ANISOTROPIC DIFFUSION EQUATIONS FOR ADAPTIVE QUADRATIC REPRESENTATIONS

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ABSTRACT

Adaptive diffusion techniques for processing time-frequency representations were first proposed by Payot and Gonçalvès in 1998 as an application of the Perona and Malik adaptive diffusion. In this communication we consider both this technique and the anisotropic diffusion of Weickert, which allows to tune orientation and shape of smoothing kernels. We propose a new adaptive diffusion scheme where the strength and the orientation of the anisotropic kernel are locally tailored to the processed time-frequency representation. We provide a comparison with other signal-dependent techniques. Finally we define a diffusion tensor that can be used to process time-frequency representations of the affine class, ensuring the preservation of their covariance properties.

1. INTRODUCTION

Because time-frequency representations (TFR) illustrate evolutions of signals with respect to both time and frequency, they have been largely used to deal with non-stationary environment. Among the host of solutions that have been proposed, Cohen class encloses bilinear TFR that are covariant with respect to time shifts and frequency shifts. Such tools lead to sharper representations of a signal than linear-based approaches, e.g., spectrograms, but at the cost of undesirable cross-terms [1]. One main goal of time-frequency smoothing is to improve readability by removing these cumbersome cross-terms while preserving the sharpness of signal terms. Here we present a way to iteratively process representations with a smoothing kernel whose shape and width can be tailored to the underlying representation. It consists of a diffusion-based technique [2, 3] extending previous works dedicated to the Cohen class [4] and the affine class [5].

In a first part we present both isotropic and anisotropic homogeneous diffusion as smoothing techniques for the Cohen class. We relate these non-adaptive techniques with spectrograms. In a second part we turn to adaptive approaches and describe the method presented by Gonçalvès and Payot in [4]. As it acts on the strength of the diffusion but not on the shape of the smoothing kernel, we then propose an anisotropic generalization. It enables to act on both the orientation and the width of the smoothing kernel depending on a priori information about the processed signal. We illustrate its performances and compare it to other signal-dependent methods. In a third part, as another illustration of the flexibility of anisotropic diffusion, we turn to the affine class and propose a diffusion tensor that garanties the preservation of the affine covariance, yielding a new technique for such a class.

2. HOMOGENEOUS DIFFUSION

Here we consider isotropic and anisotropic diffusion. In this homogeneous setting, since the action of the diffusion is homogeneous over the whole time-frequency plane, smoothing is not adapted to the underlying representation.

2.1. Isotropic diffusion

Among Cohen class, the spectrogram is a widely used tool. As the square modulus of the short-time Fourier transform, it can also be written as a convolution between the Wigner distribution of the signal and that of the analyzing window. Note that the Wigner distribution of a gaussian window is a 2D-gaussian kernel.

Interpreting a time-frequency representation as a temperature distribution one can consider its diffusion as follows:

$$\begin{cases} D_x(t,f;\tau = 0) = W_x(t,f) \\ \frac{\partial D_x(t,f;\tau)}{\partial \tau} = \mathrm{div}_{t,f}(\nabla_{t,f} D_x(t,f;\tau)), \end{cases} \quad (1)$$

where W_x is the representation to be processed, which plays the role of the initial state of the diffusion process. The

diffused representation $D_x(t, f; \tau)$ denotes the energy distribution at the time instant τ. It is well known that the fundamental solution of such classical heat equation is an isotropic gaussian. Therefore the equation (1) has the following solution:

$$D_x(t, f; \tau) = \frac{1}{4\pi\tau} \iint W_x(\eta, \nu) e^{-\frac{(t-\eta)^2 + (f-\nu)^2}{4\tau}} \, d\eta d\nu$$

(2)

Indeed the use of the heat equation on a Wigner distribution is equivalent to convolving it with a gaussian kernel whose variance increases with the diffusion time τ. This diffusion scheme is called isotropic since the kernel in (2) is isotropic. Note that the convolution form of the solution ensures the preservation of covariance with respect to time and frequency shifts.

2.2. Anisotropic diffusion

One can modifie the diffusion process described above to introduce anisotropy in the equivalent gaussian kernel. Following [3], we use a diffusion tensor that acts on the shape of the kernel. The model (1) becomes

$$\begin{cases} D_x(t, f; \tau = 0) = W_x(t, f) \\ \frac{\partial D_x(t, f; \tau)}{\partial \tau} = \text{div}_{t,f}(B \nabla_{t,f} D_x(t, f; \tau)). \end{cases}$$

(3)

Here the diffusion tensor B is a 2-by-2 matrix that allows the diffusion to be different along both time and frequency axes. More precisely, one can check that the fundamental solution of such diffusion process is a gaussian kernel with a covariance matrix depending on B^{-1} and τ. Denoting $z = (t, f)$ and $y = (t', f') \in \mathbb{R}^2$, the solution of (3) is then:

$$D_x(z; \tau) = \frac{1}{4\pi\sqrt{det(\tau B)}} \int W(y) e^{-\frac{(z-y)^T(\tau B)^{-1}(z-y)}{4}} \, dy$$

Indeed with a careful choice of the tensor B, the smoothing can be adapted independently along the time and the frequency axes. Note that choosing B as the identity matrix leads to an isotropic diffusion as presented before. The main requirement for this diffusion to be stable is the positive definiteness of the tensor B. One can also design tensors B with eigenvectors that are not aligned with the time and frequency axes, yielding distributions that can be related to spectrograms with linearly modulated gaussian windows.

3. ADAPTIVE DIFFUSION

In this section, we show that anisotropy of diffusion can be used to tailor smoothing to the local characteristics of the distribution.

3.1. Adaptive isotropic diffusion

The first mention of adaptive diffusion for time-frequency processing can be found in the work of Payot and Gonçalvès [4]. They propose to use a conductance function $b(t, f)$ to locally control the action of diffusion as follows:

$$\begin{cases} D_x(t, f; \tau = 0) = W_x(t, f) \\ \frac{\partial D_x(t, f; \tau)}{\partial \tau} = \text{div}_{t,f}(b(t, f) \nabla_{t,f} D_x(t, f; \tau)). \end{cases}$$

(4)

The selection of the conductance function depends on the application on sight and on available a priori information. In a context of signal analysis, it can be used to selectively smooth cross-terms while preserving auto-terms [4]. For the use of diffusion in a decision making context, one should refer to [6]. Following Weickert terminology in contrast with Perona and Malik, note that such a diffusion is called isotropic since b is scalar.

Using the anisotropic setting presented in Section 2, we shall now extend (4) to obtain a diffusion that is oriented along the components of the analyzed distribution and tailored via a priori information.

3.2. Anisotropic adaptive diffusion

As presented before, the anisotropic diffusion provides an oriented smoothing. Some signals, such as linearly modulated signals for example, exhibit strongly oriented time-frequency components. This suggests that including such a knowledge in the diffusion process should lead to a technique that is more suited to the signal. In order to determine the orientation needed for the smoothing, one has to estimate the orientation of time-frequency components. We propose to use the gradient of the distribution ∇D for such an estimation. Let P be the matrix whose first and second columns are the gradient vector and its orthogonal. As the mesure of orientation provided by ∇D may be very sensitive to noise, averaging and normalizing it gives a local mesure of the strongest orientation $\tilde{P}$, see [3] for more details. We then design the conductance tensor B as follows:

$$B = \tilde{P} \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \tilde{P}^T$$

(5)

with $\lambda > 1$ in order to emphasize diffusion along the direction orthogonal to the gradient.The diffusion is then applied according to

$$\begin{cases} D_x(t, f; \tau = 0) = W_x(t, f) \\ \frac{\partial D_x(t, f; \tau)}{\partial \tau} = \text{div}_{t,f}((bB)(t, f) \nabla_{t,f} D_x(t, f; \tau)), \end{cases}$$

(6)

with b a conductance function as in (4). Note that this technique preserves membership to the Cohen class since it only involves derivatives of the distribution.

(a) Wigner distribution   (b) Spectrogram   (c) Reassigned spectrogram

(d) Radially gaussian kernel   (e) Adaptive diffusion   (f) Anisotropic adaptive diffusion
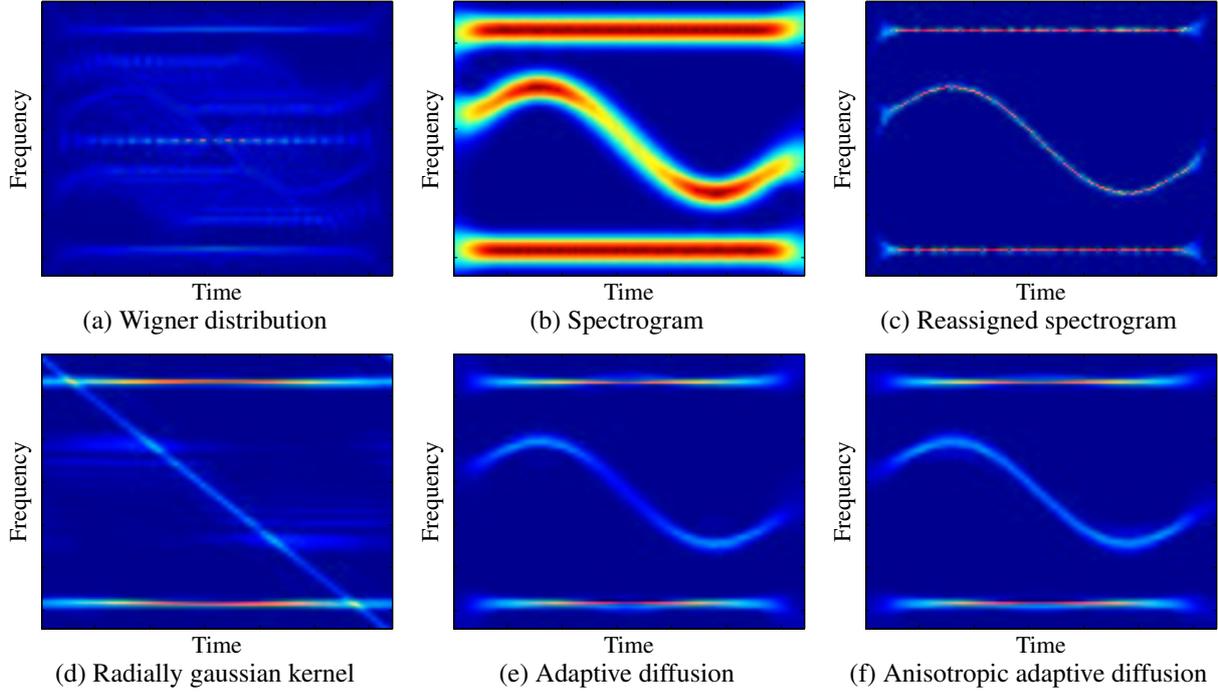
Fig. 1. Comparison between different TFR of the same signal composed of two chirps and sinusoidal modulation.

3.3. Example and comparison

To illustrate this technique we consider a signal composed of two chirps and a sinusoidal modulation in a context of readability improvement. The goal is to remove the interference terms of the Wigner distribution without loosing sharpness of the signal terms. We know that usually spectrograms do not suffer from interference terms. As suggested in [4], we then can use this information to design $b(t, f)$ as follows

$$b(t,f) = \left(1 + \left(\frac{Sp_x(t,f)}{\delta}\right)^{\alpha}\right)^{-1}, \qquad (7)$$

where Sp_x is the spectrogram of the signal. Figures (1.a) and (1.b) illustrate the trade off between sharpness of signal terms and readability. In order to highlight the performances of the anisotropic adaptive diffusion compared to other adaptive methods, we show in Fig. (1.c) a reassigned spectrogram [7] and, in Fig. (1.d), a distribution processed with the optimal radially gaussian kernel [8]. In Fig. (1.f), anisotropic diffusion ($\lambda = 20$) enforces smoothing along the time-frequency components yielding a representation of the sinusoidal modulation that is sharper than in Fig. (1.e).

4. ANISOTROPIC AFFINE DIFFUSION

One can also use the anisotropic diffusion scheme to ensure properties for the smoothed representation. As a second illustration, we then turn to the affine class. As a reminder, we first briefly present the affine operator which underlies the affine class. Next we propose a diffusion tensor B ensuring preservation of affine covariance during diffusion.

4.1. The affine class

The affine class is based on the affine operator [9], here denoted as $\mathcal{A}$. This class encompasses all the distributions that reflect the application of this operator to the signal. The affine operator acts on the set $L^2(\mathbb{R})$ as $\mathcal{A}x(t) = \sqrt{|a_0|}x\left((t - t_0)/a_0\right)$, where t_0 is the amount of time shifting and a_0 the amount of dilatation of the signal. A time-frequency representation Ω_x is covariant with respect to this operator if it obeys the following relation:

$$\Omega_{\mathcal{A}x}(t,f) = \Omega_x\left(\frac{t - t_0}{a_0}, a_0 f\right). \qquad (8)$$

As both the isotropic and the anisotropic diffusion are linear, one can check that the affine covariance, central for the affine class, is not preserved during such diffusion processes.

4.2. Affine diffusion

Using a diffusion tensor that is function of the frequency, we can design a diffusion scheme that keeps this covariance untouched. In order to design such a scheme, we propose to

use the following anisotropic affine diffusion:

$$\begin{cases} D_x(t,f;\tau=0) = W_x(t,f) \\ \frac{\partial D_x(t,f;\tau)}{\partial \tau} = \text{div}_{t,f}\left(\begin{pmatrix} f^{-2} & 0 \\ 0 & f^2 \end{pmatrix}\nabla_{t,f}D_x(t,f;\tau)\right). \end{cases}$$
(9)

Here B is no longer constant but now consists of a diagonal matrix with eigenvalues f^{-2} and f^2. The choice of this diffusion tensor ensures preservation of the affine covariance as it is now proved. At $\tau = 0$, the covariance is satisfied as the processed Wigner distribution belongs to the affine class. Moreover the diffusion term can be developed as follows:

$$\begin{aligned} &\frac{\partial D_x(t,f;\tau)}{\partial \tau} \\ &= \text{div}_{t,f}(B\nabla_{t,f}D_x(t,f;\tau)) \\ &= \text{div}_{t,f}\left(f^{-2}\frac{\partial D_x}{\partial t}(t,f)\vec{u_t} + f^2\frac{\partial D_x}{\partial f}(t,f)\vec{u_f}\right) \\ &= f^{-2}\frac{\partial^2 D_x}{\partial t^2}(t,f) + f^2\frac{\partial^2 D_x}{\partial f^2}(t,f) + 2f\,\frac{\partial D_x}{\partial f}(t,f). \end{aligned}$$

For a shifted and scaled signal $\mathcal{A}x(t)$, the diffusion reads

$$\begin{aligned} \frac{\partial D_{\mathcal{A}x}(t,f;\tau)}{\partial \tau} &= \text{div}_{t,f}\left(\frac{f^{-2}}{a_0}\frac{\partial D_x}{\partial t}\left(\frac{t-t_0}{a_0},a_0 f\right)\vec{u_t}\right. \\ &\left.\qquad + f^2 a_0\frac{\partial D_x}{\partial f}\left(\frac{t-t_0}{a_0},a_0 f\right)\vec{u_f}\right) \\ &= (fa_0)^{-2}\frac{\partial^2 D_x}{\partial t^2}\left(\frac{t-t_0}{a_0},a_0 f\right) \\ &\quad + (a_0 f)^2\frac{\partial^2 D_x}{\partial f^2}\left(\frac{t-t_0}{a_0},a_0 f\right) \\ &\quad + 2\,(a_0 f)\frac{\partial D_x}{\partial f}\left(\frac{t-t_0}{a_0},a_0 f\right) \\ &= \left.\frac{\partial A_x(t',f');\tau)}{\partial \tau}\right|_{\substack{t'=(t-t_0)a_0^{-1} \\ f'=a_0 f.}} \end{aligned}$$

Using recurrence, the relation

$$D_{\mathcal{A}x}(t,f;\tau) = D_x\left(\frac{t-t_0}{a_0},a_0 f;\tau\right)$$

completes the proof of covariance. Here we have used the degrees of freedom provided by the anisotropic diffusion (3) in order to obtain additional properties. Strictly speaking, this scheme is not homogeneous as the diffusion tensor B depends on the frequency. Finally note that adaptive affine diffusion schemes can be obtained by combining this tensor with conductance functions $b(t,f)$ as in (4).

5. CONCLUSION

In this paper, we have proposed a generalization of the adaptive diffusion introduced in [4]. We have illustrated its abil-ity in a signal analysis context, obtaining TFR that is sharp and readable even for signal terms mixed with interference terms. We then have turned to the affine class and proposed a diffusion tensor that preserves covariance with respect to time shift and dilatation. This technique is indeed an alternative to the time-scale setting proposed in [5].

6. REFERENCES

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