Adaptive Ellipse Tracking and a Convergence Proof

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I. Introduction

Good Ellipse fitting methods are desirable in pattern recognition and computer vision. Primitive shape models are often applied when a low dimensional robust model is to be extracted from high dimensional (very often noisy) data. Ellipse has its special merit in that it is the projection of circular shapes, which are common among natural and artificial objects (*e.g.*, human faces, tires, etc). It also has potential application in describing dynamical systems that exhibit semi-periodic and hysteric behaviors. Ellipse fitting can be posed as a convex optimization problem (in algebraic distance) with quadratic constraint. Its solution involves looking for the eigen vector corresponding to the largest eigenvalue in a generalized eigen decomposition problem. In this report, we develop an adaptive method to fit ellipse model and analyze the convergence of the proposed algorithm.

II. Ellipse Fitting Model For Static Data

In static data fitting, we are given a collection of 2-dimensional sample locations (x_i, y_i), $i = 1, 2 \ldots, N$, which are assumed to be noisy observations of an underlying ellipse model. The general goal is to fit an ellipse to the observed samples with respect to some distance measure.

There are several alternatives in modeling the ellipse shape and correspondingly the distance metric. Since we are interested in general ellipse fitting, there should be no presumption that the semi-major and/or minor axis aligns with the reference coordinate of observation data. The natural choice of the two-center bipolar coordinate equation:

$$||(x_i, y_i) - (x^1, y^1)||_2 + ||(x_i, y_i) - (x^2, y^2)||_2 = 2r$$

with geometric (Euclidean) distance as data fidelity metric would result in a nonlinear problem that is challenging to solve. Some authors have employed iterative methods to approximately minimize the geometric distance [1–3]. Proper transformation of coordinate and formulating the problem with an implicit function may help to simplify both constraint [4] and/or optimization. Indeed, if we relax the requirement to use geometric distance, and use the sum of squared algebraic distance as the measure of data infidelity, explicit iteration could be arguably avoided. A computational efficient ellipse fitting algorithm based on generalized eigen analysis was proposed by [5] and it is the method that we are to apply here.

Let $\boldsymbol{x} = [x^2 \; xy \; y^2 \; x \; y \; 1]^T$ and $\boldsymbol{a} = [a \; b \; c \; d \; e \; f]^T$, and write the general quadratic curve equation as:

$$F(\boldsymbol{a}, \boldsymbol{x}) = \boldsymbol{a}^T \boldsymbol{x} = ax^2 + bxy + cy^2 + dx + ey + f = 0 \tag{2.1}$$

where the superscript T denotes transpose. We know that ellipses satisfy the constraint that the discriminant is negative, *i.e.*, $b^2 - 4ac < 0$. We try to minimize the sum of squared algebraic difference: [1]

[1] Algebraic difference is in general different than the geometric distance that measures the fitting error in Euclidean terms. It can be regarded as a measure of mismatch in a transformed coordinate.

$$\begin{aligned} \mathcal{D}_A(\boldsymbol{a}) &= \sum_{i=1}^{N} F(\boldsymbol{a}, \boldsymbol{x}_i) \\ &= ||\mathbb{X}\boldsymbol{a}||^2 \\ &= \boldsymbol{a}^T \mathbb{X}^T \mathbb{X} \boldsymbol{a}, \end{aligned} \tag{2.2}$$

where $\mathbb{X} = [\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots]^T$.

Because a constant scaling applied to $\boldsymbol{a}$ does not affect the underlying ellipse, we impose the constraint that $b^2 - 4ac = -1$, or equivalently $\boldsymbol{a}^T \boldsymbol{C} \boldsymbol{a} = 1$ with

$$\boldsymbol{C} = \begin{bmatrix} \tilde{\boldsymbol{C}} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{bmatrix} \tag{2.3}$$

where $\tilde{\boldsymbol{C}} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$. Thus, the original ellipse fitting problem is a quadratic optimization problem subject to quadratic constraint. More precisely, we want to minimize $\boldsymbol{a}^T \mathbb{X}^T \mathbb{X} \boldsymbol{a}$ subject to the constraint that $\boldsymbol{a}^T \boldsymbol{C} \boldsymbol{a} = 1$.

Introducing the Lagrangian multiplier λ and differentiating: we need to solve the system of equations:

$$\begin{aligned} \boldsymbol{S}\boldsymbol{a} - \lambda \boldsymbol{C}\boldsymbol{a} &= 0 \\ \boldsymbol{a}^T \boldsymbol{C} \boldsymbol{a} = 1, \end{aligned} \tag{2.4}$$

where $\boldsymbol{S} \stackrel{\triangle}{=} \mathbb{X}^T \mathbb{X}$. Because $\boldsymbol{S}$ has the form of a covariance matrix, it is nonnegative semi-definite. We assume hereafter that there are enough data samples that $\boldsymbol{S}$ is full-rank, *i.e.*, its eigenvalues are strictly positive. In particular, by Theorem 1 in [5] [2], $\boldsymbol{S}$ has exactly one positive generalized eigenvalue and it corresponds to the unique local minimum of the Lagrangian. The corresponding eigen vector is the optimal solution to the ellipse parameter in (2.2). Let $\lambda, \boldsymbol{u}$ be the solution to the generalized eigenvalue problem $\boldsymbol{S}\boldsymbol{a} = \lambda \boldsymbol{C}\boldsymbol{a}$ with $\lambda > 0$, then $(\lambda, \boldsymbol{a}^* = \mu \boldsymbol{u})$ is the solution to the constrained minimization problem with

$$\mu = \sqrt{\frac{1}{\boldsymbol{u}^T \boldsymbol{C} \boldsymbol{u}}}.$$

For later convenience, we first provide some notations, then restate the corrected lemma 1 in [5] and prove it. We denote the generalized spectrum as $\sigma(\boldsymbol{A}, \boldsymbol{B}) \stackrel{\triangle}{=} \{\lambda : \boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{B}\boldsymbol{v}\}$. Analogous to the case of a single normal matrix, we define the condition number of a generalized eigen decomposition as $\kappa(\boldsymbol{A}, \boldsymbol{B}) = |\frac{\lambda_{max}(\boldsymbol{A},\boldsymbol{B})}{\lambda_{min}(\boldsymbol{A},\boldsymbol{B})}|$ where $\lambda_{max}(\boldsymbol{A}, \boldsymbol{B})$, $\lambda_{min}(\boldsymbol{A}, \boldsymbol{B})$ denote the maximal and minimal (by moduli) generalized eigenvalues of $(\boldsymbol{A}, \boldsymbol{B})$. The signature $i(\boldsymbol{A})$ of a real symmetric matrix $\boldsymbol{A}$ [3] is the number of positive, negative, and zero eigenvalues of the corresponding matrix [4]

Lemma 1: The number of positive, negative, and zero generalized eigenvalues of $(\boldsymbol{S}, \boldsymbol{C})$, where $\boldsymbol{S} \in \Re_{n\times n}$ is a symmetric [5] positive definite matrix and $\boldsymbol{C} \in \Re_{n\times n}$ is symmetric, are the same as the signature of the constraint matrix $\boldsymbol{C}$.

Proof: Symmetric positive definite matrix $\boldsymbol{S}$ can be decomposed as $\boldsymbol{Q}^T \boldsymbol{Q}$ with $\boldsymbol{Q}$ invertible. We can therefore rewrite the generalized eigen problem $\boldsymbol{S}\boldsymbol{u} = \lambda \boldsymbol{C}\boldsymbol{u}$ as $\boldsymbol{Q}^T \boldsymbol{Q} u = \lambda \boldsymbol{C}\boldsymbol{u}$. Apply a change of basis $\boldsymbol{v} = \boldsymbol{Q}\boldsymbol{u}$ and get $\boldsymbol{v} = \lambda (\boldsymbol{Q}^{-1})^T \boldsymbol{C} \boldsymbol{Q}^{-1} \boldsymbol{v}$. Therefore, λ is the eigenvalue of $(\boldsymbol{Q}^{-1})^T \boldsymbol{C} \boldsymbol{Q}^{-1}$. Let $i(\boldsymbol{C})$ denote the signature of $\boldsymbol{C}$, then by Sylvester's law of inertia [6], which states that the signature of $\boldsymbol{C}$ is invariant under congruence transform, we have $i(\boldsymbol{C}) = i((\boldsymbol{Q}^{-1})^T \boldsymbol{C} \boldsymbol{Q}^{-1})$. Therefore, the number of positive, negative and zeros eigenvalues of $\sigma(\boldsymbol{S}, \boldsymbol{C})$ is the same as the signature of $\boldsymbol{C}$. ■

[2] Both the statement and the proof of lemma 1 in [5] are flawed, but the result in theorem 1 is correct.

[3] Signature is often defined with respect to a symmetric bilinear (quadratic) form; they are the same objects, viewed from different perspective.

[4] In general, eigen decomposition (spectral theorem) applies to normal matrices, which may not necessarily require real symmetry. However, many applications deal with real symmetric matrices, including the one under consideration here.

[5] This is the part missing from [5]

III. Recursive Algorithm for the Generalized Eigen Problem

It is desirable to have an adaptive algorithm in the presence of system variation. In particular, a rank-1 update occurs with the data covariance matrix $\boldsymbol{S}$ every time a new observation becomes available. Assuming sufficient initial data, new data should only mildly perturb the system. Therefore, if we have a recursive algorithm for the static system with enough tolerance to the initial conditions, we shall be confident in using the last state estimate of the one-time-step-lag system as the initialization for the recursion with the new system. In this section, we provide such a recursive algorithm, provide its asymptotic convergence, and discuss the region of attraction.

We first recall some observations from before:

- $\boldsymbol{a}^*$ is identical to the generalized eigen vector $\boldsymbol{u}$ up to a normalization factor that is easy to compute. From now on, we focus on deriving a recursion on the generalized eigen vector, no longer distinguishing between $\boldsymbol{u}$ and $\boldsymbol{a}$, assuming no confusion would result from such notational convenience.
- A simple eigen decomposition of $\boldsymbol{C}$ yields that $\sigma(\boldsymbol{C}) = \{2, 0, 0, 0, -1, -2\}$. From lemma 1, and the minimization setup, we are interested in tracking only the generalized eigen vector that corresponds to the unique positive generalized eigen value of $(\boldsymbol{S}, \boldsymbol{C})$. It is equivalent to finding the eigen vector for the largest eigenvalue (or track the dominant eigen subspace).

We first discuss a method based on fixed-point and generalized Rayleigh quotient to compute the solution to generalized eigen problems $(\boldsymbol{A}, \boldsymbol{B})$ with both $\boldsymbol{A}$,$\boldsymbol{B}$ full rank.

If $\lambda, \boldsymbol{u}$ is the solution to the generalized eigen decomposition problem (proof provided in Appendix):

$$\boldsymbol{A}\boldsymbol{u} = \lambda \boldsymbol{B}\boldsymbol{u}, \tag{3.1}$$

then we can generalize the Rayleigh-Ritz theorem and show (derivation provided in appendix) that the generalized eigen vectors $\boldsymbol{u}$ correspond to the stationary points of the energy ratio function:

$$J(\boldsymbol{u}) = \frac{\boldsymbol{u}^T \boldsymbol{A}\boldsymbol{u}}{\boldsymbol{u}^T \boldsymbol{B}\boldsymbol{u}}, \tag{3.2}$$

and evaluating J at the eigenvectors result in the corresponding generalized eigenvalues.

In fact, the largest generalized eigenvalue λ is the global maximum of J. Substituting $\lambda = J(\boldsymbol{u}) = \frac{\boldsymbol{u}^T \boldsymbol{A}\boldsymbol{u}}{\boldsymbol{u}^T \boldsymbol{B}\boldsymbol{u}}$ in (3.1) yields

$$\boldsymbol{A}\boldsymbol{u} = \frac{\boldsymbol{u}^T \boldsymbol{A}\boldsymbol{u}}{\boldsymbol{u}^T \boldsymbol{B}\boldsymbol{u}} \boldsymbol{A}\boldsymbol{u}.$$

Since we assumed $\boldsymbol{A}$ to be full rank, we multiply the above expression with $\boldsymbol{A}^{-1}$ on both sides to get:

$$\boldsymbol{u} = \frac{\boldsymbol{u}^T \boldsymbol{A}\boldsymbol{u}}{\boldsymbol{u}^T \boldsymbol{B}\boldsymbol{u}} \boldsymbol{A}^{-1} \boldsymbol{B}\boldsymbol{u}. \tag{3.3}$$

Equation (3.3) indicates that the generalized eigen vector is a fixed point for the iterative map

$$f : \boldsymbol{u} \to \frac{\boldsymbol{u}^T \boldsymbol{A}\boldsymbol{u}}{\boldsymbol{u}^T \boldsymbol{B}\boldsymbol{u}} \boldsymbol{A}^{-1} \boldsymbol{B}\boldsymbol{u}. \tag{3.4}$$

Furthermore, the energy ratio function evaluated at the fixed point is exactly the generalized eigenvalue that corresponds to the fixed point $\boldsymbol{u}$.

To use (3.4) to recursively solve (2.4), we first separate the essential subspace from the nuisance ones, by decomposing the empirical correlation matrix $\boldsymbol{S} = \mathbb{X}^T \mathbb{X}$ into block form as follows:

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{E} & \boldsymbol{B} \\ \boldsymbol{B}^T & \boldsymbol{D} \end{bmatrix}.$$

We also define the Schur complement of the block $\boldsymbol{D}$ in matrix $\boldsymbol{S}$ as $\tilde{\boldsymbol{S}} \stackrel{\triangle}{=} \boldsymbol{E} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{B}^T$. The decomposition of the constraint matrix $\boldsymbol{C}$ is given by (2.3).

We provide a recursion to compute the generalized eigen vector $\boldsymbol{a}$ and prove its asymptotic convergence.

Theorem 1: Let the recursion of $\boldsymbol{a}$ be given by

$$\boldsymbol{a}_{n+1} = \eta_n \frac{\boldsymbol{a}_n^T \boldsymbol{W} \boldsymbol{a}_n}{\boldsymbol{a}_n^T \boldsymbol{C} \boldsymbol{a}_n} \boldsymbol{S}^{-1} \boldsymbol{C} \boldsymbol{a}_n + (1 - \eta_n) \boldsymbol{a}_n, \tag{3.5}$$

where

$$\boldsymbol{W} = \begin{bmatrix} \boldsymbol{E} & 0 \\ 0 & -\boldsymbol{D} \end{bmatrix},$$

and $\eta_n \in (0, 1)$ is asymptotically bounded above by $\frac{2}{\kappa+1}$ with κ being the condition number of $(\tilde{\boldsymbol{S}}, \tilde{\boldsymbol{C}})$ [6]. Then $\boldsymbol{a}_n$ converges asymptotically to the eigen vector that corresponds to the unique positive eigenvalue of $(\boldsymbol{S}, \boldsymbol{C})$.

Proof: We can decompose the state estimate $\boldsymbol{a}$ into the concatenation of two vectors $\boldsymbol{a} = [\boldsymbol{a}_1; \boldsymbol{a}_2]$. Then we rewrite (2.4) as:

$$\begin{aligned} \boldsymbol{E}\boldsymbol{a}_1 + \boldsymbol{B}\boldsymbol{a}_2 &= \lambda \tilde{\boldsymbol{C}} \boldsymbol{a}_1 \\ \boldsymbol{B}^T \boldsymbol{a}_1 + \boldsymbol{D}\boldsymbol{a}_2 &= 0. \end{aligned} \tag{3.6}$$

Notice that $\boldsymbol{E}$ is the autocorrelation matrix of the first three dimensions of the observed data, and is invertible by the assumption that $\boldsymbol{S}$ is full rank. Being full rank, $\tilde{\boldsymbol{C}}$ is invertible as well. We can solve the above equation as

$$\begin{aligned} \boldsymbol{a}_2 &= -\boldsymbol{D}^{-1} \boldsymbol{B}^T \boldsymbol{a}_1 \\ (\boldsymbol{E} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{B}^T)\boldsymbol{a}_1 &= \lambda \tilde{\boldsymbol{C}} \boldsymbol{a}_1. \end{aligned} \tag{3.7}$$

In theorem 2, we show that recursion:

$$\begin{aligned} \boldsymbol{a}_{1,n+1} &= \eta_n \frac{\boldsymbol{a}_{1,n}^T \tilde{\boldsymbol{S}} \boldsymbol{a}_{1,n}}{\boldsymbol{a}_{1,n}^T \tilde{\boldsymbol{C}} \boldsymbol{a}_{1,n}} \tilde{\boldsymbol{S}}^{-1} \tilde{\boldsymbol{C}} \boldsymbol{a}_{1,n} + (1 - \eta_n) \boldsymbol{a}_{1,n} \\ &= \eta_n h(\boldsymbol{a}_{1,n}, \tilde{\boldsymbol{S}}, \tilde{\boldsymbol{C}}) + \boldsymbol{a}_{1,n}, \end{aligned} \tag{3.8}$$

where $h(\boldsymbol{x}, \tilde{\boldsymbol{S}}, \tilde{\boldsymbol{C}}) \triangleq \frac{\boldsymbol{x}^T \tilde{\boldsymbol{S}} \boldsymbol{x}}{\boldsymbol{x}^T \tilde{\boldsymbol{C}} \boldsymbol{x}} \tilde{\boldsymbol{S}}^{-1} \tilde{\boldsymbol{C}} \boldsymbol{x} - \boldsymbol{x}$, converges asymptotically to the solution $\boldsymbol{a}_1$ [7] of (3.7). Therefore, by letting the covariant $\boldsymbol{a}_2$ evolve accordingly as:

$$\boldsymbol{a}_{2,n} = -\boldsymbol{D}\boldsymbol{B}^T \boldsymbol{a}_{1,n}, \tag{3.9}$$

we have asymptotic convergence to the only stable stationary point $\boldsymbol{a} = [\boldsymbol{a}_1; \boldsymbol{a}_2]$ of (3.7).

In Lemma 2, we show that the recursion given in (3.8) and (3.9) is identical to the update equation in (3.5). ■

Theorem 2: Recursion according to (3.8) converges to the generalized eigen vector that corresponds to the largest eigenvalue of $(\tilde{\boldsymbol{S}}, \tilde{\boldsymbol{C}})$, where $\tilde{S} = [\boldsymbol{E} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{B}^T]$ is the Schur complement of the block $\boldsymbol{D}$ in $\boldsymbol{S}$.

The second equation in (3.7) simply states that $\boldsymbol{a}_1$ is the generalized eigen vector for the pair $([\boldsymbol{E} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{B}^T], \tilde{\boldsymbol{C}})$. Observe that $[\boldsymbol{E} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{B}^T]$ is exactly the Schur complement $\tilde{\boldsymbol{S}}$ of the block $\boldsymbol{D}$ in matrix $\boldsymbol{S}$. Since $\boldsymbol{S}$ is symmetric positive definite, so is $\tilde{\boldsymbol{S}} = [\boldsymbol{E} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{B}^T]$. Since $\tilde{\boldsymbol{C}}$ is the only block in $\boldsymbol{C}$ that contributes to the nonzero spectral components and $\sigma(\tilde{\boldsymbol{C}}) = \{2, -1, -2\}$, the second equation in (3.7) captures all the nontrivial components of the original generalized eigen decomposition problem (3.1). Using the Sylvester's Law of Inertia, we see immediately that the generalized eigen spectrum of $(\tilde{\boldsymbol{S}}, \tilde{\boldsymbol{C}})$ has the form $\lambda_1 > 0 > \lambda_2 \geq \lambda_3$.

Furthermore, since the generalized spectrum has no zero component, the second equation $\tilde{\boldsymbol{S}}\boldsymbol{a}_1 = \lambda \tilde{\boldsymbol{C}} \boldsymbol{a}_1$ can be rewritten as

$$\lambda^{-1} \tilde{\boldsymbol{S}} \boldsymbol{a}_1 = \tilde{\boldsymbol{C}} \boldsymbol{a}_1,$$

[6] We will see that $\sigma(\tilde{\boldsymbol{S}}, \tilde{\boldsymbol{C}}) \subset \sigma(\boldsymbol{S}, \boldsymbol{C})$ and it is the subset that contains all non-zero generalized eigenvalues. It is often possible to obtain upper bound for κ by utilizing either prior information or proper training. The role of stepsize (gain) η_n determines the trade-off between convergence and convergence rate. The convergence behavior of vanishing gain ($\eta_n \geq 0 \quad \sum_n \eta_n = +\infty, \sum_n \eta_n^\alpha < \infty$ for some $\alpha > 1$) is commonly studied in the literature [7, 8], but asymptotic constant gain ($\eta_n \geq 0 \quad \eta := \lim_{n\to\infty} \eta_n > 0$) is more desirable in practice. The condition we have imposed includes that of the decreasing gain, but also admits cases of asymptotic constant gain.

[7] The eigen vector paired with the biggest eigenvalue when regarded as a generalized eigen decomposition problem from the second equation. The original constraint $\boldsymbol{a}^T \boldsymbol{C} \boldsymbol{a} = 1$ translates to $\boldsymbol{a}_1 \tilde{\boldsymbol{C}} \boldsymbol{a}_1 = 1$, and prevents degenerated results.

which indicates that $\sigma(\tilde{\boldsymbol{C}}, \tilde{\boldsymbol{S}}) = \{\lambda^{-1} | \lambda \in \sigma(\tilde{\boldsymbol{S}}, \tilde{\boldsymbol{C}})\}$ and the generalized eigen vector for $(\tilde{\boldsymbol{C}}, \tilde{\boldsymbol{S}})$ coincides with that for $(\tilde{\boldsymbol{S}}, \tilde{\boldsymbol{C}})$ (up to possibly some positive scaling factor) with the paring determined by the element-wise inversion relation of the spectrum. Again, since there is a unique positive eigenvalue (thus the maximum) of $(\tilde{\boldsymbol{S}}, \tilde{\boldsymbol{C}})$, the generalized eigen decomposition of $(\tilde{\boldsymbol{C}}, \tilde{\boldsymbol{S}})$ has only one positive element as well, whose corresponding generalized eigen vector is of our interest. We will make use of the above observed relationship in the proof of theorem 2.

Proof: We consider (3.8) in the framework of the generic stochastic approximation algorithm [9] $\boldsymbol{a}_{1,n+1} = \boldsymbol{a}_{1,n} + \eta_n h(\boldsymbol{a}_{1,n})$. To apply the corresponding convergence analysis technique, we need to first justify several required assumptions. Viewing (3.8) in the classic adaptive adaptive form, we know precisely the mathematical conditions relating to the objects, in particular, the gain η_n, the function h and the state $(\tilde{\boldsymbol{S}}, \tilde{\boldsymbol{C}})$ (in our case, since we start with the update equation, there is no residual perturbation involved in the evolution, as opposed to the more general form of adaptive updates). In general, the state is represented by a Markov chain controlled by the parameter to be estimated, and it is assumed that for fixed parameter, the state has to be asymptotically stationary, and its limiting behavior regular in the parameter. In our setting, static collective data $\boldsymbol{S}$ (we disregard the given constant $\boldsymbol{C}$) is used, the simple duplicate of which can be regarded as the simplest form of Markov chain if preferred. Therefore, the stationarity and regularity condition with respect to $\boldsymbol{a}_1$ [9] is trivially satisfied. Furthermore, it is easy to check that the update function $h(\boldsymbol{a}_a, \tilde{\boldsymbol{S}}, \tilde{\boldsymbol{C}})$ is continuously differentiable with respect to $\boldsymbol{a}_1$ and regular (locally Lipschitz). Therefore, we are allowed to use ODE based approach as a tool to prove asymptotic convergence. Since $\tilde{\boldsymbol{S}}$ and $\tilde{\boldsymbol{C}}$ are both constant, we omit them as argument of h for notational brevity.

We link a continuous time ODE to the discrete time algorithm (3.8) to a first order approximation by:

$$\begin{aligned} \tilde{h}(\boldsymbol{a}_1(s_n)) &= \frac{\partial}{\partial s}\boldsymbol{a}_1(s)|_{s=s_n} \quad \boldsymbol{a}_{1,n} = \boldsymbol{a}_1(s_n) \\ &\approx \frac{1}{\eta_n}(\boldsymbol{a}_{1,n+1} - \boldsymbol{a}_{1,n}) \\ &= \frac{\boldsymbol{a}_1(s)^T \tilde{\boldsymbol{S}} \boldsymbol{a}_1(s)}{\boldsymbol{a}_1(s)^T \tilde{\boldsymbol{C}} \boldsymbol{a}_1(s)} \tilde{\boldsymbol{S}}^{-1} \tilde{\boldsymbol{C}} \boldsymbol{a}_1(s) - \boldsymbol{a}_1(s). \end{aligned} \tag{3.10}$$

We represent $\boldsymbol{a}_1(s)$ as a linear combination of the generalized eigen vectors of $(\tilde{\boldsymbol{S}}, \tilde{\boldsymbol{C}})$.

$$\boldsymbol{a}_1(s) = \sum_{k=1}^{K} \theta_k(s) \boldsymbol{v}_k. \tag{3.11}$$

Use this parameterization in (3.10), and we get a coordinate-wise (with respect to $\{\boldsymbol{v}\}$) as:

$$\begin{aligned} \frac{\partial}{\partial s}\theta_k(s) &= \frac{\sum_{k=1}^{K} \theta_k(s)^2}{\sum_{k=1}^{K} \frac{1}{\lambda_k}\theta_k(s)^2} \frac{1}{\lambda_k}\theta_k(s) - \theta_k(s) \quad \forall \quad k = 1, 2, \ldots, K \\ &= [\frac{\sum_{k=1}^{K} \theta_k(s)^2}{\sum_{k=1}^{K} \frac{1}{\lambda_k}\theta_k(s)^2} \frac{1}{\lambda_k} - 1]\theta_k(s), \end{aligned} \tag{3.12}$$

where $\lambda_k, \boldsymbol{v}_k$ are the kth generalized eigenvalue and eigenvector of $(\tilde{\boldsymbol{S}}, \tilde{\boldsymbol{C}})$, and $\theta_k(s)$ is the kth time(iteration) varying projection coefficient indicating the strength of $\boldsymbol{a}_1(s)$ along direction $\boldsymbol{v}_k$.

We define a region $\Omega = \{\theta = (\theta_1, \ldots, \theta_K) || \theta_k | \leq \sqrt{\frac{-\lambda_k}{(K-1)\lambda_1}} |\theta_1| \quad \text{for} \quad k > 1\}$.In our case, $K = 3$ and $0 > \lambda_2 \geq \lambda_3$. Therefore, it holds that for any $\theta \in \Omega$, $\frac{\sum_{k=1}^{K} \theta_k(s)^2}{\sum_{k=1}^{K} \lambda_k^{-1}\theta_k(s)^2} \geq 0$. For $k > 1$, $\lambda_k < 0$, and therefore (3.12) states that

$$\frac{\partial}{\partial s}\theta_k(s) = \alpha_k(s)\theta_k(s)$$

with $\alpha_k(s) < 0$ for all $k > 1$. It follows from Lyapunov stability theorem [10] that $\theta_k(s) = 0$ as $s \to \infty$ for $k > 1$. On the other hand, since $\lambda_1 > 0 > \lambda_2 \geq \lambda_3$, we have

$$\sum_{k=1}^{K} \lambda_k^{-1}\theta_k(s)^2 < \lambda_1^{-1}\theta_1(s)^2 < \lambda_1^{-1} \sum_{k=1}^{3} \theta_k(s)^2,$$

therefore,

$$\frac{\partial}{\partial s}\theta_1(s) = [\frac{\sum_{k=1}^{K}\theta_k(s)^2}{\sum_{k=1}^{K}\frac{1}{\lambda_k}\theta_k(s)^2}\frac{1}{\lambda_1} - 1]\theta_1(s) > [\frac{\sum_{k=1}^{K}\theta_k(s)^2}{\lambda_1^{-1}\sum_{k=1}^{3}\theta_k(s)^2}\frac{1}{\lambda_1} - 1]\theta_1(s).$$

Unlike the other modes $k \neq 1$ where origin serves as a stable sink, the magnitude of $\theta_1(s)$ increases as its ODE behaves as

$$\frac{\partial}{\partial s}\theta_1(s) = \alpha_1(s)\theta_k(s),$$

with $\alpha_1(s) > 0$. Therefore, $\boldsymbol{a}_1 \to \boldsymbol{v}_1$ asymptotically.

We have thus far proved the asymptotic convergence if the update follows the ODE, in other words, when the step size η is sufficiently small. Big η corresponds to discretization of the ODE, and may cause discrepancies between the convergence properties of the ODE and the original update equation (3.8). In order to reveal this effect, we need to explore the pole structure of the dynamic system both in continuous and discrete time. We consider the behavior of $\tilde{h}(\boldsymbol{a}_1(s))$ in the neighborhood of the stationary point $\boldsymbol{a}_1(s) = \boldsymbol{v}_k$, where $\boldsymbol{v}_k$ is the k-th generalized eigen vector of $(\tilde{\boldsymbol{S}}, \tilde{\boldsymbol{C}})$. After local linearizion

$$A_k = \frac{\partial}{\partial \boldsymbol{a}_1(s)}\tilde{h}(\boldsymbol{a}_1(s))|_{\boldsymbol{a}_1=\boldsymbol{v}_k},$$

it is easy to see that the eigenvalues of A_k are given by $\sigma(A_k) = \{\frac{\lambda_k}{\lambda_j} - 1\} \quad j \in \{1, 2, \dots K\} \setminus \{k\}$. These are the Laplacian domain poles. The transformation in (3.10) defines a map to Z-domain via $z = \eta s + 1$. There are the following possibilities in mapping the pole patterns in Laplacian domain and Z-domain[8]:

1. When λ_k, λ_j are of the same sign. $s = \frac{\lambda_k}{\lambda_j} - 1 \in (-1, \kappa - 1)$.
   When $s > 0$ (s-pole in RHP), $z = \eta s + 1 > 1$ falls outside the unit circle, which corresponds to locally unstable pattern. When $s \in (-1, 0)$ (s-pole in LHP), $-1 < z < 0$ lies inside the unit circle, stable. $s = 0$ corresponds to $z = 1$, for critical stability.
2. When λ_k, λ_j are of opposite signs, and $s = \frac{\lambda_k}{\lambda_j} - 1 \in (-2, -1) \subset LHP$.
   In this case, $z = \eta s + 1 \in (-1, 0)$ is inside the unit circle, locally stable pattern.
3. When λ_k, λ_j are of opposite signs, and $s = \frac{\lambda_k}{\lambda_j} - 1 < -2 \subset LHP$.
   In this case, the s-pole lies inside the LHP, which corresponds to the local stability. To avoid discrepancy, we want the mapped z-pole to fall inside unit circle. Recall that with $\tilde{\boldsymbol{S}}$ and $\tilde{\boldsymbol{C}}$ both normal, $\kappa(\tilde{\boldsymbol{S}}, \tilde{\boldsymbol{C}}) = |\frac{\lambda_{max}(\tilde{\boldsymbol{S}}, \tilde{\boldsymbol{C}})}{\lambda_{min}(\tilde{\boldsymbol{S}}, \tilde{\boldsymbol{C}})}|$ where λ_{max} and λ_{min} are maximal and minimal (by moduli) eigenvalues of the generalized eigen decomposition. Therefore, $s \geq -\kappa - 1$, and since η is asymptotically upper-bounded by $\frac{2}{\kappa+1}$, $z \in (-1, 1)$ corresponds to local stable pattern.

Therefore, the local stability pattern of the stationary points for the ODE and the update equation (3.8) are the same. This links the convergence of the ODE to that of the discrete-time equation, and asymptotic convergence of $\boldsymbol{a}_1$ is thus proved. ■

Lemma 2: Recursion (3.5) is identical to the set of updates given in (3.8) and (3.9).

Proof: Recall that $\tilde{S}$ is the Schur complement of block $\boldsymbol{D}$ in

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{E} & \boldsymbol{B} \\ \boldsymbol{B}^T & \boldsymbol{D} \end{bmatrix}.$$

From the block form, we get inversion formula as follows:

$$\boldsymbol{S}^{-1} = \begin{bmatrix} \tilde{\boldsymbol{S}}^{-1} & -\tilde{\boldsymbol{S}}^{-1}\boldsymbol{B}\boldsymbol{D}^{-1} \\ -\boldsymbol{D}^{-1}\boldsymbol{B}^T\tilde{\boldsymbol{S}}^{-1} & \boldsymbol{D}^{-1} + \boldsymbol{D}^{-1}\boldsymbol{B}^T\tilde{\boldsymbol{S}}^{-1}\boldsymbol{B}\boldsymbol{D}^{-1} \end{bmatrix} \tag{3.13}$$

[8]This is very different than the commonly seen eigen decomposition of correlation matrices where spectrum is always positive.

If we set aside the generalized Rayleigh quotient in (3.8) and we observe that the major matrix operations involved in the update can be "extracted from (3.13) as:

$$\begin{aligned} \tilde{\boldsymbol{S}}^{-1} &= \left[\begin{array}{cc} \boldsymbol{I} & 0 \end{array}\right] \boldsymbol{S}^{-1} \left[\begin{array}{c} \boldsymbol{I} \\ 0 \end{array}\right] \\ -\boldsymbol{D}\boldsymbol{B}^T\tilde{\boldsymbol{S}}^{-1} &= \left[\begin{array}{cc} 0 & \boldsymbol{I} \end{array}\right] \boldsymbol{S}^{-1} \left[\begin{array}{c} \boldsymbol{I} \\ 0 \end{array}\right]. \end{aligned} \tag{3.14}$$

Notice that the sparse structure in $\boldsymbol{C}$ induces the following relations:

$$\begin{aligned} \boldsymbol{C}\boldsymbol{a} &= \left[\begin{array}{c} \tilde{C}\boldsymbol{a}_1 \\ 0 \end{array}\right] = \left[\begin{array}{c} \boldsymbol{I} \\ 0 \end{array}\right] \tilde{C}\boldsymbol{a}_1 \\ \boldsymbol{a}^T\boldsymbol{C}\boldsymbol{a} &= \boldsymbol{a}_1^T\tilde{\boldsymbol{C}}\boldsymbol{a}_1 \end{aligned} \tag{3.15}$$

Therefore, we can rewrite the recursion in (3.8) as:

$$\boldsymbol{a}_{:,n+1} = \eta_n\lambda_n\boldsymbol{S}^{-1}\boldsymbol{C}\boldsymbol{a}_{:,n} + (1-\eta_n)\boldsymbol{a}_{:,n}, \tag{3.16}$$

where λ_n denotes the generalized Rayleigh quotient estimated in the n-th iteration.

Furthermore, we rewrite the numerate of the generalized Rayleigh quotient as:

$$\begin{aligned} \boldsymbol{a}_1^T\tilde{\boldsymbol{S}}\boldsymbol{a}_1 &= \boldsymbol{a}_1^T[\boldsymbol{E} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{B}^T]\boldsymbol{a}_1 \\ &= \boldsymbol{a}_1^T\boldsymbol{E}\boldsymbol{a}_1 - \boldsymbol{a}_1\boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{D}\boldsymbol{D}^{-1}\boldsymbol{B}^T\boldsymbol{a}_1 \\ &= \boldsymbol{a}_1^T\boldsymbol{E}\boldsymbol{a}_1 - \boldsymbol{a}_2\boldsymbol{D}\boldsymbol{a}_2, \end{aligned} \tag{3.17}$$

by the dependence of $\boldsymbol{a}_2$ on $\boldsymbol{a}_1$ indicated in (3.9).

Putting the above ingredients back into the recursion, we end up with the compact representation as given in (3.5). ■

IV. Adaptive Ellipse Fitting

In the ellipse fitting setup, the matrix $\boldsymbol{C}$ describes the shape prior (constraint) of the data, and is therefore constant. Whenever new data becomes available, the system dynamics is reflected via changes in the empirical covariance matrix $\boldsymbol{S}$. Formulation (3.5) expresses the update of the state estimate in terms of sub-blocks and inverse of $\boldsymbol{S}$ directly (with no hidden or intermediate transformations as in (3.8)). This enables straightforward derivation for update equations when $\boldsymbol{S}$ varies upon the arrival of new data sample. Notice that $\boldsymbol{S}$ takes on the form of empirical covariance, so the diagonal sub-blocks $\boldsymbol{E}$ and $\boldsymbol{D}$ are empirical covariance matrix with respect to their own subspace and are completely decoupled as can be seen from (3.17). In practice, the update of the generalized Rayleigh quotient can be update accordingly. In essence, the only quantity of real concern in updating state estimate is $\boldsymbol{S}^{-1}$. To incorporate the time varying property of the system, we can simply extend the previous results with a hyper-level evolving time tag.

For fixed time level t (same data),

$$\boldsymbol{a}_{n+1}(t) = \eta_n\frac{\boldsymbol{a}_n(t)^T\boldsymbol{W}(t)\boldsymbol{a}_n(t)}{\boldsymbol{a}_n(t)^T\boldsymbol{C}\boldsymbol{a}_n(t)}\boldsymbol{S}(t)^{-1}\boldsymbol{C}\boldsymbol{a}_n(t) + (1-\eta_n)\boldsymbol{a}_n(t) \quad n = 0, 1, \ldots \tag{4.1}$$

When a new data becomes available ($t \to t+1$),

$$\boldsymbol{a}_0(t+1) = \eta_0\frac{\boldsymbol{a}_{N_t}(t)^T\boldsymbol{W}(t+1)\boldsymbol{a}_{N_t}(t)}{\boldsymbol{a}_{N_t}(t)^T\boldsymbol{C}\boldsymbol{a}_{N_t}(t)}\boldsymbol{S}(t+1)^{-1}\boldsymbol{C}\boldsymbol{a}_{N_t}(t) + (1-\eta_0)\boldsymbol{a}_{N_t}(t) \tag{4.2}$$

where $\boldsymbol{a}_{N_t}(t)$ is the state estimate obtained from the last inner iteration with input parameters determined by $\boldsymbol{S}(t)$. This can be looked on as initializing the new state estimate corresponding to $\boldsymbol{S}(t+1)$ with the old estimate from the "old" data. Since natural system normally involves smoothly, it is safe to assume that eigen structure of $(\boldsymbol{S}(t+1), \boldsymbol{C})$ resembles that of $(\boldsymbol{S}(t), \boldsymbol{C})$, which makes (4.2) reasonable.

Specific updates depend the definition of $\boldsymbol{S}(t)$ and correspondingly its variation over time. We discuss different adaptive methods respectively.

A. Sliding Window Adaptation

In the first adaptive method, we use a constant length sliding window to "mask out" the historical data samples except the ones that are close enough to the time instance of interest. We use $\mathbb{X}(t)$ to denote the collective data samples that are taken into account for time instant t. Without loss of generality, we assume $\mathbb{X}$ has length $L \in \mathbb{N}$ and is of the form:

$$\mathbb{X}(t) = \begin{bmatrix} \boldsymbol{x}_{t-L+1}^T \\ \boldsymbol{x}_{t-L+2}^T \\ \vdots \\ \boldsymbol{x}_t^T \end{bmatrix}. \tag{4.3}$$

Correspondingly, we define $\boldsymbol{S}(t) = \mathbb{X}(t)^T\mathbb{X}(t) = \sum_{i=t-L+1}^{t} \boldsymbol{x}_i\boldsymbol{x}_i^T$ and we aim to derive a recursion from $\boldsymbol{a}(t)$ to $\boldsymbol{a}(t+1)$. It is easy to check that the derivation for $\boldsymbol{S}(t) = \frac{1}{L}\mathbb{X}(t)^T\mathbb{X}(t)$ end up with the same update equation with L constant.

When a new sample pair $\boldsymbol{x}_{t+1}$ becomes available,

$$\begin{aligned} \boldsymbol{S}(t+1) &= \sum_{i=t+1-L+1}^{t+1} \boldsymbol{x}_i\boldsymbol{x}_i^T \\ &= \boldsymbol{S}(t) - \boldsymbol{x}_{t-L+1}\boldsymbol{x}_{t-L+1}^T + \boldsymbol{x}_{t+1}\boldsymbol{x}_{t+1}^T. \end{aligned} \tag{4.4}$$

To compute $\boldsymbol{S}(t+1)^{-1}$ from $\boldsymbol{S}(t)^{-1}$, we use two rank one updates. Let $\boldsymbol{Q}(t) = S(t) - \boldsymbol{x}_{t-L+1}\boldsymbol{x}_{t-L+1}^T$, and $\boldsymbol{S}(t+1) = \boldsymbol{Q}(t) + \boldsymbol{x}_{t+1}\boldsymbol{x}_{t+1}^T$, then

$$\begin{aligned} \boldsymbol{Q}(t)^{-1} &= (\boldsymbol{S}(t) - \boldsymbol{x}_{t-L+1}\boldsymbol{x}_{t-L+1}^T)^{-1} \\ &= \boldsymbol{S}(t)^{-1} - \boldsymbol{S}^{-1}\boldsymbol{x}_{t-L+1}(\boldsymbol{x}_{t-L+1}^T\boldsymbol{S}(t)^{-1}\boldsymbol{x}_{t-L+1} - 1)^{-1}\boldsymbol{x}_{t-L+1}^T\boldsymbol{S}^{-1} \qquad (4.5) \\ \boldsymbol{S}(t+1)^{-1} &= (\boldsymbol{Q}(t) + \boldsymbol{x}_{t+1}\boldsymbol{x}_{t+1}^T)^{-1} \\ &= \boldsymbol{Q}(t)^{-1} - \boldsymbol{Q}^{-1}\boldsymbol{x}_{t+1}(\boldsymbol{x}_{t+1}^T\boldsymbol{Q}(t)^{-1}\boldsymbol{x}_{t+1} + 1)^{-1}\boldsymbol{x}_{t+1}^T\boldsymbol{Q}^{-1}. \qquad (4.6) \end{aligned}$$

For this procedure to be executable, invertibility of $(\boldsymbol{x}_{t-L+1}^T\boldsymbol{S}(t)^{-1}\boldsymbol{x}_{t-L+1}-1)$ and $(\boldsymbol{x}_{t+1}^T\boldsymbol{Q}(t)^{-1}\boldsymbol{x}_{t+1}+1)$ are required. The second one is obvious with $\boldsymbol{Q} > 0$. We prove the first condition in appendix. For this procedure executable, We have thus provided a recursion in $\boldsymbol{S}(t)^{-1}$. A recursion in the estimation parameters $\boldsymbol{a}(t+1)$ can be obtained by plugging the result from (4.6) into (3.5).

B. Discounting Adaptation

We now consider the alternative to a fixed-length sliding window by using discount factor $\gamma \in (0, 1)$. At time instant t, the collection of $\mathbb{X}$ is of the form:

$$\mathbb{X}(t) = \begin{bmatrix} \boldsymbol{x}_1^T \\ \vdots \\ \boldsymbol{x}_{t-1}^T \\ \boldsymbol{x}_t^T \end{bmatrix}. \tag{4.7}$$

To be compliant with the assumption that the arrival of a new data point does not introduce unnecessary perturbation in the generalized eigenvector if $(\boldsymbol{S}(t), \boldsymbol{C})$, we normalize $\boldsymbol{S}$ properly. Notice that we are only interested in the Eigen vector of the decomposition, so the normalization does not change the state estimate at convergence. We define $\boldsymbol{S}(t) = \frac{1-\gamma}{1-\gamma^t}\sum_{i=1}^{t}(\gamma)^{t-i}\boldsymbol{x}_i\boldsymbol{x}_i^T$, and we can easily write $\boldsymbol{S}(t+1)$ as:

$$\frac{1-\gamma^{t+1}}{1-\gamma}\boldsymbol{S}(t+1) = \gamma\frac{1-\gamma^t}{1-\gamma}\boldsymbol{S}(t) + \boldsymbol{x}_{t+1}\boldsymbol{x}_{t+1}^T \tag{4.8}$$

Again, we invoke the matrix inversion lemma and get

$$\boldsymbol{S}(t+1)^{-1} = \frac{1-\gamma^{t+1}}{\gamma-\gamma^{t+1}}\boldsymbol{S}(t)^{-1} - \boldsymbol{S}(t)^{-1}\boldsymbol{x}_{t+1}(\frac{1-\gamma}{\gamma-\gamma^{t+1}} + \boldsymbol{x}_{t+1}^T\boldsymbol{S}(t)^{-1}\boldsymbol{x}_{t+1})^{-1}\boldsymbol{x}_{t+1}^T\boldsymbol{S}(t)^{-1}. \tag{4.9}$$

V. Applications and Experiment

Ellipse fitting algorithms have seen wide application in computer vision, especially in human face tracking

VI. Appendix

- Proof for the statement about stationary points of the energy ratio function in (3.2).
  Claim 1: The generalized eigen vectors of $(\boldsymbol{A}, \boldsymbol{B})$ correspond to the stationary points of the energy ratio function

$$J(\boldsymbol{u}) = \frac{\boldsymbol{u}^T \boldsymbol{A}\boldsymbol{u}}{\boldsymbol{u}^T \boldsymbol{B}\boldsymbol{u}}.$$

  Proof: We compute the stationary point of the energy ratio function $J(\boldsymbol{u}$, *i.e.*, we set:

$$\frac{\partial}{\partial \boldsymbol{u}} J(\boldsymbol{u}) = \mathbf{0}^T. \tag{6.1}$$

  The derivative on the LHS of (6.1) turns out to be:

$$\frac{\partial}{\partial \boldsymbol{u}} J(\boldsymbol{u}) = \frac{1}{\boldsymbol{u}^T \boldsymbol{B}\boldsymbol{u}} [\boldsymbol{A}\boldsymbol{u}\boldsymbol{u}^T \boldsymbol{B}\boldsymbol{u} - \boldsymbol{B}\boldsymbol{u}\boldsymbol{u}^T \boldsymbol{A}\boldsymbol{u}.$$

  With $\boldsymbol{A} > 0$, $\boldsymbol{B} > 0$ as previously assumed, so that $\boldsymbol{u}^T \boldsymbol{A}\boldsymbol{u}$ and $\boldsymbol{u}^T \boldsymbol{B}\boldsymbol{u}$ are simply positive scalers, setting the above expression to zero is equivalent to requiring

$$\boldsymbol{u}^T \boldsymbol{B}\boldsymbol{u}\boldsymbol{A}\boldsymbol{u} = \boldsymbol{u}^T \boldsymbol{A}\boldsymbol{u}\boldsymbol{B}\boldsymbol{u}.$$

  This is exactly the condition for generalized eigen decomposition, as

$$\boldsymbol{A}\boldsymbol{u} = \frac{\boldsymbol{u}^T \boldsymbol{A}\boldsymbol{u}}{\boldsymbol{u}^T \boldsymbol{B}\boldsymbol{u}} \boldsymbol{B}\boldsymbol{u}.$$

  Thus the generalized eigen vectors are the stationary points for the energy ratio function $J(\mu)$ and moreover, the evaluated functional values provide the corresponding generalized eigenvalues. This result can be considered as a generalization of the Rayleigh-Ritz theorem. ■
- Derivation for (3.12) We make use of the relationship between the generalized eigen decomposition $(\tilde{\boldsymbol{S}}, \tilde{\boldsymbol{C}})$ and $(\tilde{\boldsymbol{C}}, \tilde{\boldsymbol{S}})$. Up to a constant gain, the set of generalized eigen vectors of the two problems coincide, paring with element-wise inverted spectrum. Since we aim to prove the convergence of the coefficients of the eigen vectors either to zero or really large, the constant scaling can be neglected for the sake of argument clarity. We use the alternative setup of $(\tilde{\boldsymbol{C}}, \tilde{\boldsymbol{S}})$ in deriving (3.12) to take advantage of the assumed positive definiteness of $\tilde{\boldsymbol{S}}$. It is a known property that the generalized eigen decomposition of $(\boldsymbol{A}, \boldsymbol{B})$ with $\boldsymbol{B}$ being symmetric positive definite, then the generalized eigen matrix $\boldsymbol{V}$ (with columns being generalized eigen vectors) can simultaneously diagonalize $\boldsymbol{A}$ and $\boldsymbol{B}$:

$$\begin{aligned} \boldsymbol{A}\boldsymbol{V} &= \Lambda \boldsymbol{B}\boldsymbol{V} \\ \boldsymbol{V}^T \boldsymbol{B}\boldsymbol{V} &= \boldsymbol{I} \\ \boldsymbol{V}^T \boldsymbol{A}\boldsymbol{V} &= \Lambda, \end{aligned} \tag{6.2}$$

  where Λ is the diagonal matrix whose diagonal elements are the corresponding generalized eigenvalues of $(\boldsymbol{A}, \boldsymbol{B})$. Indeed, the use of two-stage conventional eigen decomposition to compute generalized eigen decomposition reflects exactly this property. We apply this to $(\tilde{\boldsymbol{C}}, \tilde{\boldsymbol{S}})$ and call their eigen matrix $\boldsymbol{V}$. Again, $\boldsymbol{V}$ is also an eigen matrix for $(\tilde{\boldsymbol{S}}, \tilde{\boldsymbol{C}})$.
  The linear representation in (3.11) can be rewritten as:

$$\boldsymbol{a}_1(s) = \boldsymbol{V}\theta(s),$$

  where $\theta(s) = [\theta_1(s), \theta_2(s), \dots]^T$.

Substituting in the relevant terms in (3.10), we have

$$\boldsymbol{a}_1(s)^T\tilde{\boldsymbol{S}}\boldsymbol{a}_1(s) = \theta(s)^T\boldsymbol{V}^T\boldsymbol{S}\boldsymbol{V}\theta(s) = \sum_{k=1}^{K}\theta_k(s)^2$$

$$\boldsymbol{a}_1(s)^T\tilde{\boldsymbol{C}}\boldsymbol{a}_1(s) = \theta(s)^T\boldsymbol{V}^T\boldsymbol{C}\boldsymbol{V}\theta(s) = \theta(s)^T\Lambda^{-1}\theta(s) = \sum_{k=1}^{K}\lambda_k^{-1}\theta_k(s)^2$$

$$\tilde{\boldsymbol{S}}^{-1}\tilde{\boldsymbol{C}}\boldsymbol{a}_1(s) = \sum_{k=1}^{K}\lambda_k^{-1}\theta_k(s) \tag{6.3}$$

The second and third line in the above derivation also uses the element-wise inversion relation between the spectra of $(\tilde{\boldsymbol{C}}, \tilde{\boldsymbol{S}})$ and $(\tilde{\boldsymbol{S}}, \tilde{\boldsymbol{C}})$.
Representing both the LHS and RHS of (3.10) with respect to the basis $\{v_k\}_{k=1,2,\ldots,K}$, and we have coordinate-wise equation (3.12).

- Proof of invertibility in (4.6).
  Claim 2: Let $\boldsymbol{Q}(t) = \boldsymbol{S}(t) - \boldsymbol{x}_{t-L+1}\boldsymbol{x}_{t-L+1}^T$, then $(\boldsymbol{x}_{t-L+1}^T\boldsymbol{S}(t)^{-1}\boldsymbol{x}_{t-L+1} - 1)$ is invertible.

  Proof: Proving the invertibility of a scaler quantity is the same as the condition that it is none-zero. We rewrite the relation between $\boldsymbol{S}$ and $\boldsymbol{Q}$ as: $\boldsymbol{S}(t) = \boldsymbol{Q}(t) + \boldsymbol{x}_{t-L+1}\boldsymbol{x}_{t-L+1}^T$. Invoking the Woodbury matrix inversion lemma, we get:

$$\boldsymbol{S}(t)^{-1} = \boldsymbol{Q}(t)^{-1} - \boldsymbol{Q}(t)^{-1}\boldsymbol{x}_{t-L+1}(\boldsymbol{x}_{t-L+1}^T\boldsymbol{Q}(t)^{-1}\boldsymbol{x}_{t-L+1} + 1)^{-1}\boldsymbol{x}_{t-L+1}^T\boldsymbol{Q}(t)^{-1}. \tag{6.4}$$

  Plugging (6.4) into $\boldsymbol{x}_{t-L+1}^T\boldsymbol{S}^{-1}\boldsymbol{x}_{t-L+1} - 1$ yields:

$$\begin{aligned}
&\boldsymbol{x}_{t-L+1}^T\boldsymbol{S}^{-1}\boldsymbol{x}_{t-L+1} - 1 \\
&= \boldsymbol{x}_{t-L+1}^T[\boldsymbol{Q}(t)^{-1} - \boldsymbol{Q}(t)^{-1}\boldsymbol{x}_{t-L+1}(\boldsymbol{x}_{t-L+1}^T\boldsymbol{Q}(t)^{-1}\boldsymbol{x}_{t-L+1} + 1)^{-1}\boldsymbol{x}_{t-L+1}^T\boldsymbol{Q}(t)^{-1}]\boldsymbol{x}_{t-L+1} + 1) - 1 \\
&= \boldsymbol{x}_{t-L+1}^T\boldsymbol{Q}(t)^{-1}\boldsymbol{x}_{t-L+1} - \boldsymbol{x}_{t-L+1}^T\boldsymbol{Q}(t)^{-1}\boldsymbol{x}_{t-L+1}\frac{1}{\boldsymbol{x}_{t-L+1}^T\boldsymbol{Q}(t)^{-1}\boldsymbol{x}_{t-L+1} + 1}\boldsymbol{x}_{t-L+1}^T\boldsymbol{Q}(t)^{-1}\boldsymbol{x}_{t-L+1} \quad (6.5)
\end{aligned}$$

  Let $p \stackrel{\triangle}{=} \boldsymbol{x}_{t-L+1}^T\boldsymbol{Q}(t)^{-1}\boldsymbol{x}_{t-L+1}$, then $p > 0$ as $\boldsymbol{Q}(t) > 0$. We rewrite the expression in (6.5) in terms of p and get

$$p - p \times \frac{1}{p+1}p - 1 = \frac{1}{p+1}[p(p+1) - p^2 - (p+1)] < 0.$$

  This result states that $\boldsymbol{x}_{t-L+1}^T\boldsymbol{S}(t)^{-1}\boldsymbol{x}_{t-L+1} - 1 < 0$, thus invertible. In fact, the negativity of this term is not accidental, but a natural result from the consistent relation stated below. If $\boldsymbol{x}_{t-L+1}^T\boldsymbol{S}(t)^{-1}\boldsymbol{x}_{t-L+1} - 1$ then with two different use of the matrix inversion lemma (one expresses $\boldsymbol{S}^{-1}$ with $\boldsymbol{Q}^{-1}$ and anther the other way around), we get:

$$\begin{aligned}
\boldsymbol{Q}(t)^{-1} - \boldsymbol{S}(t)^{-1} &= \boldsymbol{Q}(t)^{-1}\boldsymbol{x}_{t-L+1}(\boldsymbol{x}_{t-L+1}^T\boldsymbol{Q}(t)^{-1}\boldsymbol{x}_{t-L+1} + 1)^{-1}\boldsymbol{x}_{t-L+1}^T\boldsymbol{Q}(t)^{-1} \quad (6.6) \\
&= -\boldsymbol{S}(t)^{-1}\boldsymbol{x}_{t-L+1}(\boldsymbol{x}_{t-L+1}^T\boldsymbol{S}(t)^{-1}\boldsymbol{x}_{t-L+1} - 1)^{-1}\boldsymbol{x}_{t-L+1}^T\boldsymbol{S}(t)^{-1}. \quad (6.7)
\end{aligned}$$

  Because $Q > 0$, the RHS of line 1 in (6.7) is positive definite. With the minus sign in the front and its quadratic form, line 2 in (6.7) indicates that $\boldsymbol{x}_{t-L+1}^T\boldsymbol{S}(t)^{-1}\boldsymbol{x}_{t-L+1} - 1 < 0$ if it is ever nonzero (otherwise (6.7) cannot be established in the first place).

■

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