely:

$$\dfrac{A \vee B \qquad \begin{array}{c}\Gamma, [A]^u \\ \vdots \\ C\end{array} \qquad \begin{array}{c}\Gamma', [B]^u \\ \vdots \\ C\end{array}}{C}\ (\vee E) \qquad\qquad \dfrac{A \vee B \qquad \begin{array}{c}[A]^u \\ \vdots \\ C\end{array} \qquad \begin{array}{c}[B]^u \\ \vdots \\ C\end{array}}{C}\ (Q\vee E)$$

The two elimination rules give rise to different classes of theorems, *e.g.* the law of distributivity is derivable in classical logic, but not in quantum logic. Nevertheless, the introduction rules are the same in each case, and The inversion principle gives the same conversion for both rules:

$$\dfrac{\dfrac{\begin{array}{c}\Pi \\ A_i\end{array}}{A_1 \vee A_2} \qquad \begin{array}{c}[A_1]^u \\ \Pi_1 \\ C\end{array} \qquad \begin{array}{c}[A_2]^u \\ \Pi_2 \\ C\end{array}}{C}\ (u) \quad \rightsquigarrow \quad \begin{array}{c}\Pi \\ A_i \\ \Pi_i \\ C\end{array}$$

where $i \in \{1, 2\}$. Prawitz's inversion principle and Dummett's notion of proof theoretic harmony is by and large insensitive to structural properties. This is also true for most subsequent work on harmony and related constraints. My discussion will focus on *general elimination harmony* and the work of Read (2000, 2010); Francez and Dyckhoff (2012), but the general insight applies equally well to the harmony notion advocated by Tennant (1997, 2007).

## 4 General Elimination Harmony

The idea of generalizing Prawitz's inversion principle rests on an idea developed by Martin-Löf (1984), Prawitz (1978), and Schroeder-Heister (1984). The elimination rules for each connective are put into a *general elimination* form which unifies the treatment of elimination rules.[3] The general form follows the shape of the standard disjunction elimination rule, $\vee E$ (contexts now suppressed for simplicity):

$$\dfrac{A \vee B \quad \begin{matrix}[A]^u \\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B]^u \\ \vdots \\ C\end{matrix}}{C}\ (\vee E)(u)$$

Put informally, $\vee E$ says that whatever can be derived independently from each of the grounds for introducing the major premise $A \vee B$ (*i.e.* the premises $A$ and $B$ respectively) can be derived directly from the major premise itself. That is simply an instance of the inversion principle.

In fact, other connectives can also be given in a general elimination form, one for which the new rules are equivalent to the standard rules. Conjunction, for example, has the following two general elimination rules:[4]

$$\dfrac{A \wedge B \quad \begin{matrix}[A]^u \\ \vdots \\ C\end{matrix}}{C} \qquad \dfrac{A \wedge B \quad \begin{matrix}[B]^u \\ \vdots \\ C\end{matrix}}{C}$$

We now see the explicit duality with disjunction: There are two elimination rules because, as opposed to disjunction, conjunction has a single introduction rule with two premises. The standard conjunction elimination rules result from a simplification where $C = A$ and $C = B$ respectively. Indeed, they are equivalent since we can derive the generalized rule from the standard rules by a simple permutation:

$$\begin{matrix}\dfrac{A \wedge B}{A} \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}\dfrac{A \wedge B}{B} \\ \vdots \\ C\end{matrix}$$

Following Read (2000) and Read (2010) we can give schemata for harmoniously inducing a set of general elimination rules from a set of introduction rules.[5] The schemata have as instances the above examples and a number of other connectives. Furthermore, it is an immediate consequence that the elimination rule of `tonk` is not harmonious with respect to the introduction rule.

[3] Sometimes referred to as *disjunction elimination like* rules (*del*-rules) or *parallel rules.*

[4] There is a corresponding notion of *general introduction rules* in Negri (2002) that we will not discuss here.

[5] Note that Read's schemata for general elimination harmony is somewhat revised from Read (2000) to Read (2010). I return to the difference below.

The details are as follows: Each connective $\lambda$ has a finite set $\mathcal{I}$ of $n$ introduction rules, where each such rule has a finite number $m$ of premises.[6]

$$\frac{\alpha_{1_1} \quad \dots \quad \alpha_{1_m}}{\delta} \quad \dots \quad \frac{\alpha_{n_1} \quad \dots \quad \alpha_{n_m}}{\delta}$$

The introduction rule set $\mathcal{I}$ harmoniously induces a finite set $\mathcal{E}$ of $m$ elimination rules:

$$\dfrac{\delta \quad \begin{matrix}[\alpha_{1_1}]^u \\ \vdots \\ \gamma\end{matrix} \quad \dots \quad \begin{matrix}[\alpha_{n_1}]^u \\ \vdots \\ \gamma\end{matrix}}{\gamma}\,(u) \quad \dots \quad \dfrac{\delta \quad \begin{matrix}[\alpha_{1_m}]^u \\ \vdots \\ \gamma\end{matrix} \quad \dots \quad \begin{matrix}[\alpha_{n_m}]^u \\ \vdots \\ \gamma\end{matrix}}{\gamma}\,(u)$$

Here $\delta$ is the major premise (and normally a formula containing a principal occurrence of the connective $\lambda$ in question). For every premise $1 \leq i \leq m$ in the introduction rules there is an $i$th elimination rule in $\mathcal{E}$ such that it has exactly one (subderivational) minor premise for each introduction rule $1 \leq j \leq n$ in $\mathcal{I}$. For each such elimination rule $1 \leq i \leq m$, the $j$th minor premise has (an arbitrary formula) $\gamma$ as conclusion, and $\alpha_{j_i}$ as assumption. When the general elimination rule is applied, each such $\alpha_{j_i}$ assumption in the minor premises is discharged. That is simply observing the idea of the inversion principle: Whatever can be derived independently from all the possible grounds of an expression, can be be derived directly from the expression itself.

It is fairly straightforward to see that both the standard rules for $\vee$ and the general elimination rules for $\wedge$ are instances of the schemata. We will therefore say that they are *GE*-harmonious. Similarly, it is evident that the inference rules for `tonk` are not *GE*-harmonious.

The schemata and the notion of *GE*-harmony can be extended to sets of introduction rules which themselves are hypothetical. For example the standard introduction rule for the intuitionistic conditional. Since this involves some further machinery, however, I set it aside for now (but see Read, 2000 and Read, 2010 for this and other extensions). For our purposes it is sufficient to consider simple variations of the two connectives $\wedge$ and $\vee$ that we have already looked at.

## 5 Shared Contexts and Independent Contexts

Substructural connectives have received lots of attention in a sequent calculus framework, but are less frequently studied with natural deduction rules. In sequent calculus, there is a well-known distinction between *context-sharing* (additive) and *context-independent* (multiplicative) rules.[7] Let $\wedge$ and $\otimes$ be the context-sharing and context-independent variants of conjunction:

$$\frac{\Gamma, A_i \Rightarrow C}{\Gamma, A_0 \wedge A_1 \Rightarrow C}\,(L\wedge) \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\,(R\wedge)$$

[6] An infinite number of introduction rules should not in principle be excluded, but note that this will potentially yield infinitary elimination rules.

[7] The distinction originally came to prominence in Girard (1987).

$$\frac{\Gamma, A, B \Rightarrow C}{\Gamma, A \otimes B \Rightarrow C}\ (L\wedge) \qquad \frac{\Gamma_1 \Rightarrow A \quad \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow A \otimes B}\ (R\wedge)$$

Notice how the right top-most rule has the same context $\Gamma$ in each premise sequent. It is crucial that in a substructural setting the auxiliary formulae $\Gamma$ are *multisets*, *i.e.* it matters how many copies of a single formula $A$ occurs. It is therefore significant that in the conclusion sequent there is only one copy of $\Gamma$ occurring. Contrast the right bottom-most rule where each premise sequent has possibly distinct contexts $\Gamma_1$ and $\Gamma_2$; in the sequent conclusion both (multiset) contexts are preserved. The top-most, context-sharing rule smacks of *contraction*, *i.e.* the structural sequent rule which allows one to collapse copies of the same formula:

$$\frac{\Gamma, A, A \Rightarrow C}{\Gamma, A \Rightarrow C}$$

In a natural deduction setting there is no explicit contradiction rule. Instead there is a corresponding policy for discharging assumptions, namely that multiple copies of the same assumption can be discharged with a single application of a hypothetical rule.[8]

Furthermore, the left-most conjunction rules also come apart. In the top-most context-sharing rule we are forced to make a choice going from bottom to top: Either preserve the right or the left conjunct, not both. If, on the other hand, we wanted to preserve both conjuncts going upwards we would have to first apply the contraction rule to the conjunction in the conclusion sequent. With the context-independent counterpart, however, we can freely bring with us a copy of each conjunct upwards.

These features of the two rule pairs can also be found in the rules for other connectives. For disjunction, the corresponding rules are simply the duals of conjunction (albeit in multiple-succedent form rather than single-succedent). In fact, we can generalize the properties to other connectives as well, but that won't matter for our discussion.

Although less discussed in the proof theoretic literature, the distinction also exists for natural deduction rules. There are several versions of linear logic in natural deduction, but here I follow Negri (2002). She shows one way to implement the distinction while staying reasonably close to the standard Prawitz style tree presentations. Here are the context-sharing rules for conjunction:

$$\frac{\begin{array}{cc}\Gamma^{\alpha} & \Gamma^{\alpha}\\ \vdots & \vdots\\ A & B\end{array}}{A \wedge B} \qquad \frac{A \wedge B \quad \begin{array}{c}\Gamma, [A]^u\\ \vdots\\ C\end{array}}{C} \qquad \frac{A \wedge B \quad \begin{array}{c}\Gamma, [B]^u\\ \vdots\\ C\end{array}}{C}$$

The characteristic feature of the context-sharing $\wedge I$ is the labels $\alpha$ attached to the contexts of its introduction rule. This notation is used as heuristic for the fact that after the application of the additive rule, the contexts *must be treated as a single context*. Recall the additive conjunction rules in sequent calculus:

[8] The correspondence has been made formal through a translation between natural deduction and sequent calculus derivations in von Plato (2001).

There is no need for the label since after the application of $\wedge R$ the contexts are merged into one copy. The label $\alpha$ indicates that whenever a formula $A$ in one copy of $\Gamma$ is discharged, an $A$ copy in the other occurrence of $\Gamma$ is also discharged. Similarly, whenever an open assumption in one copy is substituted for a derivation ending with the same formula, an identical substitution is performed on the other copy.

As a result, when an assumption in $\Gamma$ is discharged by another rule application later in the derivation, we have a form of contraction (or multiple discharge): Two copies of the formula are discharged, one in each copy of $\Gamma$.

In contrast, consider the following multiplicative rules for conjunction:

$$\dfrac{\begin{matrix}\Gamma_0\\ \vdots\\ A\end{matrix}\quad\begin{matrix}\Gamma_1\\ \vdots\\ B\end{matrix}}{A\otimes B}\qquad\qquad \dfrac{A\otimes B\qquad \begin{matrix}\Gamma,[A,B]^u\\ \vdots\\ C\end{matrix}}{C}$$

Corresponding to sequent calculus, the multiplicative conjuntion $\otimes$ only has one elimination rule, and its introduction rule may be applied with (possibly) distinct assumptions $\Gamma_0$ and $\Gamma_1$.

It should be obvious that these two connectives are proof theoretically distinct. Yet, in the presence of vacuous and multiple discharge of assumptions (*e.g.* in classical logic) they become equivalent. In fact, we can then derive the context-sharing rules for $\otimes$ and the context-independent rules for $\wedge$. For example, using two copies of $A\wedge B$ we can derive the context-independent elimination rule for the additive connective $\wedge$:

$$\dfrac{A\wedge B\qquad \dfrac{A\wedge B\qquad \begin{matrix}[A]^1,[B]^2\\ \vdots\\ C\end{matrix}}{C}\,(1)}{C}\,(2)$$

Second, with vacuous discharge the context-sharing elimination rules are derivable for $\otimes$. Simply opt to discharge only one conjunct in each case:

$$\dfrac{A\otimes B\qquad \begin{matrix}[A]\\ \vdots\\ C\end{matrix}}{C}\qquad\qquad \dfrac{A\otimes B\qquad \begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{C}$$

Nevertheless, it should be no surprise that the rules for $\wedge$ and $\otimes$ give rise to distinct conversions via the inversion principle:

$$\dfrac{\dfrac{\begin{matrix}\Gamma^\alpha\\ \Pi_0\\ A\end{matrix}\quad\begin{matrix}\Gamma^\alpha\\ \Pi_1\\ B\end{matrix}}{A\wedge B}\qquad \begin{matrix}\Sigma,[A]\\ \Pi_2\\ C\end{matrix}}{C}\;\rightsquigarrow\;\begin{matrix}\underbrace{\Sigma\quad\begin{matrix}\Gamma\\ \Pi_0\\ A\end{matrix}}\\ \Pi_2\\ C\end{matrix}\qquad\qquad \dfrac{\dfrac{\begin{matrix}\Gamma^\alpha\\ \Pi_0\\ A\end{matrix}\quad\begin{matrix}\Gamma^\alpha\\ \Pi_1\\ B\end{matrix}}{A\wedge B}\qquad \begin{matrix}\Sigma,[B]\\ \Pi_2\\ C\end{matrix}}{C}\;\rightsquigarrow\;\begin{matrix}\underbrace{\Sigma\quad\begin{matrix}\Gamma\\ \Pi_1\\ B\end{matrix}}\\ \Pi_2\\ C\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\Gamma_0\\ \Pi_0\\ A\end{matrix}\quad\begin{matrix}\Gamma_1\\ \Pi_1\\ B\end{matrix}}{A\wedge B}\qquad \begin{matrix}\Gamma,[A,B]\\ \Pi_2\\ C\end{matrix}}{C}\qquad\rightsquigarrow\qquad \begin{matrix}\underbrace{\Gamma\quad \begin{matrix}\Gamma_0\\ \Pi_0\\ A\end{matrix}\quad\begin{matrix}\Gamma_1\\ \Pi_1\\ B\end{matrix}}\\ \Pi_2\\ C\end{matrix}$$

Notice that after the uppermost reduction step for the additive conjunction it becomes explicit that there is only one copy of Γ. In contrast, the reduction step for the multiplicative conjunction keeps both contexts.

There is nothing surprising about the above: The conversions behave differently in a way reminiscent of how cut applications are pushed for the corresponding inference rules in sequent calculus. What is interesting, however, is that *GE*-harmony as articulated by Read (2000, 2010); Francez and Dyckhoff (2012) do not contain the resources to keep these connectives apart. This is a problematic omission.

In Read (2000, pp. 130–32) the theory of *GE*-harmony induces the context-independent elimination rule from the context-independent introduction rule. That appears sensible in light of the above. However, in Read (2010), and Francez and Dyckhoff (2012) the theory of *GE*-harmony is revised, and the schemata now provides the two context-sharing rules as the elimination rules induced by the context-independent introduction rule. The reason for this change is not substructural considerations, however, but problems with other connectives.

Admittedly, the asymmetry causes no problems in the presence of standard discharge policies, and this is what the authors had in mind. My only point here is that there *are* situations in which a mismatch between context-sharing and -independent does matter. In fact, there are systems in which a lack of substructural finesse will cause considerable harm. Specifically, it will engender nonconservativeness results that will make certain legitimate connectives inconsistent extensions of the systems in question. This, I argue, is sufficient motivation to take the substructural challenge to logical inferentialism seriously.

## 6 Structural Nonconservativeness

For simplicity, let us take as our starting point a system in which neither multiple nor vacuous discharge are permissible policies for assumptions. We then introduce a conjunction which is governed by asymmetric inference rules, *i.e.* a context-sharing introduction rule, and a context-independent elimination rule:

$$\dfrac{\begin{matrix}\Gamma^{\alpha}\\ \vdots\\ A\end{matrix}\quad\begin{matrix}\Gamma^{\alpha}\\ \vdots\\ B\end{matrix}}{A\sqcap B}\qquad\qquad \dfrac{A\sqcap B\qquad \begin{matrix}\Gamma,[A,B]^{u}\\ \vdots\\ C\end{matrix}}{C}\,{\scriptstyle (u)}$$

Now let us assume that there is a derivation from two copies of $A$ to some conclusion $C$. Recall that there is no multiple discharge in the antecedent system, and therefore no immediate way in which the above derivation entails the existence of a derivation from one copy of $A$ to the same conclusion $C$:

$$\begin{array}{c} A, A \\ \vdots \\ C \end{array} \quad \rightsquigarrow \quad \begin{array}{c} A \\ \vdots \\ C \end{array}$$

Moreover, the following is an instance of the context-independent elimination rule for $\sqcap$:

$$\frac{A \sqcap A \qquad \begin{array}{c} \Gamma, [A, A]^1 \\ \vdots \\ C \end{array}}{C}\ (1)$$

With the context-sharing introduction rule, then, we can extend the above to the following perilous derivation:

$$\frac{\dfrac{\begin{array}{c} \Gamma^\alpha \\ \vdots \\ A \end{array} \quad \begin{array}{c} \Gamma^\alpha \\ \vdots \\ A \end{array}}{A \sqcap A} \qquad \begin{array}{c} \Gamma, [A, A]^1 \\ \vdots \\ C \end{array}}{C}\ (1)$$

If we let $\Gamma$ be simply $A$, both the premises of the introduction rule are trivial, and after its application they are treated as one premise. The latter fact is merely the characteristic feature of the context-sharing introduction rule. What this means is that there is now a derivation directly from a single copy of $A$ to the conclusion $C$. Contraction is admissible.

We can drive the point home with two concrete examples.

EXAMPLE 1. Assume that prior to the introduction of the conjunction $\sqcap$ the system only contained a conditional $\to$ governed by the two standard rules:

$$\frac{\begin{array}{c} [A]^u \\ \vdots \\ B \end{array}}{A \to B}\ (\to I)(u) \qquad\qquad \frac{A \to B \quad A}{B}\ (\to E)$$

As per our hypothesis that the antecedent system is contraction-free, the conditional rules do not allow multiple discharge of assumptions. But with the introduction of the conjunction, we have the return of explicit multiple discharge. We can now extend the previous derivation with an application of the conditional rules:

$$\frac{\dfrac{\dfrac{\dfrac{\begin{array}{c} [A^\alpha]^2 \\ \vdots \\ A \end{array} \quad \begin{array}{c} [A^\alpha]^2 \\ \vdots \\ A \end{array}}{A \sqcap A} \qquad \begin{array}{c} \Gamma, [A, A]^1 \\ \vdots \\ C \end{array}}{C}\ (1)}{A \to C}\ (2) \qquad A}{C}$$

The derivation is now blatantly from one copy of $A$ to $C$, with the original two assumptions of $A$ treated as one discharged assumption in the application of $\rightarrow I$ (with index 2). Since both $A$ and $C$ are arbitrary we can now make this move with any formulae in the language. This is a form of structural nonconservativeness—multiple discharge becomes admissible with the introduction of a new connective.

EXAMPLE 2. A more startling example of structural nonconservativeness results if we instead consider a less standard connective, $\bullet$, read 'bullet' (see Read, 2000, p. 141):

$$\dfrac{\begin{array}{c}[\bullet]^u \\ \vdots \\ \bot\end{array}}{\bullet}\ (\bullet I)(u) \qquad \dfrac{\bullet \quad \bullet \quad \begin{array}{c}[\bot]^u \\ \vdots \\ C\end{array}}{C}\ (\bullet E)(u)$$

The inference rules for $\bullet$ are *GE*-harmonious by the standards of Read (2000). Yet Read acknowledges that not only do the rules fail to normalize, they lead straight to triviality in the presence of multiple discharge and *ex falso quodlibet* for $\bot$. In short, contrary to what Dummett and Prawitz assumed, harmony does not entail consistency.

It is all the more remarkable that a connective like $\bullet$ does indeed normalize in the absence of multiple discharge of assumptions. The inference rules are perfectly consistent in contraction-free systems. All the more problematic, then, that when the conjunction $\sqcap$ is added to the system, the result is inconsistency.[9] In other words, the upshot is more than just the admissibility of contraction; the resulting system can derive anything.

It is important to note that the type of nonconservativeness result in question is a direct result of mixing context-sharing and context-independent rules. If we were to combine the context-independent introduction and elimination rules, the problematic derivation would not be possible. The context-independent introduction rule would still leave us with two separate copies of the assumption $A$, in which case no contraction has occurred. Similarly, using exclusively context-sharing rules would not lead to nonconservativeness. It is only because of the reckless combination of inference rules that the policies for discharging assumptions are indirectly altered.

The admissibility of contraction is nothing new in sequent calculus. The above sort of derivation has a more familiar sequent counterpart:

$$\dfrac{\dfrac{A \Rightarrow A \quad A \Rightarrow A}{A \Rightarrow A \sqcap A}\ (R\sqcap) \qquad \dfrac{A, A \Rightarrow C}{A \sqcap A \Rightarrow C}\ (L\sqcap)}{A \Rightarrow C}\ (Cut)$$

Again, the $R\sqcap$ application is context-sharing (note how two copies of $A$ are collapsed into one), while the $L\sqcap$ rule application is context-independent. After an application of Cut we have transformed a sequent of the form $A, A \Rightarrow C$ into one of the form $A \Rightarrow C$. A case of contraction, if anything is. Indeed,

[9] See Read (2012) for details on the derivation that leads to inconsistency.

the particular inference rules for conjunction is used in **G3** systems of classical logic precisely to make contraction admissible. In contrast, the more common **G1** systems have an explicit contraction rule.[10]

## Conclusion

For all that is said so far, perhaps one might want to reply with a shrug: The examples are artificial, and have little if any impact on philosophical debates. But that contention is simply wrong, and the attitude is unfortunate. Setting aside the principle that proof theoretic harmony should apply generally to connectives of all persuasions, there are still ample reasons to take them seriously in the philosophy of logic.[11] Substructural connectives are already commonplace in the relevant logic literature, but have now permeated a number of philosophical discussions. In formal epistemology, substructural systems have been used by van Benthem (2008) and Sequoiah-Grayson (167); in debates about vagueness substructural systems have been advocated by Zardini (2008) and Cobreros et al. (2012); and in formal semantics Barker (2010) has used substructural connectives to model free choice permission. The example I am mostly interested in strikes closer to home, however.

EXAMPLE 3. Contraction-free systems have received a lot of attention in discussions of set-theoretic and semantic paradoxes. Two recent substructural approaches are Petersen (2000) and Zardini (2011), but the strategy goes back to Curry (1942) and a number of works by Ross Brady. The observation common to their work is that unrestricted comprehension (for set theory) and unrestricted truth predicates (for theories of truth) can be consistently added to contraction-free systems. These logical constants can be given a natural deduction form as follows:

$$\frac{A}{T\langle A\rangle}\ (TI) \qquad \frac{T\langle A\rangle}{A}\ (TE)$$

$$\frac{\Phi(t)}{t \in \lambda x\Phi(x)}\ (\lambda E) \qquad \frac{t \in \lambda x\Phi(x)}{\Phi(t)}\ (\lambda E)$$

It is obviously a controversial philosophical issue whether such unrestricted rules are appropriate for our theories of truth and properties respectively. But precisely because this an ongoing philosophical concern, we cannot have it short-circuited by a harmony constraint which excludes them. On the contrary, the logical inferentialist ought not impose constraints which fail to preserve the substructural nuances required for these philosophical theories.

A theory of proof theoretic harmony that is insensitive to substructural differences is in danger of permitting structural nonconservativeness, *e.g.* admissibility of contraction. The result is a theory which renders inconsistent connectives whose inference rules are philosophically motivated and perfectly well-behaved in the appropriate context of deducibility. What we are left with

[10] See Troelstra and Schwichtenberg (2000, p. 77) for details.

[11] Full disclosure: My position is that structural properties are indeed part of the meaning of logical connectives, and that proof theoretic semantics therefore has to reflect it (see Hjortland, 2012a). For the opposite view, see *e.g.* Paoli (2003).

is what I call the *substructuralist challenge* to logical inferentialism: A re-articulation of proof theoretic harmony in substructural terms.[12]

# BIBLIOGRAPHY


Barker, C. (2010). Free choice permission as resource-sensitive reasoning. *Semantics and Pragmatics*, 3(10):1–38.

Belnap, N. D. (1962). Tonk, plonk and plink. *Analysis*, 22(6):130–134.

Brandom, R. B. (1994). *Making it Explicit: Reasoning, Representing and Discursive Commitment*. Harvard University Press, Cambridge, MA.

Cobreros, P., Egré, P., Ripley, D., and van Rooij, R. (2012). Tolerant, classical, strict. *Journal of Philosophical Logic*, 41:347–385.

Cook, R. (2005). What's wrong with tonk(?). *Journal of Philosophical Logic*, 34:217–226.

Curry, H. B. (1942). The inconsistency of certain formal logics. *Journal of Symbolic Logic*, 7(3):115–117.

Dummett, M. A. E. (1991). *The Logical Basis of Metaphysics*. Duckworth, London.

Francez, N. and Dyckhoff, R. (2012). A note on harmony. *Journal of Philosophical Logic*, 41:613–628.

Girard, J.-Y. (1987). Linear logic. *Theoretical Computer Science*, 50:1–102.

Hjortland, O. T. (2012a). Logical pluralism, meaning-variance, and verbal disputes. *Australasian Journal of Philosophy*. To appear.

Hjortland, O. T. (2012b). Proof theoretic semantics in the substructural era. Unpublished manuscript.

Martin-Löf, P. (1984). *Intuitionistic Type Theory*. Bibliopolis, Napoli.

Milne, P. (1994). Classical harmony: Rules of inference and the meaning of the logical constants. *Synthese*, 100:49–94.

Negri, S. (2002). A normalizing system of natural deduction for intuitionistic linear logic. *Archive for Mathematical Logic*, 41(8):789–810.

Paoli, F. (2003). Quine and Slater on paraconsistency and deviance. *Journal of Philosophical Logic*, 32:531–548.

Petersen, U. (2000). Logic without contraction as based on inclusion and unrestricted abstraction. *Studia Logica*, 64(3):365–403.

Prawitz, D. (1971). Ideas and results in proof theory. In Fenstad, J. E., editor, *Proceedings of the 2nd Scandinavian Logic Symposium*, pages 235–308. North Holland, Amsterdam.

Prawitz, D. (1978). Proofs and the meanings and completeness of the logical constants. In Hintikka, J., editor, *Essays on Mathematical and Philosophical Logic*, pages 25–40. Reidel, Dordrecht.

Prior, A. N. (1961). The runabout inference ticket. *Analysis*, 21:38–39.

Read, S. (2000). Harmony and autonomy in classical logic. *Journal of Philosophical Logic*, 29:123–154.

Read, S. (2010). General-elimination harmony and the meaning of the logical constants. *Journal of Philosophical Logic*, 39(5):557–576.

Read, S. (2012). Harmony and modality. In C. Dégremont, L. K. and Rückert, H., editors, *On Dialogues, Logics and other Strange Things*. Kings College Publications. Forthcoming.

Restall, G. (2007). Proof theory and meaning: On the context of deducibility. In *Proceedings of Logic Colloquium 2007*. Cambridge University Press, Cambridge.

Rumfitt, I. (2000). "Yes" and "No". *Mind*, 109:781–820.

Schroeder-Heister, P. (1984). A natural extension of natural deduction. *Journal of Symbolic Logic*, 49(4):1284–1300.

Schroeder-Heister, P. (2004). On the Notion of Assumption in Logical Systems. In Bluhm, R. and Nimtz, C., editors, *Selected Papers Contributed to the Sections of GAP5, Fifth International Congress of the Society for Analytical Philosophy, Bielefeld, 22-26 September 2003*, pages 27–48. Mentis, Paderborn. `www.gap5.de/proceedings`.

Schroeder-Heister, P. (2007). Generalized definitional reflection and the inversion principle. *Logica Universalis*, 1(2):355–376.

Schroeder-Heister, P. (2012). Definitional reflection and basic logic. In Miaetti, M. E., Palmgren, E., and Rathjen, M., editors, *Advances in Constructive Topology and Logical Foundations*. To appear.


[12]Schroeder-Heister (2012) offers a substructural version of his definitional reflection. My own framework is outlined in Hjortland (2012b).

Sequoiah-Grayson, S. (167). A positive information logic for inferential information. *Synthese*, 2(409–431).
Tennant, N. (1997). *The Taming of the True*. Oxford University Press, Oxford.
Tennant, N. (2002). Ultimate normal forms for parallelized natural deductions. *Logic Journal of the IGPL*, 10(3):299–337.
Tennant, N. (2007). Inferentialism, logicism, harmony, and a counterpoint. In Miller, A., editor, *Essays for Crispin Wright: Logic, Language, and Mathematics, vol. 2*. OUP, Oxford.
Troelstra, A. S. and Schwichtenberg, H. (2000). *Basic Proof Theory*. Cambridge University Press, Cambridge.
van Benthem, J. (2008). Logical dynamics meets logical pluralism? *Australasian Journal of Logic*, 6:182–209.
von Plato, J. (2001). Natural deduction with general elimination rules. *Archive for Mathematical Logic*, 40(7):541–567.
Weir, A. (1986). Classical harmony. *Notre Dame Journal of Formal Logic*, 27(4):459–482.
Zardini, E. (2008). A model of tolerance. *Studia Logica*, 90(3):337–368.
Zardini, E. (2011). Truth without contra(di)ction. *The Review of Symbolic Logic*, 4:498–535.