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Dynamical analysis of a two-wheeled vehicle with constant pitch angle

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Introduction

The bicycle is one of the most common means of transport in the world. In particular, the history of modern bicycles started in the 19th century in Europe and the shape of this vehicle, which has changed little since the first chain-driven model was developed around 1885, descends from the so-called *safety bicycle*. Nevertheless, materials and design have been improved, especially since the 21st century with the introduction of new technologies. In addition to this, the invention of the bicycle has had also an enormous effect on society, changing deeply the culture and favouring the advancement of modern industrial methods: several components that eventually played a key role in the development of cars were invented for the bicycle.

Therefore, it is not surprising that since the end of the nineteenth century many authors have been aiming to find accurate equations to describe the motion of this system (see [Whi99], [Bou99]). However, due to the complexity of the problem related mainly to the nonholonomic constraints of the system, the majority of researchers who studied this argument introduced simplified or linearised models in order to handle the problem, for instance [LM82], [MS06]. Others considered a nonlinear model, but only considering particular motions, as in [BMCP07]. A more detailed review of the nonlinear and linearised models developed for studying the bicycle dynamics can be found in [MPRS11] and [MPRS07], respectively.

Recently, much theoretical research has focused on bicycle self-stability, that is the capability of the system to reach equilibrium again asymptotically if initially perturbed. It is common knowledge that the rideability of a bicycle may be related to its self-stability. The problems behind the self-stability of this system are not very clear, even if it is widely believed that gyroscopic and caster trail effects play an important role in such stability. Nevertheless, Kooijmann et al. recently demonstrated that a riderless two-wheeled vehicle can be self-stable without trail or gyroscopic effects. In order to do this, they introduced a simplified bicycle model, composed by only two masses and called two-mass-skate (TMS), [MPRS07].

In this thesis we aim to solve the problem of whether or not a bicycle can be self-stable considering a sufficiently accurate model. However, as stated above, the complexity of the system is a hindrance to this achievement. Although a bicycle has a structural simplicity, its geometry is complicated and presents considerable difficulties in expressing the position of the front and rear frames. For example, if the front wheel is tilted, a variation in both the steering angle and the yaw angle can occur. This is the reason most of the models for bicycle dynamics available in the literature always have a certain number of approximations.

Instead of having considered a linearised model, we have preferred to simplify the model geometry and take into account the nonlinearity of the bicycle. We believe that bicycle self-stability is closely related to the nonlinear equations for the system. Thus, we focused our attention on the element of the general model which causes the equations of motion to be complex, that is the *pitch angle*, which is the angle between the local rear frame x-axis and the line of intersection of the symmetry plane with the ground. In particular, as shown in [RF12] and [Cos06], this angle depends on two other angles, and, consequently, finding the solution of the whole system requires us to evaluate a differential-algebraic equation (DAE, see [Ric11]). Hence, the pitch angle is usually approximated (see [MS06], [Cos06]) or considered constant.

Starting from a model of a general bicycle with toroidal wheel, the aim of this work is to find the minimum set of assumptions such that the pitch angle is constant, and subsequently to study the dynamics of this new model. It will be shown that the most particular feature of this model is a spherical front wheel.

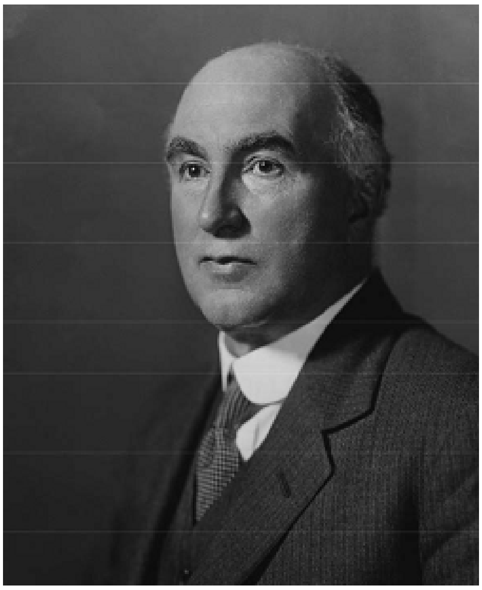

Figure 0.1: Francis J. W. Whipple (1876-1943), who proposed one of the first mathematical models for a bicycle.

The paper is organized as follows. First we present a brief review of the main mathematical notions needed to study the problem. Then, we turn our attention to the bicycle model. In particular, we first consider the geometry of a bicycle with toroidal wheels, by defining the geometric parameters which characterize the system itself. After having chosen a proper set of generalised coordinates, we study the pitch angle and find the algebraic equation which defines this angle. Subsequently, the hypothesises which guarantee that this angle does not depend on time are determined.

Therefore, we define a new bicycle model, called *Constant Pitch Angle* bicycle, and study its geometry and kinematics. In particular, we derive the linear and angular velocities for this new system and introduce the nonholonomic constraints to model the rear and front contact points of the wheels with the ground plane. By a slightly lengthy calculation, we express the constraints with respect to the generalised coordinates, but, due to the length of the equations, we need to define certain nonlinear functions to write all the constraints in a clear, concise form.

Finally, the dynamics of this system is considered. We write the bicycle's kinetic energy and its potential. Moreover, we use the artifice of nonlinear functions to easily handle both these expressions. Then, following the geometric approach proposed in [BBCM03], the equations of motion are derived before studying the self-stability of the system and particular motions, such as the circular one.

Chapter 1

Background from differential geometry

Modern analytical mechanics is naturally discussed in the mathematical language of differential geometry. In this chapter we give an introduction to the basic elements of differential geometry and then we will use them in the study of mechanical systems from a geometric point of view. However, a more comprehensive introduction to this subject may be found in [Boo75], [GPV95], [AMR88] and [War71].

1.1 Differentiable manifolds

Roughly speaking, a differentiable manifold is a topological space which locally looks like an Euclidean space, even if it differs from an Euclidean space globally. In the following, Q is a paracompact connected Hausdorff space.

Definition 1.1. Let U be an open set of Q and $\varphi\colon U \to \mathbb{R}^n$ be a homeomorphism of U onto $\varphi(U)$ with the induced topology of Q in $\mathbb{R}^n$ through $\mathbb{F}$. Then, we call the pair (U, φ) a *coordinate chart* (or coordinate system) of Q of dimension n.

Definition 1.2. An n-dimensional *atlas* on Q is given by a collection $\{(U_j, \varphi_j)\}_{j\in J}$ of coordinate charts of dimension n on Q, such that:

i. $\bigcup_{j\in J} U_j = Q$, that is, $\{U_j\}_{j\in J}$ is an open cover of Q;

ii. for each nonempty intersection $U_j \cap U_i$, the mapping $\varphi_j \circ \varphi_i^{-1}\colon \varphi_i(U_j \cap U_i) \to \varphi_j(U_j \cap U_i)$ is a diffeomorphism.

Remark 1.1. Since the topological space is connected, we do not have atlas with different dimensions; even if it is not necessary, this assumption results in a well-defined dimension.

Definition 1.3. Let $\mathscr{U} = \{(U_j, \varphi_j)\}_{j\in J}$ be a differentiable atlas of dimension n on Q. Then, a coordinate chart (U, φ) is said *compatible* with $\mathscr{U}$ if, for each intersection $U \cap U_j \neq \varnothing$,

$$\varphi_j \circ \varphi^{-1}\colon \varphi(U \cap U_j) \to \varphi_j(U \cap U_j)$$

is a diffeomorphism.

Therefore, it is possible to order the atlases by inclusion. In particular, if $\mathscr{U}$ and $\mathscr{V}$ are two n-dimensional atlases on Q, where $\mathscr{V}$ is obtained adding compatible charts to $\mathscr{U}$, then $\mathscr{U} < \mathscr{V}$. The notation $<$ represents a partial order, and by Zorn's Lemma, we can state that there exists a maximal element of the inclusion sequence which contains an initial atlas. In other words, we can choose a family of coordinate charts is *maximal.*

Definition 1.4 (Differentiable manifold)**.** A *differentiable manifold* of dimension n is paracompact connected Hausdorff space Q together with an n-dimensional maximal differentiable atlas. Moreover, the maximal atlas is also called an n-dimensional *differentiable structure* on Q.

We note that the definition of differentiable manifold is independent of the choice of atlas because the collection of coordinate charts is *maximal*.

Example 1.1. A trivial example of differentiable manifold is given by $Q = \mathbb{R}$ together with the atlas $\mathscr{U} = \{(\mathbb{R}, \mathrm{Id})\}$. Also, the circle S^1 is another example of a differentiable manifold. Indeed, it can be covered with two charts that are each locally $\mathbb{R}^1$.

Definition 1.5. Let Q and N be two differentiable manifolds of dimension n and m, respectively. Then, let U an open set of Q, and consider $x \in U$. We say that the mapping $f\colon U \to N$ is *differentiable in $x \in U$* if there exists a chart (U_j, φ_j) in a neighbourhood of x in Q and a chart (V_k, ψ_k) in a neighbourhood of $f(x)$ in N such that

$$\psi_k \circ f \circ \phi_j^{-1} \colon \phi_j \left(U_j \cap U \cap f^{-1}(V_k) \right) \to \psi_k(V_k) \subset \mathbb{R}^m$$

is differentiable in x. Furthermore, if f is differentiable for all $x \in U$, then we say that f is *differentiable in U*.

Remark 1.2. We note that the definition above it given by means of the representation of the manifold in $\mathbb{R}^n$ through the coordinates charts. Moreover, it does not depends on the choice of the chart φ_j and ψ_k. Indeed, given two other charts φ_i and ψ_h, we have

$$\psi_h \circ f \circ \phi_i^{-1} = \underbrace{\psi_h \circ \psi_k^{-1}}_{\text{diffeomorphism}} \circ \overbrace{\psi_k \circ f \circ \phi_j^{-1}}^{\text{differentiable}} \circ \underbrace{\phi_j \circ \phi_i^{-1}}_{\text{diffeomorphism}} ,$$

hence it is well defined.

Definition 1.6. Given two differentiable manifolds as above, a one-to-one and invertible mapping $f\colon Q \to N$ is called a *diffeomorphism* (between manifolds) if f is differentiable in Q and also the inverse $f^{-1}\colon N \to Q$ is differentiable.

Definition 1.7. Let Q be a differentiable manifolds. A pair (N, f) is a *submanifold* of Q if $f\colon N \to Q$ is injective and its differential is injective for each point in N. If f is also a homeomorphism, then we say that f is an *embedding.*

Let Q be a differentiable manifold of dimension n and $q \in Q$ is a point in the manifold. Then we consider the space

$$C^\infty(Q, q) = \{\, f \in C^\infty \text{ real-valued function defined in a neighbourhood } U_f \text{ of } q \,\},$$

where the neighbourhood of q depends on function. This is clearly a vector space, and defining a multiplication as

$$f \cdot g(q) = f(q) \cdot g(q),$$

for each function f, g of class C^∞ in $C^\infty(Q, q)$. Therefore, the set $C^\infty(Q, q)$ is a real algebra. We consider the *equivalence relation* $\sim$ in $C^\infty(Q, q)$ such that two elements f and g in $C^\infty(Q, q)$ are equivalent if they coincide in a neighbourhood of q, that is,

$$f \sim g \iff f \equiv g \text{ in a neighbourhood of } q.$$

We define the equivalence class of the *germs of functions* $C^\infty(Q,q)/_\sim := C^\infty_q(Q) = C^\infty_q$, which defines an algebra on $\mathbb{R}$. Then, we consider the dual space

$$(C^\infty_q)^* = \left\{ v \;\middle|\; v\colon C^\infty_q \to \mathbb{R} \text{ is a linear form} \right\},$$

and its vector subspace

$$X(q) = \left\{ v \in (C^\infty_q)^* \;\middle|\; v(f_q g_q) = f(q)v(g_q) + v(f_q)g(q),\ \forall\, f_q, g_q \in C^\infty_q \right\},$$

that is, we consider the subspace of the elements of the dual space which follow the product rule above.

Definition 1.8. The vector subspace $X(q)$ of $(C^\infty_q)^*$ is called the *derivation space* in q, and the linear form v is a *derivation* of the C^∞_q algebra.

Let $C^\infty_q(0) = \left\{ f_q \in C^\infty_q \;\middle|\; f_q(q) = 0 \right\}$ be the *ideal* of C^∞_q. It is possible to prove that the derivation space $X(q)$ is canonically isomorph to $\left[C^\infty_q(0)/(C^\infty_q(0))^2\right]^*$, which is n-dimensional.[1]

Definition 1.9. The vector space $X(q)$ is called the *tangent space* to Q at q, and it is denoted by T_qQ. Each element $v \in T_qQ$ is said the *tangent vector* to Q at q. The dual space $X(q)^*$ is called the *cotangent space* of Q at q, and it is denoted by T^*_qQ.

Once we have defined the notion of differentiable manifold, we can quickly review other elements of differential geometry we will need later. The *tangent bundle* of a manifold Q is the disjoint union of the tangent spaces to Q at the points $q \in Q$; that is,

$$TM = \mathop{\overset{\circ}{\bigcup}}_{q\in Q} T_qM.$$

Thus, a point of TM is a vector v which is tangent to M at some point $q \in M$. The natural projection on the tangent bundle is the mapping $\tau_Q\colon TQ \to Q$ which assigns to each vector its base point. We note that the inverse image $\tau_Q(q)$ of a point $q \in Q$ under the natural projection is the tangent space T_qQ. This space is also called the *fibre* of the tangent bundle over the point $q \in Q$.

Likewise, the *cotangent bundle* T^*Q of a manifold Q is the vector bundle over Q formed by the collection of all the dual spaces T^*_qQ. Elements $\omega \in T^*_qQ$ are called dual vectors or *covectors*. The cotangent bundle projection, which assigns to each covector its base point, is denoted by $\pi_Q\colon T^*Q \to Q$.

Let $f\colon Q \to N$ be a diffeomorphism between manifolds Q and N. We define the *differential* of f the map $Tf\colon TQ \to TN$. There are other notations such as f_* and df. The set of all diffeomorphism from Q to N will be denoted by $C^\infty(Q,N)$. When $N = \mathbb{R}$, we shall denote the set of smooth real-valued functions on Q by $C^\infty(Q)$.

Example 1.2 (Level sets)**.** For a given set of smooth functions $f_i(x)\colon \mathbb{R}^n \to \mathbb{R}$, $i = 1,\dots,k$, manifolds often arise as level sets $Q = \{\, x \mid f_i(x) = 0, i = 1,\dots,k \,\}$. If the gradients ∇f_i are linearly independent, or more generally if the rank of $\{\nabla f(x)\}$ is a constant r for all x, then Q is a smooth manifold of dimension $n-r$. The proof uses the implicit function theorem to show that an $(n-r)$-dimensional coordinate chart may be defined in a neighbourhood of each point on Q. In this situation, the set Q is called an *implicit submanifold* of R^n.

[1] More details about the definition of the derivation space and the results needed to define the tangent space can be found in [GPV95]

Definition 1.10. Let Q and N be two differentiable manifolds and $f\colon Q \to N$ be a differentiable map. The map f is a *submersion at a point* $q \in Q$ if its differential is a surjective linear map. In this case q is called a *regular point* of the map f. Moreover, a point $p \in N$ is a *regular value* of f if all points q in the pre-image $f^{-1}(p)$ are regular points. A differentiable map f that is a submersion at each point is called a *submersion*. Equivalently, f is a submersion if its differential Tf has constant rank equal to the dimension of N.

Remark 1.3. If we consider the manifold in Example 1.2 and assume that $r = k = \dim Q$, then the map $f\colon \mathbb{R}^n \to Q$ is a *submersion*.

A *vector field* X on Q is a smooth mapping $X\colon Q \to TQ$ which assigns to each point $q \in Q$ a tangent vector $X(q) \in T_qQ$, or, in other words, $\tau_Q \circ X = \mathrm{Id}_Q$. The set of all vector fields over Q is denoted by $\mathfrak{X}(Q)$. We note that a vector field is a *section* of the tangent bundle. An *integral curve* of a vector field X is a curve satisfying $\dot{c}(t) = X(c(t))$. Given $q \in Q$, let $\phi_t(q)$ denote the maximal integral curve of X, that is, $c(t) = \phi_t(q)$, with $c(0) = q$. In this case, "maximal" means that the interval of definition of $c(t)$ is maximal. It is easy to verify that $\phi_0 = \mathrm{Id}$ and

$$\phi_{t+s} = \phi_t \circ \phi_s,$$

whenever the composition is defined. The *flow* of a vector field X is then determined by the collection of mappings $\phi_t\colon Q \to Q$. From the definition, they satisfies

$$\frac{d}{dt}(\phi_t(q)) = X(\phi_t(q)), \qquad t \in (-\varepsilon_1(q), \varepsilon_2(q)), \quad \forall\, q \in Q.$$

Similarly, a *one-form* α on Q is a smooth mapping $\alpha\colon Q \to T^*Q$ which associates to each point $q \in Q$ a covector $\alpha(q) \in T_q^*Q$, that is, $\pi_Q \circ \alpha = \mathrm{Id}$. The set of all one-forms over Q is denoted by $\Lambda^1(Q)$.

Both notions, vector fields and one-forms, are special cases of a more general geometric object, called *tensor field*. A tensor field t of controvariant order r and covariant order s is a C^∞-section of T^r_sQ, that is, it associates to each $q \in Q$ a multilinear map

$$t(q)\colon \underbrace{T_q^*Q \times \cdots \times T_q^*Q}_{r \text{ times}} \times \underbrace{T_qQ \times \cdots \times T_qQ}_{s \text{ times}} \to \mathbb{R}.$$

It is common to say that t is a (r,s)-tensor field. The *tensor product* of a (r,s)-tensor field t and a (r',s')-tensor field t' is the $(r+r', s+s')$-tensor field $t \otimes t'$ defined by

$$\begin{aligned} t \otimes t'(q)(\omega_1, \dots, \omega_r, \mu_1, \dots, \mu_{r'}, v_1, \dots, v_s, w_1, \dots, w_{s'}) =& \\ = t(q)(\omega_1, \dots, \omega_r, v_1, \dots, v_s) \cdot t'(q)(\mu_1, \dots, \mu_{r'}, w_1, \dots, w_{s'}),& \end{aligned}$$

where $q \in Q$, while $v_i, w_i \in T_qQ$ and $\omega_j, \mu_j \in T_q^*Q$.

A special subspace of tensor fields is $\Lambda^k(Q) \subset T^0_kQ$, the set of all $(0,k)$ skew-symmetric tensor field . The elements of $\Lambda^k(Q)$ are called k-forms. If we consider a $(0,k)$-tensor field t, the *alternation map* $A\colon T^0_kQ \to \Lambda^k(Q)$ is defined by

$$A(t)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in \Sigma_k} \mathrm{sign}(\sigma) t(v_{\sigma(1)}, \dots, v_{\sigma(k)}),$$

where $\mathrm{sign}(\sigma)$ is the *sign* of the permutation σ,

$$\mathrm{sign}(\sigma) = \begin{cases} +1, & \text{if } \sigma \text{ is even,} \\ -1, & \text{if } \sigma \text{ is odd,} \end{cases}$$

and Σ_k is the set of all k-permutations. A permutation is called *odd* if it can be written as the product of an odd number of transposition (that is, a permutation that interchanges just two objects) and otherwise is *even*. Thus, the operator A skew-symmetrises k-multilinear maps. It is easy to see that A is linear, $A|_{\Lambda^k(Q)} = \mathrm{Id}$ and $A \circ A = A$.

Definition 1.11. The *wedge* or *exterior product* between $\alpha \in \Lambda^k(Q)$ and $\beta \in \Lambda^l(Q)$ is the form $\alpha \wedge \beta \in \Lambda^{k+l}(Q)$ defined by

$$\alpha \wedge \beta = \frac{(k+l)!}{k!\, l!} A(\alpha \otimes \beta).$$

We note that the numerical factor in this definition agrees with the convention of [AMR88] and [Mon02], but not that of [GPV95]. For example, let us consider α and β one-forms, then

$$(\alpha \wedge \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1), \qquad \forall\, v_1, v_2 \in T_qQ,$$

while if $\alpha \in \Lambda^2(Q)$ and $\beta \in \Lambda^1(Q)$, we have

$$(\alpha \wedge \beta)(v_1, v_2, v_3) = \alpha(v_1, v_2)\beta(v_3) + \alpha(v_3, v_1)\beta(v_2) + \alpha(v_2, v_3)\beta(v_1).$$

Proposition 1.1. *The wedge product has the following properties:*

1. $\alpha \wedge \beta$ *is* associative*, that is,* $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$*;*
2. $\alpha \wedge \beta$ *is* bilinear *in* α *and* β*, i.e.,*

$$(a\alpha_1 + b\alpha_2) \wedge \beta = a(\alpha_1 \wedge \beta) + b(\alpha_2 \wedge \beta),$$
$$\alpha \wedge (c\beta_1 + d\beta_2) = c(\alpha \wedge \beta_1) + d(\alpha \wedge \beta_2);$$

3. $\alpha \wedge \beta$ *is* anticommutative*, that is,* $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$*, where* $\alpha \in \Lambda^k(Q)$ *e* $\beta \in \Lambda^l(Q)$*.*

The *algebra of exterior differential forms* $\Lambda(Q)$ is given by the direct sum $\bigoplus_{k=0}^n \Omega^k(Q)$, together with its structure as an infinite-dimensional real vector space and with the multiplication $\wedge$.

When dealing with exterior differential forms, another important geometric object is the *exterior derivative* $\mathbf{d}$. In particular, the exterior derivative $\mathbf{d}\alpha$ of a k-form α on the manifold Q is the $(k+1)$-form on Q determined by the following proposition:

Proposition 1.2. *There exists a unique mapping* $\mathbf{d}$ *from* k*-forms on* Q *to* $(k+1)$*-forms on* Q *such that:*

1. *if* α *is a* 0*-form, that is,* $\alpha = f \in C^\infty(Q)$*, then* $\mathbf{d}f$ *is the one-form that is the differential of* f*;*
2. $\mathbf{d}\alpha$ *is* linear *in* α*; that is,for all real numbers* c_1 *and* c_2

$$\mathbf{d}(c_1\alpha_1 + c_2\alpha_2) = c_1\mathbf{d}\alpha_1 + c_2\mathbf{d}\alpha_2;$$

3. $\mathbf{d}\alpha$ *satisfies the* chain rule

$$\mathbf{d}(\alpha \wedge \beta) = \mathbf{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbf{d}\beta,$$

where α *is a* k*-form and* β *an* l*-form (that is, it is a* $\wedge$*-antiderivation);*

4. $\mathbf{d}^2 = 0$, *that is,* $\mathbf{d}(\mathbf{d}\alpha) = 0$ *for any k-form* α*;*

5. $\mathbf{d}$ *is a* local operator*; that is,* $\mathbf{d}\alpha(x)$ *depends only on* α *restricted to any open neighbourhood of q; in fact, id $U \subset V \subset Q$ are open, then*

$$\mathbf{d}(\alpha|_U) = (\mathbf{d}\alpha)|_U\,,$$

where $\alpha \in \Omega^k(V)$*. We also say that* $\mathbf{d}$ *is natural with respect to inclusions.*

A k-form is called *closed* if $\mathbf{d}\alpha = 0$ and *exact* if there exits a $(k-1)$-form β such that $\alpha = \mathbf{d}\beta$. By the definition of exterior derivative, it follows that every exact form is closed, whilst, by Poincaré lemma, a closed form is locally exact.

Definition 1.12. Let $f\colon Q \to N$ be a smooth mapping from the manifold Q to the manifold N, and let α be a k-form on N. Then we define the *pull back* $f^*\alpha$ of α by ω to be the k-form on Q given by

$$f^*\alpha(q)(v_1,\ldots,v_k) = \alpha(f(q))(T_q f(v_1),\ldots,T_q f(v_k)),$$

where $v_i \in T_qQ$. Furthermore, if f is a diffeomorphism between manifolds, we can also define the *push forward* f_* as $f_* = (f^{-1})^*$.

We note that the pull back defines a mapping $f^*\colon \Lambda^k(N) \to \Lambda^k(Q)$.

Proposition 1.3. *The* pull back *has the following properties:*

1. $(g\circ f)^* = f^*\circ g^*$*, where* $f \in C^\infty(Q,N)$ *and* $g \in C^\infty(N,W)$*, with* Q*,* N *and* W *differentiable manifolds;*

2. $(\mathrm{Id}_Q^*)\big|_{\Lambda^k(Q)} = \mathrm{Id}_{\Lambda^k(Q)}$*;*

3. *the pull back of a wedge product is the wedge product of the pull back, that is,*

$$f^*(\alpha\wedge\beta) = f^*\alpha\wedge f^*\beta,$$

where $f \in C^\infty(Q,N)$*,* $\alpha \in \Lambda^k(Q)$ *and* $\beta \in \Lambda^l(N)$*;*

4. *the exterior derivative* $\mathbf{d}$ *commutes with the pull back, that is, for any* $f \in C^\infty(Q,N)$ *we have* $\mathbf{d}(f^*\alpha) = f^*(\mathbf{d}\alpha)$*, con* $\alpha \in \Lambda^k(N)$*. We also say that the* $\mathbf{d}$ *is natural with respect to the pull back.*

There is also another natural operation associated with a vector field X, which allow us to decrease the dimension of a k-form.

Definition 1.13. Let $\alpha \in \Lambda^k(Q)$ be a k-form on the manifold Q, and let X be a vector field. The *contraction* or *interior product* is defined by

$$\mathbf{i}_X\alpha = X \lrcorner\, \alpha = X^j \alpha_{ji_2\ldots i_k} dx^{i_2}\wedge\cdots\wedge dx^{i_k}.$$

Proposition 1.4. *Let α be a k-form and β an l-form on a manifold Q. Then the contraction is $\mathbb{R}$-linear and satisfies*

$$\mathbf{i}_X(\alpha\wedge\beta) = (\mathbf{i}_X\alpha)\wedge\beta + (-1)^k\alpha\wedge(\mathbf{i}_X\beta),$$

that is, it is a $\wedge$-antiderivation.

1.1.1 Lie derivatives and Jacobi-Lie bracket

Tensor field and differential forms can be differentiated with respect to a vector field. The resulting derivative is known as the Lie derivative and may be defined in two equivalent ways. We begin with its dynamical definition.

Definition 1.14 (Dynamic definition of Lie derivative)**.** Let α be a k-form on a manifold Q and let X be a vector field with flow ϕ_t on Q. The Lie derivative of α along X is given by

$$\mathcal{L}_X\alpha = \lim_{t\to 0}\frac{1}{t}\left[(\phi_t^*\alpha) - \alpha\right] = \left.\frac{d}{dt}\right|_{t=0}\phi_t^*\alpha.$$

Theorem 1.1 (Lie derivative theorem)**.** *Using the above notation, we have*

$$\left.\frac{d}{dt}\right|_{t=0}\phi_t^*\alpha = \phi_t^*\mathcal{L}_X\alpha,$$

which holds also for time-dependent *vector fields.*

Example 1.3. Let $X(x,y) = (x,y)$ and $\alpha = (x^2+y^2)dx$. The time-t flow of X is given by $\phi_t(x,y) = (e^t x, e^t y)$, so

$$\begin{aligned}(\phi_t^*\alpha)(x,y) &= \left((e^t x)^2 + (e^t y)2\right)\phi_t^* dx = \\ &= e^{2t}(x^2+y^2)d(x\circ\phi_t) = \\ &= e^{2t}(x^2+y^2)(e^t dx) = \\ &= e^{3t}\alpha(x,y).\end{aligned}$$

Therefore,

$$\mathcal{L}_X\alpha = \left.\frac{d}{dt}\right|_{t=0} e^{3t}\alpha = 3\alpha.$$

The other definition of the Lie derivative is given by following an algebraic approach and requiring that the Lie derivative is a derivation.

Definition 1.15 (Cartan's formula for Lie derivative)**.** Cartan's formula defines the Lie derivative of a k-form α with respect to a vector field X in terms of the exterior derivative $\mathbf{d}$ and the contraction $\lrcorner$ as

$$\mathcal{L}_X\alpha = X \lrcorner\, \mathbf{d}\alpha + \mathbf{d}(X \lrcorner\, \alpha).$$

It can be proved by straightforward computation that the two definitions of Lie derivative given above are equivalent.

Let us consider the case when f is a real-valued function on a manifold Q and X is a vector field on Q; then, the Lie derivative of f along X is indeed the *directional derivative*

$$(\mathcal{L}_X f)(q) = X_q(f) := \mathbf{d}f_q(X(q)),$$

which in coordinates on Q has the expression

$$\mathcal{L}_X f = X^i\frac{\partial f}{\partial x^i}.$$

Moreover, given two vector fields $X, Y \in \mathfrak{X}(Q)$, we can define the Lie derivative of Y with respect to X. However, it is useful to introduce the Jacobi-Lie bracket before.

Definition 1.16 (Jacobi-Lie bracket)**.** The *Jacobi-Lie bracket* on $\mathfrak{X}(Q)$ is defined in local coordinates by

$$[X, Y] = (DY) \cdot X - (DX) \cdot Y,$$

which, in finite dimensions, is equivalent to

$$[X, Y] = (X \cdot \nabla)Y - (Y \cdot \nabla)X.$$

It is easy to prove that the vector field determined by the Jacobi-Lie bracket is unique and that the map $f \mapsto X(Y(f)) - Y(X(f))$ is a derivation; for further details, see [AMR88], [HSCS09]. Furthermore, there is and interesting link between the Jacobi-Lie bracket and the Lie derivative as follows.

Theorem 1.2. *Given $X, Y \in \mathfrak{X}(Q)$, the Lie derivative of Y along X is equal to the Jacobi-Lie bracket of the vector fields, that is,*

$$\mathcal{L}_X Y = [X, Y].$$

Thus, theorem 1.1 holds with α replaced by the vector field Y.

Remark 1.4. The Lie bracket of two vector fields has a geometric meaning in terms of successive applications of the flows of the two vector fields in forward and reverse direction. The case for two vector fields in $\mathbb{R}^n$ is given in [BBCM03].

If a set of vector fields X_i is such that there exist functions γ_{ijk} such that

$$[X_i, X_k] = \gamma_{ijk} X_k$$

then the set is called *involutive*. In this case no new directions are generated by bracketing, and this is an impediment to show controllability.

Proposition 1.5. *Given a diffeomorphism $\phi \colon Q \to N$, the Jacobi-Lie bracket satisfies*

$$[\phi^* X, \phi^* Y] = \phi^*[X, Y].$$

Proposition 1.6. *Given a function $f \in C^\infty(Q)$ and two vector fields $X, Y \in \mathfrak{X}(Q)$, the Lie derivative satisfies*

$$\mathcal{L}_X(F \cdot Y) = \mathcal{L}_X f \cdot Y + f \cdot \mathcal{L}_X Y.$$

Finally, we conclude by stating some relevant properties which involve the exterior derivative, the contraction and the Lie derivative.

Proposition 1.7. *Let us consider $X, Y \in \mathfrak{X}(Q)$ two arbitrary vector fields, $f \in C^\infty(Q)$ and $\alpha \in \Lambda^k(Q)$, then we have:*

1. $\mathbf{d}\mathcal{L}_X \alpha = \mathcal{L}_X \mathbf{d}\alpha$;

2. $\mathcal{L}_{fX} \alpha = f\mathcal{L}_X \alpha + \mathbf{d}f \wedge (X \lrcorner\, \alpha)$;

3. $[X, Y] \lrcorner\, \alpha = \mathcal{L}_X(Y \lrcorner\, \alpha) - Y \lrcorner\, (\mathcal{L}_X \alpha)$.

1.2 Distributions and the Frobenius theorem

We introduce here the notion of distributions. This will be the key notion in the geometrical modelling of nonholonomic dynamical systems.

Definition 1.17. Let Q be an n-dimensional manifold and let c be an integer such that $1 \leq c \leq n$. A c-dimensional *distribution* $\mathcal{D}$ on Q is a family of linear c-dimensional subspaces $\{ \mathcal{D}_q \}$ of the tangent spaces T_qQ for each $q \in Q$. A distribution $\mathcal{D}$ is called *smooth* (or *differentiable*) if for each $q \in Q$ there exists a neighbourhood U of q there are c vector fields $X_1, \dots, X_c$ of class C^∞ on U which span $\mathcal{D}$ at each point of U.

Likewise, it is possible to define *codistributions* on Q as a family of linear subspace of the cotangent spaces T_q^*Q. A more complete exposition can be found in [Mon02].

We define the *rank* of $\mathcal{D}$ at q as the dimension of the linear subspace $\mathcal{D}_q$, that is, the mapping $\rho\colon Q \to \mathbb{R}$ such that $\rho(q) = \dim \mathcal{D}_q$. For any $q_0 \in Q$, if $\mathcal{D}$ is differentiable, it is clear that $\rho(q) \geq \rho(q_0)$ in a neighbourhood of q_0. Therefore, *rho* is a lower semicontinuous function. If ρ is a constant function, then $\mathcal{D}$ is called a *regular* distribution. Henceforth we will consider regular distributions, unless otherwise stated.

For a distribution $\mathcal{D}$, a point $q \in Q$ is called *regular* if q is a local maximum of ρ, that is, ρ is constant on an open neighbourhood of q. Otherwise, q is called a *singular* point of $\mathcal{D}$. The set R of regular point of $\mathcal{D}$ is open. Moreover, it is dense, because if $q_0 \in S = Q \setminus R$, and U is a neighbourhood of q_0, U necessarily contains regular points of $\mathcal{D}$, because $\rho|_U$ must have a maximum being integer valued and bounded. Consequently, $q_0 \in \overline{R}$. We observe that in general R will not be connected.

Definition 1.18. A vector field X on Q is said to *belong to* (or *lie in*) the distribution $\mathcal{D}$ if $X(q) \in \mathcal{D}_q$ for each $q \in Q$. A smooth distribution $\mathcal{D}$ is called *involutive* (or *completely integrable*) if $[X, Y] \in \mathcal{D}$ whenever X and Y are smooth vector fields lying in $\mathcal{D}$.

Definition 1.19. A submanifold N of Q is an *integral manifold* of a distribution $\mathcal{D}$ on Q if

$$d\psi(T_nN) = \mathcal{D}(\psi(n)), \qquad \forall\, n \in N,$$

where $\psi\colon Q \to N$.

Definition 1.20. Given a distribution $\mathcal{D}$, we define its *annihilator* $\mathcal{D}^o \subset T^*Q$ as the codistribution given by

$$\begin{aligned} \mathcal{D}^o\colon \operatorname{Dom}\mathcal{D} \subset Q &\to T^*Q \\ q &\mapsto \mathcal{D}^o_q = (\mathcal{D}_q)^o = \left\{ \alpha \in T_q^*Q \;\middle|\; \alpha(v) = 0,\ \forall\, v \in \mathcal{D}_q \right\}, \end{aligned}$$

where α is a one-form.

We note that $\mathcal{D}^o$ is differentiable if and only if $\mathcal{D}$ is a regular distribution.

Remark 1.5. We note that an immersed submanifold N of Q is an integral submanifold of $\mathcal{D}$ if T_nN is annihilated by $\mathcal{D}_n$ at each point $n \in N$. Furthermore, N is an integral submanifold of maximal dimension if $(T_nN)^o = \mathcal{D}_n$ for all $n \in N$. In particular, this implies that the rank of $\mathcal{D}$ is constant along N.

A *leaf* L of $\mathcal{D}$ is a connected integral submanifold of maximal dimension such that every connected integral submanifold of maximal dimension of $\mathcal{D}$ which intersects L is an open submanifold of L. The distribution $\mathcal{D}$ is said *partially integrable* if for every regular point $q \in R$, there exists a leaf passing through q. Moreover, $\mathcal{D}$ is called a *completely integrable* distribution if there exists a leaf passing through q for every $q \in Q$. In this case, the collection of all these leaves defines a *foliation* of Q.

Proposition 1.8. *Let $\mathcal{D}$ be a smooth distribution on Q such that through each point of Q there passes an integral manifold of $\mathcal{D}$. Then $\mathcal{D}$ is involutive.*

Theorem 1.3 (Frobenius)**.** *Let $\mathcal{D}$ be a c-dimensional, involutive smooth distribution on Q. Let $q \in Q$. Then there exists an integral manifold of $\mathcal{D}$ passing through q, that is, if $\mathcal{D}$ id involutive then it is also integrable.*

1.3 Fibre bundles and connections

In this section we give a brief introduction of fibre bundles and related concepts such as connections, which will be useful for studying the geometric structure of mechanics. In particular, fibre bundles provide a basic geometric structure for the understanding of many mechanical and control problems, in particular for nonholonomic problems. Roughly speaking, a fibre bundle consists of a given space, named the base, together with another space, called the fibre, attached at each point, and certain compatibility conditions.

Definition 1.21. A *fibre bundle* is a space Q for which the following are given: a space B called the *base space*, a *projection* $\pi\colon Q \to B$ with fibres $\pi^{-1}(b)$, $b \in B$, homeomorphic to a space F, a *structure group* G of homeomorphisms of F into itself, and a covering of B by open sets U_j, satisfying

1. the bundle is locally trivial, that is, $\pi^{-1}(U_j)$ is homeomorphic to the product space $U_j \times F$;
2. if h_j is the map giving the homeomorphism on the fibres above the set U_j, then $h_j(h_k^{-1})$ is an element of the structure group G for any $x \in U_j \cap U_k \neq \varnothing$.

If the fibres of the bundle are homeomorphic to the structure group, we call the bundle a *principal bundle*. If the fibres of the bundle are homeomorphic to a vector space, we call the bundle a *vector bundle*.

Example 1.4. A basic example of a vector bundle is TS^1, that is the tangent bundle of the circle. The base is S^1, the fibres are homeomorphic to $\mathbb{R}$, and since the tangent space can be represented by any nonzero real number, the structure group is ratios of nonzero real numbers and may be identified with $\mathbb{R} \setminus \{0\}$.

The frame bundle of a manifold Q has the same structure group as TQ, but the fibres are the set of all bases for the tangent space. Therefore, for TS^1, the fibres of the frame bundle are homeomorphic to its structure group $\mathbb{R} \setminus \{0\}$, and hence the frame bundle is a principal bundle. In fact, all frame bundles are principal.

An important additional structure on a bundle is a (Ehresmann) *connection*. Intuitively, suppose we have a bundle and consider (locally) a section of this bundle, i.e., a choice of a point in the fibre over each point in the base. We call such a choice a *field*.

The idea is to single out fields that are "constant". For vector fields on the plane, for example, such fields are literally constant. For vector fields on a manifold or an arbitrary bundle, we have to specify this notion. Such fields are called *horizontal* and are also key to defining a notion of derivative, or rate of change of a vector field along a curve.[2] A connection is used to single out horizontal fields, and is chosen to have other desirable properties, such as linearity. For instance, the sum of two constant fields should still be constant. As we shall see below, we can specify horizontality by taking a class of fields

[2] The Lie derivative does not give a way of differentiating vector fields along curves.

that are the kernel of a suitable form. Note that we do not in general have a metric; given one, there is a natural choice of connection and horizontality on the tangent bundle.

More formally, let us consider a bundle with projection map $\pi\colon Q \to R$, where the manifold R is the base space, and let $T_q\pi$ denote its differential at any point. We call the kernel of $T_q\pi$ at any point the *vertical space*, which is denoted by V_q.

Definition 1.22. An *Ehresmann connection* A is a vector-valued one-form on Q such that

1. A is *vertical valued*, that is, the map $A_q\colon T_qQ \to V_q$ is linear for each point $q \in Q$;

2. A is a *projection*, that is, $A(v_q) = v_q$ for all $v_q \in V_q$.

The key property of the connection is the following: if we denote by H_q or $\operatorname{hor} q$ the kernel of A_q and call it the *horizontal space*, the tangent space to Q is the direct sum of the horizontal and vertical spaces, that is, $T_qQ = V_q \oplus H_q$. For instance, we can project a tangent vector onto its vertical part using the connection. Note that the vertical space at Q is tangent to the fibre over q. When nonholonomic systems will be discussed, we shall choose the connection so that the constraint distribution is the horizontal space of the connection.

Now define the bundle coordinates $q^i = (r^\alpha, s^a)$ for the base and fibre spaces. The coordinate representation of the projection π is just projection onto the factor r, and the connection A can be represented locally by a vector-valued differential form ω^a, that is,

$$A = \omega^a \frac{\partial}{\partial s^a},$$

where $\omega^a(q) = ds^a + A^a_\alpha(r, s)dr^\alpha$. Henceforth, the summation on repeated indices is understood. In particular, since the tangent space is is the direct sum of H_q and V_q, every vector $v_q \in T_qQ$ can be written as

$$v_q = \dot{r}^\beta \frac{\partial}{\partial r^\beta} + \dot{s}^b \frac{\partial}{\partial s^b};$$

then $\omega^a(v_q) = \dot{s}^a + A^a_\alpha \dot{r}^\alpha$ and

$$A(v_q) = (\dot{s}^a + A^a_\alpha \dot{r}^\alpha)\frac{\partial}{\partial s^a}.$$

This clearly demonstrates that A is a projection, since when A acts again only ds^a results in a nonzero term, and this has a unitary coefficient.

Definition 1.23. Given an Ehresmann connection A, a point $q \in Q$, and a vector $v_r \in T_rR$ tangent to the base at a point $r = \pi(q) \in R$, we can define the *horizontal lift* of v_r to be the unique vector v_r^h in H_q that projects to v_r under $T_q\pi$. If we have a vector $X_q \in T_qQ$, we shall also write its *horizontal part* as

$$\operatorname{hor} X_q = X_q - A(q) \cdot X_q.$$

Remark 1.6. In coordinates, the vertical projection is given by the map

$$(\dot{r}^\alpha, \dot{s}^a) \mapsto (0, \dot{s}^a + A^a_\alpha(r, s)\dot{r}^\alpha),$$

while the horizontal projection is the map

$$(\dot{r}^\alpha, \dot{s}^a) \mapsto (\dot{r}^\alpha, -A^a_\alpha(r, s)\dot{r}^\alpha).$$

Definition 1.24. The *curvature* of A is the vertical-vector-valued two-form B on Q defined by its action on two vector fields $X, Y \in \mathfrak{X}(Q)$ by

$$B(X,Y) = -A\left([\operatorname{hor} X, \operatorname{hor} Y]\right),$$

where the bracket $[\,\cdot\,,\,\cdot\,]$ on the right hand side is the Jacobi-Lie bracket of vector fields.

Remark 1.7. This definition shows that the curvature exactly measures the failure of the horizontal distribution to be integrable, because the right hand side is equal to zero if and only if the horizontal subbundle is Frobenius integrable. In particular, the curvature measures the lack of integrability of a (constraint) distribution.

For the exterior derivative of a one-form α, which could be vector-space valued, on a manifold Q acting on two vector fields $X, Y \in \mathfrak{X}(Q)$, we have the following useful identity

$$\mathbf{d}\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y])$$

which shows that *in coordinates*, the curvature may be evaluated by writing the connection as a one-form ω^a in coordinates, computing its exterior derivative (component by component), and restricting the result to horizontal vectors, that is, to the constraint distribution. In other words,

$$B(X,Y) = \mathbf{d}\omega^a(\operatorname{hor} X, \operatorname{hor} Y)\frac{\partial}{\partial s^a},$$

so that the local expression for curvature is given by

$$B(X,Y)^a = B^a_{\alpha\beta} X^\alpha Y^\beta,$$

where the coefficients $B^a_{\alpha\beta}$ are given by

$$B^b_{\alpha\beta} = \left(\frac{\partial A^b_\alpha}{\partial r^\beta} - \frac{\partial A^b_\beta}{\partial r^\alpha} + A^a_\alpha \frac{\partial A^b_\beta}{\partial s^a} - A^a_\beta \frac{\partial A^b_\alpha}{\partial s^a}\right).$$

Example 1.5 (Connections on $T\mathbb{R}^1$)**.** We are going to illustrate the idea of connection by considering the simplest possible example. Let us consider a connection on the bundle $TQ = T\mathbb{R}^1$ with coordinates $(x, \dot{x})$. We may define the horizontal space to be the kernel of the form

$$d\dot{x} + A^1_1(x,\dot{x})dx,$$

where A^1_1 is a smooth function of x and $\dot{x}$. More specifically, we can choose a connection that is linear in the velocities, that is

$$d\dot{x} + a(x)\dot{x}dx.$$

In this case, A is the $\mathbb{R}$-valued form

$$(d\dot{x} + a(x)\dot{x}dx)\frac{\partial}{\partial \dot{x}},$$

and elements of T_qQ are

$$v_q = \dot{x}\frac{\partial}{\partial x} + \ddot{x}\frac{\partial}{\partial \dot{x}},$$

and their projection onto the vertical space is

$$A(v_q) = (d\dot{x} + a(x)\dot{x}dx)\frac{\partial}{\partial \dot{x}}\left(\dot{x}\frac{\partial}{\partial x} + \ddot{x}\frac{\partial}{\partial \dot{x}}\right) = (\ddot{x} + a(x)\dot{x}^2)\frac{\partial}{\partial \dot{x}}.$$

The kernel of A, i.e., the horizontal vectors, is generated by

$$\operatorname{Span}\left\{ \frac{\partial}{\partial x} - a(x)\dot{x}\frac{\partial}{\partial \dot{x}} \right\}.$$

Note that the standard choice is $a(x) = 0$, that is, the standard horizontal space is the span of the vectors $\partial/\partial x$.

Chapter 2

Basic of geometric mechanics

In this chapter we develop the basic concepts in the geometric mechanics of holonomic and nonholonomic systems, which provide the elements for studying the dynamics of the bicycle in the next chapters. More detail about this subject can be found in [AKN02], [Hol08a], [Hol08b], [NF67], [BBCM03] and references therein.

2.1 Constrained systems

The affine space E^3 where the motion takes place is three-dimensional and Euclidean, that is, the *distance* between two elements of the affine space is defined as

$$\rho(a,b) := \|a-b\| = \sqrt{\langle a-b, a-b\rangle}, \qquad \forall\, a,b, \in E^3,$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the metric pairing which defines the scalar (or inner) product on $\mathbb{R}^3$. Then, we fix some point $O \in E^3$ called the *origin of reference* of the reference frame, so that the affine space E^3 inherits a vector space structure. The position of every point $S \in E^3$ is uniquely determined by its *position vector* $\overrightarrow{OS} = r$, whose initial point is O and end point is S. The set of all position vectors forms the three-dimensional vector space $\mathbb{R}^3$, which is equipped with the scalar product defined in E^3.

Time is one-dimensional and it is denoted by t. The set $\mathbb{R} = \{t\}$ is called the *time axis.* A moment in time occurs at $t \in \mathbb{R}$.

The *motion* (or *path*) of the point S is a smooth map $\mathbb{R} \supset \Delta \to E^3$, where Δ is an interval of the time axis. We say that the motion is defined on the interval Δ. If the origin O is fixed, then every motion is uniquely determined by a smooth vector function $r\colon \Delta \to \mathbb{R}^3$.

Definition 2.1. The image of the interval Δ under the map $t \mapsto r(t)$ is called the *trajectory* or *orbit* of the point S.

Given the position of the point S in an *inertial reference frame*, that is, a coordinate systems in uniform rectilinear motion relative to absolute space, its *velocity* v at an instant $t \in \Delta$ is defined as

$$\frac{dr}{dt} = \dot{r}(t) \in \mathbb{R}^3.$$

Clearly, the velocity is independent of the choice of the origin. Likewise, the *acceleration* of the point is by definition the vector $a = \dot{v} = \ddot{r} \in \mathbb{R}^3$.

We note that the space E^3 can be viewed as a differentiable manifold which is also called *configuration space* for a single point, and the velocity of the point is an element of the tangent space to this configuration manifold.

Now, let us consider a set of n particles in E^3 each having constant mass m_i and located at position $r_i = (x_i, y_i, z_i)$, $i = 1, \ldots, n$, where the triple (x_i, y_i, z_i) denotes the standard Cartesian coordinates. In general, the system of n point masses is moving under the influence of external and internal forces and it may be that there are certain functional relations among some of the coordinate components. In this case we say that the motion of the point masses is subject to certain *constraints*.

Definition 2.2. Given a set of n particles in the physical space E^3, we define a *bilinear* and *limiting constraint* as a relation of the form

$$f(x_1, y_1, z_1, \ldots, z_n, \dot{x}_1, \dot{y}_1, \ldots, \dot{z}_n, t) = 0, \tag{2.1}$$

which does not depend on the acceleration of the points. We say the constraint is *scleronomic* if it is independent of time, otherwise it is called *rheonomic*. The general constraint (2.1) is also called a *kinematic constraint*, whilst we say a constraint is *geometric* if it can be expressed in the form

$$f(x_1, y_1, z_1, \ldots, z_n, t) = 0. \tag{2.2}$$

In general we have a number C of constraints imposed on the n particles.

In the following, we consider only scleronomic constraints. A kinematic constraint is *integrable* if the functional relation (2.1) is integrable, that is, it may be expressed as a geometric constraint. Using the terminology introduced by Hertz in [Her94], we have the following definition.

Definition 2.3. An integrable kinematic constraint is called *holonomic*, that is, it is a functional relation of the form (2.2), while a nonintegrable kinematic constraint is called *nonholonomic*.

We are now considering two simple examples to better understand the difference between holonomic and nonholonomic constraints.

Example 2.1 (Pure rolling)**.** Ideal rolling motion combines rotation and translation of an object with respect to a surface, such that, the two are in contact with each other without sliding. In particular, let us consider a disc of radius R which is rolling (without sliding) smoothly on a horizontal surface along the X-axis of an inertial reference frame such that its centre of mass translates with a velocity $\vec{v}(C) = \dot{x}\vec{\imath}$.

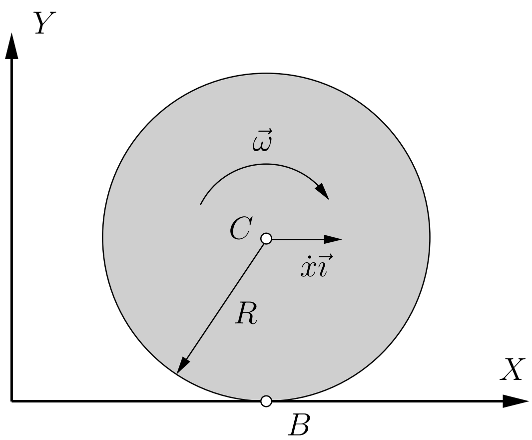

Figure 2.1: Pure rolling of a disc along the X-axis.

Because the disc is a rigid body, the velocity of the point C can be written with respect to the velocity of B as

$$\vec{v}(C) = \vec{v}(B) + \vec{\omega} \wedge (C - B) = \vec{\omega} \wedge (C - B),$$

where $\vec{v}(B) = 0$ since the particle at contact has zero instantaneous velocity resulting from equal and opposite linear velocities due to pure rotation and pure translation. Thus, the pure rolling motion is characterized by the kinematic constraint

$$\dot{x} = R\dot{\varphi},$$

where φ is the angle subtended by the arc, such that $\dot{\varphi} = \omega$. However, we note that this relation is integrable, and we have

$$x = R\varphi,$$

that is, the pure rolling constraint is a holonomic.

This example shows that holonomic constraints, given as constraints on the velocity, may be integrated and expressed as relations on the configuration variables. Therefore, holonomic constraints impose restrictions on both the positions and the velocities of a system. On the other hand, nonholonomic constraints restrict types of motion but not position, that is, they impose no restrictions on the possible values of the coordinates of the system. This statement should become clearer with the following example.

Example 2.2 (Falling disc)**.** Let us consider a disc of radius R rolling without sliding on the π-plane. As shown in Figure 2.2, the position of the disc at any time can be determined by the contact point coordinates (x, y), the self-rotation angle χ, the angle α between the plane of the disc and the vertical axis, and the heading angle θ.

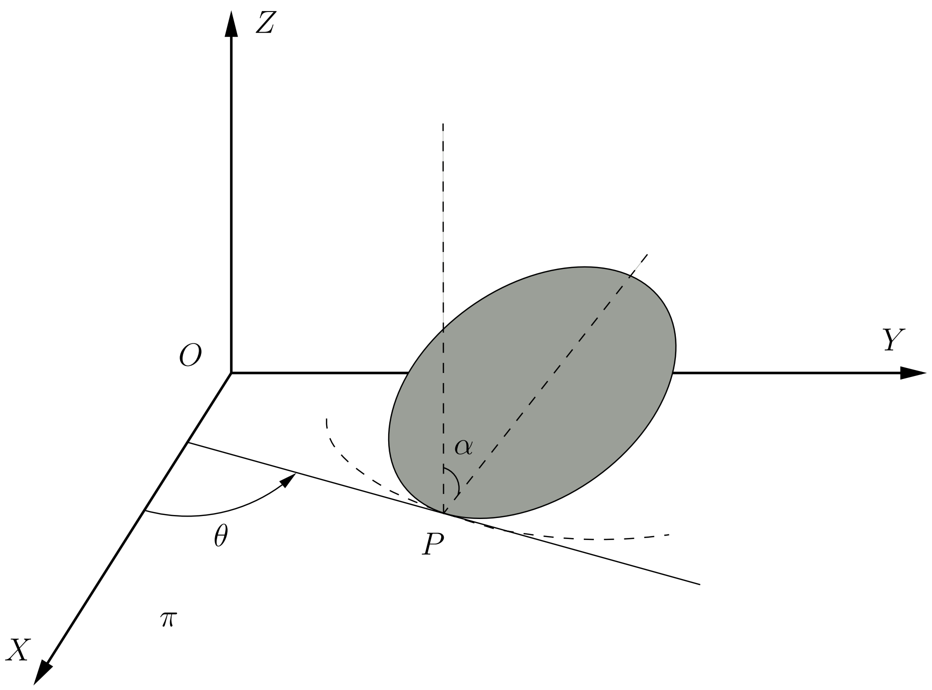

Figure 2.2: The geometry for the rolling falling disc.

The condition that the disc rolls without sliding on the plane π means that the instantaneous velocity of the contact point of the disc is equal to zero at any time. An arbitrary small displacement of the disc can be characterised by the variations of all the coordinates, which we shall denote by dx, dy, $d\theta$, $d\chi$ and $d\alpha$. Let R be the radius of the disc. Since the system rolls without sliding, the variations of the five coordinates that determine the position of the disc must satisfy the conditions

$$\begin{cases} dx = Rd\chi\cos\theta, \\ dy = Rd\chi\sin\theta. \end{cases}$$

Thus, the condition of rolling without sliding is a nonintegrable kinematic constraint expressed by constraints

$$\begin{cases} \dot{x} = R\dot{\chi}\cos\theta, \\ \dot{y} = R\dot{\chi}\sin\theta. \end{cases} \tag{2.3}$$

We note that, although these conditions must be satisfied, the five coordinates may take all sets of values as the disc rolls on the plane, that is, the disc can take up any position relative to the plane.

2.1.1 Generalised coordinates

We assume that the number of constraints imposed on the system is equal to $C = m + p$, where the holonomic constraints are m, whilst p is the number of nonholonomic constraints. Furthermore, we require that the m holonomic constraints are *independent*, that is, the Jacobian matrix of the vector function $F(x_1, y_1, \ldots, z_n) = 0$ which defines all the holonomic constraints

$$J = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial y_1} & \cdots & \dfrac{\partial f_1}{\partial z_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \dfrac{\partial f_m}{\partial y_1} & \cdots & \dfrac{\partial f_m}{\partial z_n} \end{pmatrix}$$

satisfies the full rank condition. Consequently, the system can take only configurations which satisfy these constraints, that is, all the possible positions of the system belong to

$$Q = \{ (x_1, y_1, \ldots, z_m) \mid F(x_1, y_1, \ldots, z_m) = 0 \}, \tag{2.4}$$

where $F(x_1, y_1, \ldots, z_m) = 0$ is the vector function of the constraints. We note that Q has a differentiable manifold structure; in particular, since the holonomic constraints are independent, it is an l-dimensional *embedded submanifold* of $\mathbb{R}^{3n}$, with $l = 3n - m$. Therefore, following the formalism introduced by Lagrange, we introduce a new set of coordinates, called generalised coordinates, which may be interpreted as coordinates for the system's configuration space. Let us formalise these notions.

Definition 2.4. The l-dimensional manifold Q defined by the set (2.4) is called the *configuration space* of the system, which is parametrised by a set of l *generalised coordinates*. A motion (or trajectory) of the system is given by the curve $q(t) \in Q$ parametrised by time in some interval $t \in (t_1, t_2)$. The tangent vector at any point $q \in Q$ defines the *generalised velocity* $\dot{q} \in T_qQ$. The *phase* (or *state*) *space* is the tangent bundle TQ with coordinates $(q, \dot{q})$.

We note that the generalised coordinates are the smallest number of variables needed to determine the position of the system at a give moment of time.

Example 2.3. With reference to Example 2.2, the generalised coordinates for the falling disc are given by $q = (x, y, \theta, \chi, \alpha)$. As a result, the corresponding configuration space is $Q = \mathrm{SE}(2) \times S^1 \times S^1$, where $\mathrm{SE}(2) \simeq \mathbb{R}^2 \times S^1$ (as a set) is the Euclidean group in the plane, that is, the group of rigid motions in the plane.

The introduction of generalised coordinates allows us to consider the constrained motion of the system in $\mathbb{R}^{3n}$ as a free motion a reduced space, namely the configuration space Q. In this space, we have only the remaining nonholonomic constraints, which can be written as

$$f_i(q^1, \dots, q^l, \dot{q}^1, \dots, \dot{q}^l) = 0, \qquad i = 1, \dots, p.$$

However, in mechanics we usually have nonholonomic constraints which are *linear* in the velocity field, that is,

$$\sum_{k=1}^{l} a_{ki}(q^j, t)\dot{q}^k + b_i(q^j, t) = 0, \qquad i = 1, \dots, p. \tag{2.5}$$

If $b \equiv 0$, then the constraints are called *homogeneous*, while they are scleronomic whenever both a_k and b do not depend on time. In this thesis, we consider only systems with scleronomic homogeneous constraints, which are also called *Pfaffian constraints*, expressed as

$$A(q)\dot{q} = 0,$$

where $A(q)$ is a $p \times l$ linear matrix and $\dot{q}$ is a column vector. Moreover, we assume that $\mathrm{rank}(A) = p$, so that also the nonholonomic constraints are linearly independent.

Example 2.4. If we consider constraints (2.3) introduced in Example 2.2, we can easily prove that these are Pfaffian. In fact, they can be written in the form

$$\begin{cases} \dot{x}\cos\theta + \dot{y}\sin\theta = R\dot{\chi}, \\ \dot{x}\sin\theta - \dot{y}\cos\theta = 0, \end{cases} \tag{2.6}$$

which is obtained by requiring that velocity is along the direction of the motion, that is,

$$\begin{cases} \langle v_P, \vec{\imath}_1 \rangle = R\dot{\chi}, \\ \langle v_P, \vec{\jmath}_1 \rangle = 0, \end{cases}$$

where $\vec{\imath}_1$ and $\vec{\jmath}_1$ are the versors of the local reference frame. In conclusion, we have

$$\begin{pmatrix} \cos\theta & \sin\theta & -R \\ -\sin\theta & \cos\theta & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\chi} \end{pmatrix} = 0, \tag{2.7}$$

where the matrix A has rank 2.

2.1.2 Virtual displacements and degrees of freedom

As stated before, an arbitrary motion of a system can be represented in its configuration space by a curve parametrised by time. If the system is holonomic, then the converse is also true: any curve in the configuration space represents a motion of the system in the physical space. However, this is not true for nonholonomic systems. In particular, only certain curves in the configuration space correspond to motions of the system compatible with its (nonholonomic) constraints, since a point of the configuration space, which represents the position of the system at a given instant of time, cannot move in an arbitrary direction. This is because the generalised velocities, and possibly the time, defining a displacement must satisfy the linear nonholonomic constraints (2.5).

If we consider scleronomic homogeneous nonholonomic constraints, for each configuration q, all the possible displacement compatible with the constraints lie on the tangent space T_qQ to the configuration space.

Definition 2.5. At any configuration q, the set of all possible *virtual displacements* is defined to be the subspace of the tangent space to the configuration manifold at q consisting of vectors δq that satisfy the constraints, that is, the subspace $\mathcal{D}_q$ defined by

$$\mathcal{D}_q = \{ \delta q \in T_qQ \mid A(q)\delta q = 0 \} .$$

Thus, the kinematic constraints are described by a distribution $\mathcal{D}$ which is the collection of the linear subspaces $D_q \subset T_qQ$, for each $q \in Q$. Then, a curve $q(t) \in Q$ is said to satisfy the constraints if $\dot{q}(t) \in \mathcal{D}_{q(t)}$ for all t in a certain interval. In general, this distribution will be nonintegrable in the sense of Frobenius theorem, that is, the constraints are, in general, nonholonomic.

Definition 2.6. The number of linearly independent virtual displacements of a system is called the number of its *degrees of freedom*.

We note that for a holonomic system the number of degrees of freedom is equal to the number of generalised coordinates. On the other hand, for a nonholonomic system we have $g = l - p$, where l is the number of generalised coordinates, whilst p is the number of independent nonholonomic constraints.

Example 2.5. If we consider the falling disc of Example 2.2, the system is composed by a single rigid body whose position in the three-dimensional space is described by means of six coordinates. By requiring that the disc has a contact point with the plane π, we introduce a holonomic constraint, hence, we need five generalised coordinates to parametrise the configuration space. Finally, we have two nonholonomic constraints, and the number of degrees of freedom is equal to three. For instance, we can naturally choose θ, χ and α as degrees of freedom, while x and y are determined by constraints (2.3).

2.2 Hamilton's principle

In this section we give a brief account of the variational principles involved in the derivation of the equations of motion for holonomic systems. Let Q be the configuration space of a system with generalised coordinates q^i, $i = 1, \dots, l$. Then, let us consider a family of C^2 curves $q(t,s)\colon [t_1,t_2] \times (-\varepsilon,\varepsilon) \to Q$ which connect two given point q_1 and q_2 in the configuration space, such that $q(t,0) = q(t)$ for all $t \in [t_1,t_2]$, while $q(t_1,s) = q_1$ and $q(t_2,s) = q_2$ for all $s \in (-\varepsilon,\varepsilon)$.

Next, we consider a real-valued function $L\colon TQ \to \mathbb{R}$, called *Lagrangian*. For a mechanical system, L is often chosen to be

$$L(q,\dot{q}) = K(q,\dot{q}) + U(q),$$

where $K\colon TQ \to \mathbb{R}$ is the *kinetic energy* of the system and $U\colon Q \to \mathbb{R}$ is the *potential*. The *action* S is defined as the integral over the time interval (t_1,t_2) of the Lagrangian, that is,

$$S = \int_{t_1}^{t_2} L(q^i(t), \dot{q}^i(t))dt.$$

Theorem 2.1 (Hamilton's principle of stationary action)**.** *The Euler-Lagrange equations*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0, \qquad i = 1, \ldots, l, \tag{2.8}$$

follow from the stationarity of the action S, that is, Hamilton's principle singles out particular curves $q(t)$ by the condition

$$\delta S = 0,$$

where the variation is over smooth curves in Q with fixed endpoints q_1 and q_2.

Proof. The *variational derivative* in the statement of Hamilton's principle is defined as

$$\delta S = \delta \int_{t_1}^{t_2} L(q^i(t), \dot{q}^i(t))dt := \left.\frac{d}{ds}\right|_{s=0} \int_{t_1}^{t_2} L(q^i(t,s), \dot{q}^i(t,s))dt.$$

Differentiating under the integral sign, denoting

$$\delta q^i(t) := \left.\frac{d}{ds}\right|_{s=0} q^i(t,s),$$

and integrating by parts, we have

$$\begin{aligned}\delta S &= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i \right) dt = \\ &= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} \right) \delta q^i dt + \left[\frac{\partial L}{\partial \dot{q}^i}\delta q^i \right]_{t_1}^{t_2},\end{aligned}$$

where $\delta \dot{q}^i = \frac{d}{dt}\delta q^i$. Because the endpoints are fixed, the variations vanish and so the last term, hence we obtain

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} \right) \delta q^i dt$$

Therefore, the action S is stationary, that is, $\delta S = 0$, for an arbitrary C^1 function $\delta q^i(t)$ if and only if the Euler-Lagrange equations (2.8) hold. □

Thus, a motion of the Lagrangian system extremises the functional S among all its possible variations.

A critical aspect of the Euler-Lagrange equations is that they may be regarded as a way to write Newton's second law in a way that makes sense in arbitrary curvilinear and even moving coordinate systems. That is, the Euler-Lagrange formalism is *covariant*. This is of enormous benefit, not only theoretically, but for practical problems as well.

In the presence of *external forces* F_i, $i = 1, ;l$ we must consider the total work done by these forces along the motion, which is given by

$$W = \sum_{j=1}^{l} \int_{t_1}^{t_2} F_j dt.$$

We note that if these forces are derivable from a potential U, in the sense that $F_i = -\partial U/\partial q^i$, then these forces can be incorporated into the Lagrangian by adding to potential to L. Thus, this way of adding forces is consistent with the Euler-Lagrange equations themselves. In general, we derive the equations of motion from a variational-like principle, namely the *Lagrange-d'Alembert principle* for system with external forces, which states that

$$\delta S = \delta W,$$

where

$$\delta W = \sum_{j=1}^{l} \int_{t_1}^{t_2} F_j \delta q^j dt = \int_{t_1}^{t_2} F \cdot \delta q dt$$

is the *virtual work* done by the force field F with a virtual displacement δq as defined above. Taking the variations as before, the equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = F_i, \qquad i = 1, \ldots, l. \tag{2.9}$$

An advantage of Lagrangian models of mechanical system dynamics is their manifest *invariance* with respect to coordinate changes. Moreover, it is possible to extend these models to include dissipation by defining a *dissipation function* $D(q, \dot{q})$ such that

$$\dot{q}^T D \dot{q} = \textit{rate of dissipation of energy per second.}$$

We generally assume that dissipation functions are quadratic, symmetric, and positive definite with respect to the generalised velocity variables $\dot{q}$. This type of rate-dependent dissipation is often called *Rayleigh dissipation.* The dissipative equations of motion are given locally by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} + \frac{\partial D}{\partial \dot{q}} = 0. \tag{2.10}$$

Theorem 2.2. *If $E(q, \dot{q})$ is the total energy of the system, then*

$$\frac{d}{dt}E(q, \dot{q}) = -\dot{q}^T \frac{\partial D}{\partial \dot{q}}.$$

Theorem 2.3. *The dissipative Lagrangian system is invariant under a change of coordinates $q = Q(q)$. In particular, if the system dynamics is given by a Lagrangian $L(q, \dot{q})$ and dissipation function $D(q, \dot{q})$, with corresponding equation of motion* (2.10)*, then the same system dynamics is prescribed in terms of Q-coordinates by a Lagrangian $\mathcal{L}(Q, \dot{Q})$, dissipation function $\mathcal{D}(Q, \dot{Q})$ and equations of motion*

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{Q}} - \frac{\partial \mathcal{L}}{\partial Q} + \frac{\partial \mathcal{D}}{\partial \dot{Q}} = 0.$$

2.3 Nonholonomic mechanics

An important issue about the equations of motion for nonholonomic systems is whether the constraints are to be imposed before or after taking variations. Imposing the constraints on the class of curves considered before taking the variations, we get equations that are variational in the usual sense, that is, an action functional defined on a space of curves id extremised. This type of approach is certainly appropriate for optimal control problems. However, for mechanics, the correct approach is to impose the constraints after taking variations, that is, the correct equations of motion for nonholonomic mechanical systems are given by the Lagrange-d'Alembert principle.

Let us consider a mechanical system described by generalised coordinates $q = (q^1, \ldots, q^l)$ and Lagrangian $L = K + U$. Then, we assume that the mechanical system is subjected to p linear nonholonomic constraints which in generalized coordinates can be expressed as

$$A(q)\dot{q} = 0,$$

where $A(q)$ is a $p \times l$ matrix. At any configuration q in the configuration space Q, we know that the set of all possible virtual displacements is defined to be the subspace $\mathcal{D}_q \subset T_qQ$ of the tangent space to the configuration manifold at q consisting of vectors δq that satisfy the constraints, i.e., the subspace $\mathcal{D}_q$ defined by

$$\mathcal{D}_q = \{ \, \delta q \in T_qQ \mid A(q)\delta q = 0 \, \} \, .$$

Therefore, the nonholonomic constraints are introduced by means of a distribution $\mathcal{D}$ on Q. The (generalized) constraint force, which is regarded as a cotangent vector at q, is assumed to lie in the annihilator of the distribution $\mathcal{D}$. Thus, F has to be a linear combination of the rows of $A(q)$:

$$F = \lambda A(q),$$

where λ is a row vector whose elements are called *Lagrange multipliers*. This assumption is usually named as the *nonholonomic principle*. We observe that these multipliers are related to the forces that have to be exerted by the constraints in order that the system satisfy the nonholonomic constraints. Intrinsically, λ is a section of the cotangent bundle at the point $q(t)$ which annihilates the constraint distribution. We can summarise this situation with the assumption: in any virtual displacement consistent with the constraints, the constraint forces F_i do no work, that is, we assume that the identity

$$F_1\delta q^1 + F_2\delta q^2 + \cdots + F_l\delta q^l = 0$$

holds for all virtual displacements $\delta q^i \in \mathcal{D}_q$. In this case, the nonholonomic constraints are said to be *ideal*. Therefore, the system is equivalent to a holonomic one with applied appropriate external forces. Recalling equations (2.9), we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} + \frac{\partial L}{\partial q^i} = \lambda A(q), \qquad A(q)\dot{q} = 0. \tag{2.11}$$

In particular, we have l second order differential equations, and p constraint equations.

Definition 2.7. We call equations (2.11) the *nonholonomic equations* or the Lagrange-d'Alembert equations for a mechanical system with Pfaffian constraints.

A problem with the above classical derivation of the Lagrange-d'Alembert equations is that no adequate justification is given for the nonholonomic principle, i.e., the assertion that the vector of generalized forces always has to annihilate all possible virtual displacements (in the case of ideal constraints). With this assumption, the total energy of the system is *conserved*, and conservation of energy indeed holds for many systems with nonholonomic constraints. The rate of change of the total energy of the system is equal to the rate of work done by the generalized forces, which is $\langle F, \dot{q} \rangle$, therefore, conservation of energy requires only that the work done by the generalized forces at each instant be zero, that is, that $\langle F, \dot{q} \rangle = 0$. The constraints ensure that the vector $\dot{q}$ does lie in $\mathcal{D}_q$, but conservation of energy in itself does not explain why the generalized force vector should annihilate all the possible virtual displacements.

It has long been the general consensus in the mechanics community that the Lagrange-d'Alembert equations do indeed provide an accurate model of the observed behaviour of constrained physical systems. However, what the confusion over the equations mentioned above did do was, quite properly, to highlight the inadequacies in the classical derivation of the Lagrange-d'Alembert equations.

To solve this situation, we now derive the equations of motion for nonholonomic system by means of *variational problems*, although we remark that the equations for nonholonomic mechanical systems are not literally variational. This formulation, in general, leads to equations that are different from the Lagrange-d'Alembert equations (2.11), though in the case of holonomic constraints, both formulations obviously yield the same equations. Let us consider again a configuration space Q and a distribution $\mathcal{D}$ that describes the kinematic constraints. A curve $q(t) \in Q$ will be said to *satisfy the constraints* if $\dot{q}(t) \in \mathcal{D}_q(t)$ for all t. This distribution will, in general, be nonintegrable in the sense of Frobenius theorem, that is, the constraints are nonholonomic.

Definition 2.8 (Lagrange-d'Alembert principle)**.** Given a system with nonholonomic constraints defined by a distribution $\mathcal{D}$ and Lagrangian $L\colon TQ \to \mathbb{R}$, the *Lagrange-d'Alembert equations* of motion for the system are determined by

$$\delta \int_{t_1}^{t_2} L(q^i(t), \dot{q}^i(t))dt = 0,$$

where the variations $\delta q(t)$ of the curve $q(t)$ must satisfy $\delta q(t) \in \mathcal{D}_{q(t)}$ for each $t \in [t_1, t_2]$, and $\delta q(t_1) = \delta q(t_2) = 0$.

This principle is supplemented by the condition that the curve $q(t)$ itself satisfy the constraints.

As explained before, in such a principle we take the variation δq *before* imposing the constraints, that is, we do not impose the constraints on the family of curves defining the variation. The usual arguments in the calculus of variations show that this constrained variational principle is equivalent to the equations

$$-\delta L = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \right) \delta q^i = 0, \tag{2.12}$$

for all variations δq such that $\delta q \in \mathcal{D}_q$ at each point of the underlying curve $q(t)$. To explore the structure of the equations determined by (2.12) in more detail, let $\{\omega^a\}$ be a set of p independent one-forms whose vanishing describes the constraints on the system, that is, the constraints on $\delta q \in TQ$ are defined by the p conditions $\omega^a \cdot v = 0$, for $a = 1, \dots, p$. Using the fact that these p one-forms are independent, it is possible to choose local coordinates such that the one-forms ω a have the form

$$\omega^a(q) = ds^a + A^a_\alpha(r, s)dr^\alpha, \qquad a = 1, \dots, p, \tag{2.13}$$

where $q = (r, s) \in \mathbb{R}^{n-p} \times \mathbb{R}^p$. In other words, we are locally writing the distribution as

$$\mathcal{D} = \{\, (r, s, \dot{r}, \dot{s}) \in TQ \mid \dot{s} + A^a_\alpha \dot{r}^\alpha = 0 \,\}.$$

With this choice, the constraints on $\delta q = (\delta r, \delta s)$ are given by the conditions

$$\delta s^a + A^a_\alpha \delta r^\alpha = 0. \tag{2.14}$$

The equations of motion for the system are given by (2.12), where the variations satisfy the constraints. Substituting (2.14) into (2.12) and using the fact that δr is arbitrary, we get

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^\alpha} - \frac{\partial L}{\partial r^\alpha} \right) = A^a_\alpha \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{s}^a} - \frac{\partial L}{\partial s^a} \right), \qquad \alpha = 1, \dots, n-p. \tag{2.15}$$

These equations combined with the constraint equations

$$\dot{s}^a = -A^a_\alpha \dot{r}^\alpha, \qquad a = 1, \dots, p, \tag{2.16}$$

give a complete description of the *equations of motion* of the system.

Definition 2.9. We define the *constrained Lagrangian* by substituting the constraint (2.16) into the Lagrangian, that is

$$L_c(r^\alpha, s^a, \dot{r}^\alpha) = L(r^\alpha, s^a, \dot{r}^\alpha, -A^a_\alpha(r,s)\dot{r}^\alpha).$$

Consequently, the equations of motion (2.15) can be written in terms of the constrained Lagrangian in the following way. Let us consider the relations

$$\begin{aligned}
\frac{\partial L_c}{\partial \dot{r}^\alpha} &= \frac{\partial L}{\partial \dot{r}^\alpha} - A^b_\alpha \frac{\partial L}{\partial \dot{s}^b},\\
\frac{\partial L_c}{\partial r^\alpha} &= \frac{\partial L}{\partial r^\alpha} - \frac{\partial L}{\partial \dot{s}^b}\left(\frac{\partial A^b_\beta}{\partial r^\alpha}\dot{r}^\beta\right),\\
\frac{\partial L_c}{\partial s^a} &= \frac{\partial L}{\partial s^a} - \frac{\partial L}{\partial \dot{s}^b}\left(\frac{\partial A^b_\beta}{\partial s^a}\dot{r}^\beta\right).
\end{aligned}$$

Substituting the constrained Lagrangian L_c into equation (2.12), we have

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L_c}{\partial \dot{r}^\alpha} - \frac{\partial L_c}{\partial r^\alpha} + A^a_\alpha \frac{\partial L_c}{\partial s^a} &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}^\alpha} - A^b_\alpha\frac{\partial L}{\partial \dot{s}^b}\right) - \frac{\partial L}{\partial r^\alpha} + \frac{\partial L}{\partial \dot{s}^b}\frac{\partial A^b_\beta}{\partial r^\alpha}\dot{r}^\beta +\\
&\quad + A^a_\alpha\frac{\partial L}{\partial s^a} - A^a_\alpha \frac{\partial L}{\partial \dot{s}^b}\frac{\partial A^b_\beta}{\partial s^a}\dot{r}^\beta =\\
&= \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^\alpha} - \frac{\partial L}{\partial r^\alpha}\right) - A^a_\alpha\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{s}^a} - \frac{\partial L}{\partial s^a}\right) - \frac{d}{dt}A^b_\alpha\frac{\partial L}{\partial \dot{s}^b} +\\
&\quad + \frac{\partial L}{\partial \dot{s}^b}\frac{\partial A^b_\beta}{\partial r^\alpha}\dot{r}^\beta - A^a_\alpha\frac{\partial L}{\partial \dot{s}^b}\frac{\partial A^b_\beta}{\partial s^a}\dot{r}^\beta =\\
&= \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^\alpha} - \frac{\partial L}{\partial r^\alpha}\right) - A^a_\alpha\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{s}^a} - \frac{\partial L}{\partial s^a}\right) +\\
&\quad + \frac{\partial L}{\partial \dot{s}^b}\left(\frac{\partial A^b_\beta}{\partial r^\alpha} - \frac{\partial A^b_\alpha}{\partial r^\beta} + A^a_\beta\frac{\partial A^b_\alpha}{\partial s^a} - A^a_\alpha\frac{\partial A^b_\beta}{\partial s^a}\right)\dot{r}^\beta.
\end{aligned}$$

Therefore, the equations of motion for nonholonomic systems can be written as

$$\frac{d}{dt}\frac{\partial L_c}{\partial \dot{r}^\alpha} - \frac{\partial L_c}{\partial r^\alpha} + A^a_\alpha\frac{\partial L_c}{\partial s^a} = -\frac{\partial L}{\partial \dot{s}^b}B^b_{\alpha\beta}\dot{r}^\beta, \tag{2.17}$$

where

$$B^b_{\alpha\beta} = \left(\frac{\partial A^b_\alpha}{\partial r^\beta} - \frac{\partial A^b_\beta}{\partial r^\alpha} + A^a_\alpha\frac{\partial A^b_\beta}{\partial s^a} - A^a_\beta\frac{\partial A^b_\alpha}{\partial s^a}\right). \tag{2.18}$$

Remark 2.1. We note that the coefficients $B^b_{\alpha\beta}$ are such that $B^b_{\alpha\alpha} = 0$ and $B^b_{\alpha\beta} = -B^b_{\beta\alpha}$.

Letting $\mathbf{d}\omega^b$ be the exterior derivative of ω^b, we have

$$\begin{aligned}
\mathbf{d}\omega^b &= \mathbf{d}(ds^b + A^b_\alpha dr^\alpha) =\\
&= \frac{\partial A^b_\alpha}{\partial r^\beta}dr^\beta \wedge dr^\alpha - \frac{\partial A^b_\alpha}{\partial s^a}A^a_\beta dr^\beta \wedge dr^\alpha,
\end{aligned}$$

and contracting $\mathbf{d}\omega^b$ with $\dot{q}$, we obtain

$$\begin{aligned}
\mathbf{d}\omega^b(\dot{q},\,\cdot\,) &= \frac{\partial A^b_\alpha}{\partial r^\beta}\dot{r}^\beta dr^\alpha - \frac{\partial A^b_\alpha}{\partial s^a}A^a_\beta\dot{r}^\beta dr^\alpha - \frac{\partial A^b_\alpha}{\partial r^\beta}\dot{r}^\alpha dr^\beta + \frac{\partial A^b_\alpha}{\partial s^a}A^a_\beta\dot{r}^\alpha dr^\beta =\\
&= \left(\frac{\partial A^b_\beta}{\partial r^\alpha} - \frac{\partial A^b_\alpha}{\partial r^\beta} + A^a_\beta\frac{\partial A^b_\alpha}{\partial s^a} - A^a_\alpha\frac{\partial A^b_\beta}{\partial s^a}\right)\dot{r}^\beta dr^\alpha =\\
&= B^b_{\alpha\beta}\dot{r}^\beta dr^\alpha.
\end{aligned}$$

Therefore, the equations of motion for the constrained system have the form

$$\frac{d}{dt}\frac{\partial L_c}{\partial \dot{r}^\alpha} - \frac{\partial L_c}{\partial r^\alpha} + A^a_\alpha \frac{\partial L_c}{\partial s^a} = -\frac{\partial L}{\partial \dot{s}^b}\mathbf{d}\omega^b\left(\dot{q}, \frac{\partial}{\partial r^\alpha}\right). \tag{2.19}$$

This form of the equations isolates the effects of the constraints. In fact, the left-hand side may be checked to be the variational derivative of the constrained Lagrangian, while the right-hand side consists of the forces that maintain the constraints. In the special case that the constraints are holonomic, $\mathbf{d}\omega^b = 0$, since $\mathbf{d}\omega^b$ represents the curvature, that is, the lack of integrability of the constraints.

2.3.1 Intrinsic formulation of the equations

We can now rephrase our coordinate computations in the language of the Ehresmann connections that we discussed in Chapter 1. Suppose that we have chosen a bundle and an *Ehresmann connection* A on that bundle such that the constraint distribution $\mathcal{D}$ is given by the horizontal subbundle associated with A. In other words, we assume that the connection A is chosen such that the constraints are written as $A \cdot \dot{q} = 0$.

Note that this is an intrinsic way of writing the constraints and a way of thinking of the collection of one-forms that we used in the coordinate description. In those coordinates, it is possible to choose the bundle $Q \to R$ to be given in coordinates by $(r, s) \mapsto r$, and the connection is, in this choice of bundle, defined by the constraints. It is clear that this choice of bundle is not unique; sometimes this sort of ambiguity is removed for systems with symmetry.

Example 2.6. If we consider the physical example of the falling disc given in Example 2.2, it is natural to choose $r = (\theta, \chi, \alpha)$ and $s = (x, y)$ Then the connection given by the constraints can be written as

$$\omega^1 = dx - \cos\theta d\chi$$

and

$$\omega^2 = dy - \sin\theta d\chi.$$

Definition 2.10. Let L be a Lagrangian on TQ and let $\mathbb{F}L \colon TQ \to T^*Q$ be defined in coordinates by

$$(q^i, \dot{q}^j) \mapsto (q^i, p_j),$$

where $p_j = \partial L/\partial \dot{q}^j$. We call $\mathbb{F}L$ the *fibre derivative* of L.

In the language of connections, the *constrained Lagrangian* can be written as

$$L_c(q, \dot{q}) = L(q, \operatorname{hor}\dot{q}),$$

and we have the following theorem.

Theorem 2.4. *The Lagrange-d'Alembert equations may be written as the equations*

$$\delta L_c = \langle \mathbb{F}L, B(\dot{q}, \delta q)\rangle,$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the pairing between a vector and a dual vector, and where

$$\delta L_c = \left\langle \delta q^\alpha, \frac{\partial L_c}{\partial q^\alpha} - \frac{d}{dt}\frac{\partial L_c}{\partial \dot{q}^\alpha}\right\rangle,$$

in which δq is a horizontal variation, that is, it takes values in the horizontal space, and B is the curvature regarded as a vertical-valued two-form, in addition to the constraint equations

$$A(q)\dot{q} = 0.$$

Chapter 3

Introduction of a new bicycle model

Once we have quickly introduced the theory of nonholonomic systems, we turn the attention to the mathematical description of the bicycle model we want to study. It is common knowledge that in the literature there exist many different models for describing the bicycle dynamics, although the mechanical system is usually handled introducing approximation or simplification of the geometric structure. In a general and comprehensive case, we pointed out that problems arise from the pitch angle, that is, the angle between the local rear frame x-axis and the line of intersection of the symmetry plane with the ground. In particular, because this angle depends on other coordinates, the motion of the bicycle is obtained by solving a set of Differential-Algebraic equations. Therefore, we are going to define a new model for the bicycle in which this angle remains constant during the motion, still considering a nonlinear system.

3.1 Geometry of a general bicycle

We start considering a general bicycle model with toroidal wheel. In particular, we refer to the notation introduced in [RF12]. According to the model in Figure 3.1, we assume that the the riderless bicycle, which moves on a horizontal plane rolling without slipping, is composed by the following rigid bodies: the rear and front wheels, the rear frame, and the front frame.

Furthermore, the rear wheel and the rear frame identify the so-called *rear assembly*, while the front wheel and the front frame the *front assembly*. Introduce a plane of symmetry, named rear plane, which contains the rear assembly, we can define the *contact line* as the intersection between the rear plane and the ground. If the front assembly lies in the front plane of symmetry, we say the bicycle is in the *trivial configuration* when the rear and the front plans of symmetry are parallel and both are normal to the horizontal ground plane.

Referring to the trivial configuration, we define the geometric parameters which characterise the whole bicycle. The major radii of the rear and the front wheel are R and R_f, respectively, whereas r and r_f are the crown radii.

Moreover, the *wheelbase* w is defined as the distance between the two contact points in the trivial configuration, while the *caster angle* λ is the angle between the vertical axis and the steering axis. We also identify the segment $\overline{BC}$ as the length l, the trivial pitch angle φ, the *fork lower* b and the *fork offset* d, that is, the perpendicular distance between the steering axis and the centre of the front wheel. Hence, in the trivial configuration, the

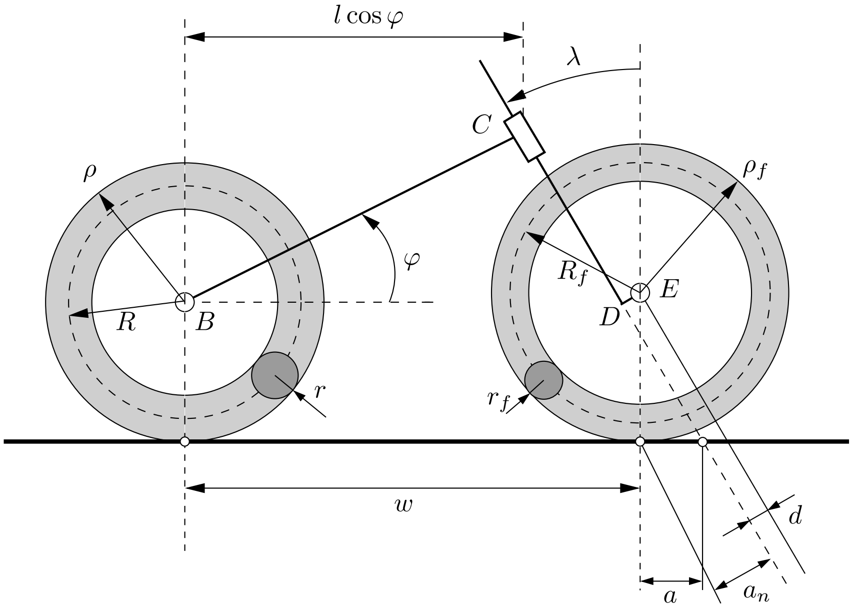

Figure 3.1: Model for general bicycle.

relation

$$w = l\cos\varphi + b\sin\lambda + d\cos\lambda$$

holds. We note that the trivial pitch angle can be set to zero because of its arbitrariness; in the following, we will choose it such that the centre of mass of the rear frame is characterized by only one coordinate in a proper reference frame. It is also useful to introduce the *reduced caster angle* as $\varepsilon = \lambda - \varphi$.

With reference to Figure 3.1, we can easily define the trail by means of geometric considerations. In particular, if the bicycle is in the trivial configuration, the trail is given by

$$a = \rho_f \tan\lambda - \frac{d}{\cos\lambda},$$

and the normal trail is

$$a_n = \rho_f \sin\lambda - d,$$

where $\rho_f = R_f + r_f$. These two parameters has been widely studied in the literature because of its stability effects on the bicycle, [Cos06] and [CLM11].

In conclusion, we have fifteen different geometric parameters characteristic of the bicycle, and, according to the situation, one singles out the *nine* independent which can be measured easier in order to define the bicycle geometry.

3.1.1 Reference frames

In order to identify the bicycle in a generic configuration, we define one inertial reference frame and different local reference frames for each of the bodies which compose the system. Let $\Sigma = (O; X, Y, Z)$ be the inertial reference frame, where the Z-axis is perpendicular to the ground in the direction opposite to gravity, and the X-axis is parallel with the contact line in the trivial configuration. The right-hand rule is used to determine the direction of the Y-axis as usual.

The local reference frames are introduced as in [RF12]. The first one is centred in the rear contact point $S_{A'} = (A'; x_{A'}, y_{A'}, z_{A'})$, with $z_{A'}$ normal to the ground, passing through A' and A, while $x_{A'}$ is parallel with the X-axis in the trivial configuration. Another moving frame, labelled as $S_A = (A; x_A, y_A, z_A)$, is attached in A, being x_A always parallel with X, but z_A directed through A and B. The third frame is introduced for the rear assembly is $S_B = (B; x_B, y_B, z_B)$, with origin in the rear wheel centre B and x_B passing through B and C, while z_B is normal to x_B and lie on the rear symmetry plane. We note that, due to the hinge, S_B is not sensitive to the wheel rotation, therefore it is useful to introduce an addition frame $S_{Br} = (B; x_{Br}, y_{Br}, z_{Br})$ which takes such a rotation into account.

With reference to the front assembly, we have the reference frame $S_D = (D; x_D, y_D, z_D)$ centred in D, with z_D directed parallel to the steering axis, whereas x_D lying on the front symmetry plane and normal to the steering axis itself. In addition to this, we introduce four more reference frames like those used for the rear wheel: $S_E = (E; x_E, y_E, z_E)$ and $S_{Er} = (E; x_{Er}, y_{Er}, z_{Er})$ attached in E, the former uninfluenced by the wheel rotation, while the latter sensitive to it; $S_F = (F; x_F, y_F, z_F)$ with z_F passing through F and E, whilst $S_{F'} = (F'; x_{F'}, y_{F'}, z_{F'})$ with $z_{F'}$ passing through F' and F, both having the abscissa directed parallel to X in the trivial configuration.

After having defined the local reference frames, their orientations can be related to the inertial frame by means of proper Euler angles and rotation matrices. However, we first introduce some transformations which relate each local frame to the following one.

Considering the bicycle in a generic configuration, the coordinates of the rear contact point are $A' = (x, y, 0)^T$ in the inertial reference frame, because we have the holonomic (geometric) constraint $z = 0$. On the other hand, the contact line is parallel with X no more, but they form an angle θ, named *yaw angle* and taken about the vertical Z-direction. We adopt the right-hand rule, so the angle is positive for counter-clockwise rotations. Therefore, the $S_{A'}$ orientation with respect to the inertial frame is described by the rotation matrix

$$\mathcal{R}_{A'} = \mathcal{R}_1(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Remark 3.1. We choose the *alias* approach to represent rotation, that is, the change in vector coordinates is due to a turn of the coordinate system, instead of a turn of the vector itself.

The orientation of S_A with respect to $S_{A'}$ can be obtained by a rotation about $x_{A'}$ of the *roll angle* α, which the bicycle's rear plane of symmetry makes with the vertical direction.[1] We remark that, due to physical reasons, this angle can assume values in the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Moreover, in this paper we take α positive for clockwise rotations, in order to have a positive angle when the bicycle leans to the left. Hence, the rotation matrix becomes

$$\mathcal{R}_2(-\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}.$$

Passing to the rear frame, we note that the hinge B allows a rotation about the y_B-axis. The *pitch angle* μ takes this rotation into account, being defined as the angle between x_B and the contact line, both belonging to the rear plane of symmetry. As the roll angle, even

[1] It can be proved that the S_A orientation is obtained by means of two rotations, and an additional rotation about the y_A-axis is useless.

the pitch angle is taken positive for clockwise rotations, thus

$$\mathcal{R}_3(-\mu) = \begin{pmatrix} \cos\mu & 0 & \sin\mu \\ 0 & 1 & 0 \\ -\sin\mu & 0 & \cos\mu \end{pmatrix}.$$

Remark 3.2. We define the pitch angle such that it includes the constant angle φ. In particular, we write $\mu(t) = \overline{\mu}(t) + \varphi$, where $\overline{\mu}(t)$ is the *effective* pitch angle.

Furthermore, we introduce the *steering angle* $\psi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ as the rotation about z_D, that is, the steering axis, which is tilted backward with respect to z_B by the reduced caster angle ε, therefore we have

$$\mathcal{R}'(-\varepsilon) = \begin{pmatrix} \cos\varepsilon & 0 & \sin\varepsilon \\ 0 & 1 & 0 \\ -\sin\varepsilon & 0 & \cos\varepsilon \end{pmatrix},$$

and

$$\mathcal{R}_4(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We observe that, due to the previous choice about the roll angle, the angles ψ and α have the same sign.

The overall rotation characteristic of every frame can be expressed as a sequence of partial rotations, each defined with respect to the preceding one. The frame with respect to which the rotation occurs is termed *current frame*. Because we have chosen the alias approach to describe rotations, composition of successive rotations is then obtained by multiplication of the rotation matrices following the inverse order of rotations, that is

$$R_n^0 = R_n^{n-1} R_{n-1}^{n-2} \dots R_2^1 R_1^0,$$

where R_i^{i-1}, $i = 1, \dots n$, denotes the rotation matrix of frame i with respect to frame $i-1$. Therefore, the direct transformation from Σ to S_A is given by the rotation matrix

$$\mathcal{R}_A = \mathcal{R}_2(-\alpha)\mathcal{R}_1(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\cos\alpha\sin\theta & \cos\alpha\cos\theta & -\sin\alpha \\ -\sin\alpha\sin\theta & \sin\alpha\cos\theta & \cos\alpha \end{pmatrix}.$$

By recalling the meaning of a rotation matrix in terms of the orientation of a current frame with respect to a fixed frame, it can be recognized that its rows are the direction cosines of the axes of the current frame with respect to the fixed frame, whilst its columns are the direction cosines of the axes of the fixed frame with respect to the current frame.

Let χ be the rear wheel rotation angle, which we assume to be zero in the trivial configuration. Then, the resulting orientation of the S_{Br} frame is obtained by[2]

$$\begin{aligned} \mathcal{R}_{Br} &= \mathcal{R}_{y_B}(\chi)\mathcal{R}_2(-\alpha)\mathcal{R}_1(\theta) = \\ &= \begin{pmatrix} \mathrm{c}_\chi\,\mathrm{c}_\theta + \mathrm{s}_\alpha\,\mathrm{s}_\chi\,\mathrm{s}_\theta & \mathrm{c}_\chi\,\mathrm{s}_\theta - \mathrm{s}_\alpha\,\mathrm{s}_\chi\,\mathrm{c}_\theta & -\mathrm{c}_\alpha\,\mathrm{s}_\chi \\ -\mathrm{c}_\alpha\,\mathrm{s}_\theta & \mathrm{c}_\alpha\,\mathrm{c}_\theta & -\mathrm{s}_\alpha \\ \mathrm{s}_\chi\,\mathrm{c}_\theta - \mathrm{s}_\alpha\,\mathrm{c}_\chi\,\mathrm{s}_\theta & \mathrm{s}_\chi\,\mathrm{s}_\theta + \mathrm{s}_\alpha\,\mathrm{c}_\chi\,\mathrm{c}_\theta & \mathrm{c}_\alpha\,\mathrm{c}_\chi \end{pmatrix}, \end{aligned}$$

[2] The notations c and s are the abbreviations for cos and sin, respectively.

where the rotation $\mathcal{R}_{y_B}(\chi)$ is taken about the y_B-axis. Similarly, for the S_B orientation we have

$$\mathcal{R}_B = \mathcal{R}_3(-\mu)\mathcal{R}_2(-\alpha)\mathcal{R}_1(\theta) = \\ = \begin{pmatrix} \mathrm{c}_\mu\,\mathrm{c}_\theta - \mathrm{s}_\alpha\,\mathrm{s}_\mu\,\mathrm{s}_\theta & \mathrm{c}_\mu\,\mathrm{s}_\theta + \mathrm{s}_\alpha\,\mathrm{s}_\mu\,\mathrm{c}_\theta & \mathrm{c}_\alpha\,\mathrm{s}_\mu \\ -\,\mathrm{c}_\alpha\,\mathrm{s}_\theta & \mathrm{c}_\alpha\,\mathrm{c}_\theta & -\,\mathrm{s}_\alpha \\ -\,\mathrm{s}_\mu\,\mathrm{c}_\theta - \mathrm{s}_\alpha\,\mathrm{c}_\mu\,\mathrm{s}_\theta & -\,\mathrm{s}_\mu\,\mathrm{s}_\theta + \mathrm{s}_\alpha\,\mathrm{c}_\mu\,\mathrm{c}_\theta & \mathrm{c}_\alpha\,\mathrm{c}_\mu \end{pmatrix}.$$

Then, we draw the rotation matrix characteristic of the front assembly, which describes the S_D orientation with respect to Σ. This is given by the composition of four different matrices, that is,

$$\mathcal{R}_D = \mathcal{R}_4(\psi)\mathcal{R}_3(-\mu-\varepsilon)\mathcal{R}_2(-\alpha)\mathcal{R}_1(\theta) = \\ = \begin{pmatrix} \mathrm{c}_\psi(\mathrm{c}_{\mu+\varepsilon}\,\mathrm{c}_\theta - \mathrm{s}_\alpha\,\mathrm{s}_{\mu+\varepsilon}\,\mathrm{s}_\theta) - \mathrm{c}_\alpha\,\mathrm{s}_\psi\,\mathrm{s}_\theta \\ \mathrm{s}_\psi(\mathrm{s}_\alpha\,\mathrm{s}_{\mu+\varepsilon}\,\mathrm{s}_\theta - \mathrm{c}_{\mu+\varepsilon}\,\mathrm{c}_\theta) - \mathrm{c}_\alpha\,\mathrm{c}_\psi\,\mathrm{s}_\theta \\ -\,\mathrm{s}_{\mu+\varepsilon}\,\mathrm{c}_\theta - \mathrm{s}_\alpha\,\mathrm{c}_{\mu+\varepsilon}\,\mathrm{s}_\theta \end{pmatrix. \\ \begin{matrix}. \mathrm{c}_\psi(\mathrm{c}_{\mu+\varepsilon}\,\mathrm{s}_\theta + \mathrm{s}_\alpha\,\mathrm{s}_{\mu+\varepsilon}\,\mathrm{c}_\theta) + \mathrm{c}_\alpha\,\mathrm{s}_\psi\,\mathrm{c}_\theta & \mathrm{c}_\alpha\,\mathrm{s}_{\mu+\varepsilon}\,\mathrm{c}_\psi - \mathrm{s}_\alpha\,\mathrm{s}_\psi \\ \mathrm{c}_\alpha\,\mathrm{c}_\psi\,\mathrm{c}_\theta - \mathrm{s}_\psi(\mathrm{c}_{\mu+\varepsilon}\,\mathrm{s}_\theta + \mathrm{s}_\alpha\,\mathrm{s}_{\mu+\varepsilon}\,\mathrm{c}_\theta) & -\,\mathrm{c}_\alpha\,\mathrm{s}_{\mu+\varepsilon}\,\mathrm{s}_\psi - \mathrm{s}_\alpha\,\mathrm{c}_\psi \\ -\,\mathrm{s}_{\mu+\varepsilon}\,\mathrm{s}_\theta + \mathrm{s}_\alpha\,\mathrm{c}_{\mu+\varepsilon}\,\mathrm{c}_\theta & \mathrm{c}_\alpha\,\mathrm{c}_{\mu+\varepsilon} \end{pmatrix}.$$

Now, in order to overcome the complexity of $\mathcal{R}_D$, we believe convenient to introduce three more auxiliary rotation angles which define the same transformation. It is common knowledge that three independent rotations are sufficient to describe the orientation of a rigid body in space. Thus, let $\widetilde{\theta}$, $\widetilde{\alpha}$, and $\widetilde{\mu}$ be the front yaw angle, roll or camber angle, and pitch angle, taking the signs in accordance with the previous conventions, so we have the rotation matrix

$$\widetilde{\mathcal{R}}_D = \widetilde{\mathcal{R}}_3(-\widetilde{\mu})\widetilde{\mathcal{R}}_2(-\widetilde{\alpha})\widetilde{\mathcal{R}}_1(\widetilde{\theta}) = \\ = \begin{pmatrix} \mathrm{c}_{\widetilde{\mu}}\,\mathrm{c}_{\widetilde{\theta}} - \mathrm{s}_{\widetilde{\alpha}}\,\mathrm{s}_{\widetilde{\mu}}\,\mathrm{s}_{\widetilde{\theta}} & \mathrm{c}_{\widetilde{\mu}}\,\mathrm{s}_{\widetilde{\theta}} + \mathrm{s}_{\widetilde{\alpha}}\,\mathrm{s}_{\widetilde{\mu}}\,\mathrm{c}_{\widetilde{\theta}} & \mathrm{c}_{\widetilde{\alpha}}\,\mathrm{s}_{\widetilde{\mu}} \\ -\,\mathrm{c}_{\widetilde{\alpha}}\,\mathrm{s}_{\widetilde{\theta}} & \mathrm{c}_{\widetilde{\alpha}}\,\mathrm{c}_{\widetilde{\theta}} & -\,\mathrm{s}_{\widetilde{\alpha}} \\ -\,\mathrm{s}_{\widetilde{\mu}}\,\mathrm{c}_{\widetilde{\theta}} - \mathrm{s}_{\widetilde{\alpha}}\,\mathrm{c}_{\widetilde{\mu}}\,\mathrm{s}_{\widetilde{\theta}} & -\,\mathrm{s}_{\widetilde{\mu}}\,\mathrm{s}_{\widetilde{\theta}} + \mathrm{s}_{\widetilde{\alpha}}\,\mathrm{c}_{\widetilde{\mu}}\,\mathrm{c}_{\widetilde{\theta}} & \mathrm{c}_{\widetilde{\alpha}}\,\mathrm{c}_{\widetilde{\mu}} \end{pmatrix}.$$

Because $\mathcal{R}_D$ and $\widetilde{\mathcal{R}}_D$ describe the same transformation, the two matrices are identical. So equating the direction cosines we obtain nine expressions of the auxiliary angles, used in the following, such as

$$\sin\widetilde{\alpha} = \cos\alpha\sin(\mu+\varepsilon)\sin\psi + \sin\alpha\cos\psi, \tag{3.1}$$

although they are not all independent, due to the orthogonality conditions between the direction cosines.

Finally, introducing the front wheel rotation angle χ_f, which is set to zero in the trivial configuration, the direct transformation from Σ to S_{Er} is given by the rotation matrix $\mathcal{R}_{Er} = \mathcal{R}_{y_E}(\chi_f)\widetilde{\mathcal{R}}_2(-\widetilde{\alpha})\widetilde{\mathcal{R}}_1(\widetilde{\theta})$, and by analogy with the rear wheel, the $S_{F'}$ and S_F orientations are obtained by $\mathcal{R}_{F'} = \widetilde{\mathcal{R}}_1(\widetilde{\theta})$ and $\mathcal{R}_F = \widetilde{\mathcal{R}}_2(-\widetilde{\alpha})\widetilde{\mathcal{R}}_1(\widetilde{\theta})$, respectively.

3.1.2 Configuration space

Once we have geometrically characterised the general bicycle model in its trivial configuration, we need to introduce a certain number of coordinates for describing the behaviour of the system. We observe that the bicycle is composed by four rigid bodies, therefore, without any constraints, it should have 24 degrees of freedom. However, imposing the holonomic constraints on the system, that is, frames are connected by three hinges and each wheel has one contact point with the flat ground plane, the minimum number of generalised coordinates needed to describe the configuration space is equal to seven.

Hence, the motion of the system is characterised by orbits on a 7-dimensional manifold. We remark that the number of generalised coordinates for nonholonomic systems is different from the number of degrees of freedom of the system itself. Afterwards, as the nonholonomic constraints will be imposed on the system, we will see that our bicycle has three degrees of freedom.

In principle, any set of generalised coordinates is good as another, hence, according to [RF12], we choose our seven independent generalised coordinates as follows:

1. the triple (x, y, θ), which gives the translational position of the rear contact point together with its rotational position;
2. the *roll angle* α;
3. the *steering angle* ψ;
4. the rear and front wheel rotation angles χ_r and χ_f, respectively.

In summary, identifying the *Euclidean group* in the plane SE(2) as the group of translations and rotations in the plane, that is, the group of rigid motions in the plane, the configuration space of the bicycle is given by

$$Q = \mathrm{SE}(2) \times S^1 \times S^1 \times S^1 \times S^1,$$

which we parametrize with the coordinate vector $q = (x, y, \theta, \chi_f, \alpha, \psi, \chi)^T$. We note that, by definition, all the generalised coordinates are equal to zero in the trivial configuration.

We also remark that the pitch angle, as well as the coordinates x_f and y_f of the point F', are not independent of the generalized coordinates, therefore it is possible to express them as a functions of these coordinates. In the next section, we will derive the relations of both this angle and the front contact point coordinates with respect to the coordinate vector q.

3.1.3 Pitch angle and front contact point.

We have mentioned that both the pitch angle and the front wheel contact point depend on the generalized coordinates chosen. Their expressions can be derived by writing the vector $(F' - A')$ explicitly. It is clear that, in the trivial configuration, this vector has magnitude w and direction parallel to the X-axis, but in general its length and direction are not constant. First of all, we observe that

$$\begin{aligned}(F' - A') = (F' - F) + (F - E) + (E - D) +& \\ + (D - C) + (C - B) + (B - A) + (A - A'),&\end{aligned} \tag{3.2}$$

and then we express each of the vectors on the right hand side with respect to the inertial reference frame. Because the wheels are toroidal, we have

$$(A - A') = r\vec{k}, \qquad\qquad (F' - F) = -r_f\vec{k},$$

where $\{\vec{\imath}, \vec{\jmath}, \vec{k}\}$ are the unit vectors of the coordinate system Σ. Then, using the direction cosines of the rotation matrix $\mathcal{R}_A$, the position of the rear hub B in the inertial reference frame is given by

$$(B-A)_\Sigma = -R\sin\alpha\sin\theta\vec{\imath} + R\sin\alpha\cos\theta\vec{\jmath} + R\cos\alpha\vec{k}.$$

Likewise, we proceed with the other vectors. In particular, we have

$$\begin{aligned}(C-B) &= l(\cos\mu\cos\theta - \sin\alpha\sin\mu\sin\theta)\,\vec{\imath} + \\ &\quad + l(\cos\mu\sin\theta + \sin\alpha\sin\mu\cos\theta)\vec{\jmath} + l\cos\alpha\sin\mu\,\vec{k},\end{aligned}$$

$$\begin{aligned}(D-C) &= b(\sin\widetilde{\mu}\cos\widetilde{\theta} + \sin\widetilde{\alpha}\cos\widetilde{\mu}\sin\widetilde{\theta})\,\vec{\imath} + \\ &\quad + b(\sin\widetilde{\mu}\sin\widetilde{\theta} - \sin\widetilde{\alpha}\cos\widetilde{\mu}\cos\widetilde{\theta})\vec{\jmath} - b\cos\widetilde{\alpha}\cos\widetilde{\mu}\,\vec{k},\end{aligned}$$

$$\begin{aligned}(E-D) &= d(\cos\widetilde{\mu}\cos\widetilde{\theta} - \sin\widetilde{\alpha}\sin\widetilde{\mu}\sin\widetilde{\theta})\,\vec{\imath} + \\ &\quad + d(\cos\widetilde{\mu}\sin\widetilde{\theta} + \sin\widetilde{\alpha}\sin\widetilde{\mu}\cos\widetilde{\theta})\vec{\jmath} + d\cos\widetilde{\alpha}\sin\widetilde{\mu}\,\vec{k},\end{aligned}$$

and

$$\begin{aligned}(F-E) &= -R_f\sin\widetilde{\mu}\vec{\imath}_{S_D} - R_f\cos\widetilde{\mu}\vec{k}_{S_D} = \\ &= R_f\sin\widetilde{\alpha}\sin\widetilde{\theta}\vec{\imath} - R_f\sin\widetilde{\alpha}\cos\widetilde{\theta}\vec{\jmath} - R_f\cos\widetilde{\alpha}\vec{k},\end{aligned}$$

where the auxiliary angles introduced above are used for the vectors characterising the front assembly. Therefore, from relation (3.2), we obtain three scalar equations. The first and the second give the coordinates of the front contact point, that is,

$$\begin{aligned}x_f &= x - R\sin\alpha\sin\theta + l(\cos\mu\cos\theta - \sin\alpha\sin\mu\sin\theta) + \\ &\quad + b(\sin(\mu+\varepsilon)\cos\theta + \sin\alpha\cos(\mu+\varepsilon)\sin\theta) + \\ &\quad + d\cos\psi(\cos(\mu+\varepsilon)\cos\theta - \sin\alpha\sin(\mu+\varepsilon)\sin\theta) + \\ &\quad - d\cos\alpha\sin\psi\sin\theta + R_f\sin\widetilde{\alpha}\sin\widetilde{\theta}\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}y_f &= y + R\sin\alpha\cos\theta + l(\cos\mu\sin\theta + \sin\alpha\sin\mu\cos\theta) + \\ &\quad + b(\sin(\mu+\varepsilon)\sin\theta - \sin\alpha\cos(\mu+\varepsilon)\cos\theta) + \\ &\quad + d\cos\psi(\cos(\mu+\varepsilon)\sin\theta + \sin\alpha\sin(\mu+\varepsilon)\cos\theta) + \\ &\quad + d\cos\alpha\sin\psi\cos\theta - R_f\sin\widetilde{\alpha}\cos\widetilde{\theta},\end{aligned} \tag{3.4}$$

whereas the third one provides an algebraic equation for the pitch angle:

$$\begin{aligned}&(r - r_f) + R\cos\alpha + l\cos\alpha\sin\mu - b\cos\alpha\cos(\mu+\varepsilon) + \\ &\quad + d(\cos\alpha\sin(\mu+\varepsilon)\cos\psi - \sin\alpha\sin\psi) - R_f\cos\widetilde{\alpha} = 0,\end{aligned} \tag{3.5}$$

where we have used the auxiliary angles for writing these expressions easily.

3.2 Definition of the CPA bicycle

Because a closed-form solution for equation (3.5) is difficult to be found, now we have derived the algebraic equation which defines the pitch angle μ we want to find the minimum conditions such that this angle does not depend on the generalised coordinates, that is, it is constant in time. Assuming that R_f and d are equal to zero, and $r = r_f$, we obtain

$$R\cos\alpha + l\cos\alpha\sin\mu - b\cos\alpha\cos(\mu+\varepsilon) = 0,$$

and dividing by $\cos\alpha$, which is always different from zero, we have

$$R + l\sin\mu - b\cos(\mu+\varepsilon) = 0. \tag{3.6}$$

Thus, in this situation, the effective pitch angle is constant and equal to the trivial pitch angle φ. From a physical point of view, the assumptions introduced above correspond to a zero fork offset and a spherical front wheel, which has the radius equal to the crown radius r, that is, $r_f = r$, as in Figure 3.2. We will call this particular bicycle as *Constant Pitch Angle* (CPA) bicycle.

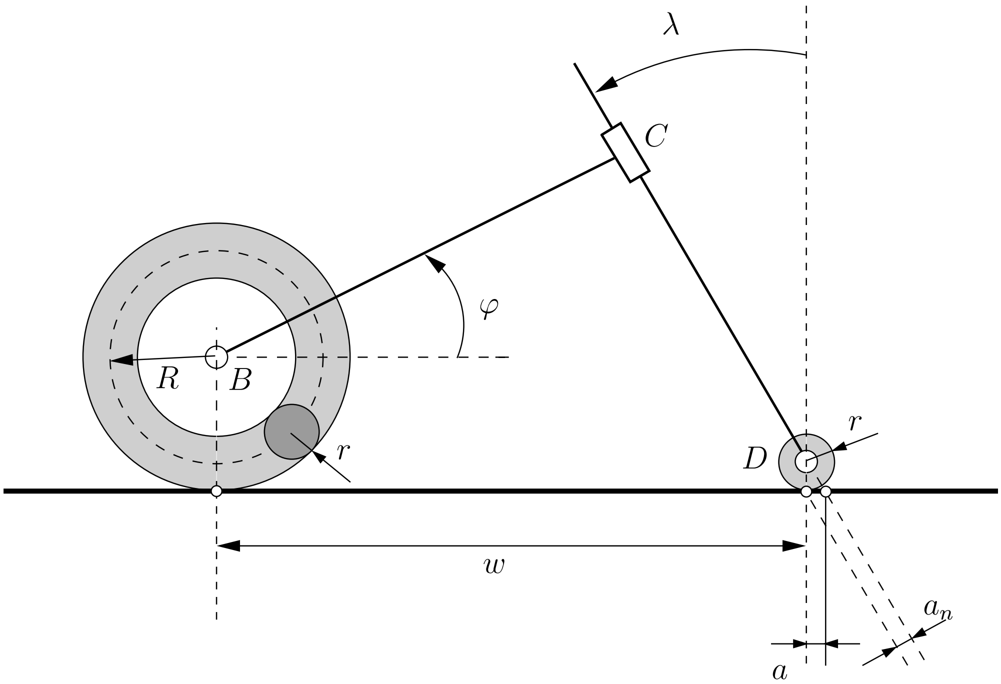

Figure 3.2: Model for the CPA bicycle.

Remark 3.3. Because the trivial pitch angle is arbitrary, we remark that it will be chosen such that the centre of mass of the rear frame can be characterized by only one coordinate in the local reference frame.

As a result of this simplification, also the expressions (3.3) and (3.4) for the front wheel contact point are simpler. In particular, we have

$$\begin{aligned} x_f &= x - [R + l\sin\varphi - b\cos(\varphi+\varepsilon)]\sin\alpha\sin\theta + \\ &\quad + [l\cos\varphi + b\sin(\varphi+\varepsilon)]\cos\theta = \\ &= x + w\cos\theta \end{aligned}$$

and

$$\begin{aligned} y_f &= y + [R + l\sin\varphi - b\cos(\varphi+\varepsilon)]\sin\alpha\sin\theta + \\ &\quad + [l\cos\varphi + b\sin(\varphi+\varepsilon)]\sin\theta = \\ &= y + w\sin\theta, \end{aligned}$$

respectively, where we remind that $w = l\cos\varphi + b\sin(\varphi+\varepsilon)$ is the wheelbase.

Furthermore, requiring that the pitch angle has to be constant, the wheelbase in a general configuration becomes constant. Indeed, the distance between the rear and front contact points is given by

$$\sqrt{(x_f - x)^2 - (y_f - y)^2} = \sqrt{w^2\cos^2\theta + w^2\sin^2\theta} = w.$$

However, even if both the pitch angle and the wheelbase do not depend on the generalised coordinates, the general configuration of the CPA bicycle is given by means of the rotational matrices we have introduced in Section 3.1.1.

3.3 Kinematics of the CPA bicycle

We now turn our attention to the kinematics of CPA bicycle. The linear velocities of each point of the system with respect to the inertial reference frame by differentiating the expression of its positions with respect to time. We write down the linear velocities of the four centres of mass in the inertial reference frame, which will be useful in the next chapter. Therefore, the rear wheel centre of mass has velocity

$$\begin{aligned} \vec{v}(B) = &(\dot{x} - R\dot{\alpha}\cos\alpha\sin\theta - R\dot{\theta}\sin\alpha\cos\theta)\vec{\imath} + \\ &+ (\dot{y} + R\dot{\alpha}\cos\alpha\cos\theta - R\dot{\theta}\sin\alpha\sin\theta)\vec{\jmath} + \\ &- R\dot{\alpha}\sin\alpha\vec{k}, \end{aligned} \tag{3.7}$$

while for we rear frame we have

$$\begin{aligned} \vec{v}(G) = \vec{v}(B) &+ \Big[l_2\left(-\dot{\alpha}\cos\alpha\sin\varphi\sin\theta - \dot{\theta}(\sin\alpha\sin\varphi\cos\theta + \cos\varphi\sin\theta)\right) + \\ &+ d_2\left(-\dot{\alpha}\sin\alpha\sin\theta - \dot{\theta}\cos\alpha\cos\theta\right)\Big]\vec{\imath} + \\ &+ \Big[l_2\left(\dot{\alpha}\cos\alpha\sin\varphi\cos\theta + \dot{\theta}(\cos\varphi\cos\theta - \sin\alpha\sin\varphi\sin\theta)\right) + \\ &+ d_2\left(-\dot{\alpha}\sin\alpha\cos\theta - \dot{\theta}\cos\alpha\sin\theta\right)\Big]\vec{\jmath} + \\ &+ (l_2\dot{\alpha}\sin\alpha\sin\varphi - d_2\dot{\alpha}\cos\alpha)\,\vec{k}, \end{aligned} \tag{3.8}$$

where $(G - B)_{S_B} = (l_2, d_2, 0)^T$; the velocity of the point C is obtain from equation (3.8) noting that $(C - B)_{S_B} = (l, 0, 0)^T$, hence the front frame centre of mass velocity is

$$\begin{aligned} \vec{v}(H) = \vec{v}(C) &+ \Big[h_3\Big(-\dot{\alpha}\cos\alpha\cos\lambda\sin\theta + \dot{\theta}(\sin\lambda\sin\theta - \sin\alpha\cos\lambda\cos\theta)\Big) + \\ &+ d_3\Big(\dot{\alpha}(\sin\alpha\cos\psi\sin\theta + \cos\alpha\sin\lambda\sin\psi\sin\theta) + \\ &\quad + \dot{\theta}(\sin\alpha\sin\lambda\sin\psi\cos\theta - \cos\alpha\cos\psi\cos\theta + \cos\lambda\sin\psi\sin\theta) + \\ &\quad + \dot{\psi}(\cos\alpha\sin\psi\sin\theta + \sin\alpha\sin\lambda\cos\psi\sin\theta - \cos\lambda\cos\psi\cos\theta)\Big) + \\ &+ l_3\Big(\dot{\alpha}(\sin\alpha\sin\psi\sin\theta - \cos\alpha\sin\lambda\cos\psi\sin\theta) + \end{aligned}$$

$$
\begin{aligned}
&\quad + \dot\theta(-\cos\lambda\cos\psi\sin\theta - \sin\alpha\sin\lambda\cos\psi\cos\theta - \cos\alpha\sin\psi\cos\theta)+\\
&\quad +\dot\psi(\sin\alpha\sin\lambda\sin\psi\sin\theta - \cos\alpha\cos\psi\sin\theta - \cos\lambda\sin\psi\cos\theta)\Big)\Big]\,\vec\imath+\\
&+\Big[h_3\Big(\dot\alpha\cos\alpha\cos\lambda\cos\theta + \dot\theta(-\sin\lambda\cos\theta - \sin\alpha\cos\lambda\sin\theta)\Big)+\\
&\;+ d_3\Big(\dot\alpha(-\sin\alpha\cos\psi\cos\theta - \cos\alpha\sin\lambda\sin\psi\cos\theta)+\\
&\quad + \dot\theta(\sin\alpha\sin\lambda\sin\psi\sin\theta - \cos\alpha\cos\psi\sin\theta - \cos\lambda\sin\psi\cos\theta)+\\
&\quad + \dot\psi(-\cos\alpha\sin\psi\cos\theta - \sin\alpha\sin\lambda\cos\psi\cos\theta - \cos\lambda\cos\psi\sin\theta)\Big)+\\
&\;+ l_3\Big(\dot\alpha(\cos\alpha\sin\lambda\cos\psi\cos\theta - \sin\alpha\sin\psi\cos\theta)+\\
&\quad + \dot\theta(\cos\lambda\cos\psi\cos\theta - \sin\alpha\sin\lambda\cos\psi\sin\theta - \cos\alpha\sin\psi\sin\theta)+\\
&\quad +\dot\psi(\cos\alpha\cos\psi\cos\theta - \sin\alpha\sin\lambda\sin\psi\cos\theta - \cos\lambda\sin\psi\sin\theta)\Big)\Big]\,\vec\jmath+\\
&+\Big[-h_3\dot\alpha\cos\lambda\sin\alpha + d_3\Big(\dot\alpha(\sin\alpha\sin\lambda\sin\psi - \cos\alpha\cos\psi)+\\
&\quad + \dot\psi(\sin\alpha\sin\psi - \cos\alpha\sin\lambda\cos\psi)\Big)+\\
&\;+ l_3\Big(\dot\alpha(-\sin\alpha\sin\lambda\cos\psi - \cos\alpha\sin\psi)+\\
&\quad +\dot\psi(-\cos\alpha\sin\lambda\sin\psi - \sin\alpha\cos\psi)\Big)\Big]\,\vec k,
\end{aligned}
$$

whilst the velocity of the front wheel is simply

$$\vec v(E) = \dot x_f\vec\imath + \dot y_f\vec\jmath = (\dot x - w\dot\theta\sin\theta)\vec\imath + (\dot y + w\dot\theta\cos\theta)\vec\jmath.$$

Moreover, the angular velocities of each body can be easily obtained in the body-fixed frame by the rule

$$\vec\Omega_S = \left\langle \frac{d\vec\jmath_S}{dt}, \vec k_S\right\rangle \vec\imath_S + \left\langle \frac{d\vec k_S}{dt}, \vec\imath_S\right\rangle \vec\jmath_S + \left\langle \frac{d\vec\imath_S}{dt}, \vec\jmath_S\right\rangle \vec k_S,$$

where the pairing $\langle\cdot,\cdot\rangle$ is the scalar product, while the versors are obtained as the rows of the rotation matrices introduced above. Therefore, by a slightly lengthy but straightforward calculation, all the kinematic quantities of interest shall be provided. In particular, we write down the angular velocities in the local reference frame, that is,

$$
\begin{aligned}
\vec\omega_1 = &-(\dot\alpha\cos\chi + \dot\theta\cos\alpha\sin\chi)\vec\imath_{S_{Br}}+\\
&+(\dot\chi - \dot\theta\sin\alpha)\vec\jmath_{S_{Br}} + (\dot\theta\cos\alpha\cos\chi - \dot\alpha\sin\chi)\vec k_{S_{Br}}
\end{aligned}
$$

for the rear wheel, considering the proper rotation,

$$
\begin{aligned}
\vec\omega_2 = &(-\dot\alpha\cos\varphi + \dot\theta\cos\alpha\sin\varphi)\vec\imath_{S_B}+\\
&-\dot\theta\sin\alpha\vec\jmath_{S_B} + (\dot\alpha\sin\varphi + \dot\theta\cos\alpha\cos\varphi)\vec k_{S_B}
\end{aligned}
$$

for the rear frame,

$$
\begin{aligned}
\vec\omega_3 = &\Big[-\dot\alpha\cos\lambda\cos\psi + \dot\theta(\cos\alpha\sin\lambda\cos\psi - \sin\alpha\sin\psi)\Big]\,\vec\imath_{S_D}+\\
&+\Big[\dot\alpha\cos\lambda\sin\psi - \dot\theta(\cos\alpha\sin\lambda\sin\psi + \sin\alpha\cos\psi)\Big]\,\vec\jmath_{S_D}+\\
&+\Big[\dot\alpha\sin\lambda + \dot\theta\cos\alpha\cos\lambda + \dot\psi\Big]\,\vec k_{S_D}
\end{aligned}
$$

for the front frame, and, using the auxiliary angles, the angular velocity of the front wheel is

$$\begin{aligned}\vec{\omega}_4 = &-(\dot{\tilde{\alpha}}\cos\chi_a + \dot{\tilde{\theta}}\cos\tilde{\alpha}\sin\chi_a)\vec{\imath}_{S_{Er}} + \\ &+ (\dot{\chi}_a - \dot{\tilde{\theta}}\sin\tilde{\alpha})\vec{\jmath}_{S_{Er}} + (\dot{\tilde{\theta}}\cos\tilde{\alpha}\cos\chi_a - \dot{\tilde{\alpha}}\sin\chi_a)\vec{k}_{S_{Er}},\end{aligned}$$

by analogy with the rear wheel. Then, we have to impose the nonholonomic constraints on the two contact points.

3.3.1 Nonholonomic constraints

In general, the most interesting aspect of the bicycle is probably related to the nonholonomic constraints on the velocities of the two contact points. Even for our simplified model, we require that both the wheels roll on the plane without slipping. It is common knowledge that this particular constraints are not integrable in the sense of Frobenius's theorem. For deriving the constraint equations, we consider the infinitesimal displacement of the wheel, as shown in Figure 3.3.

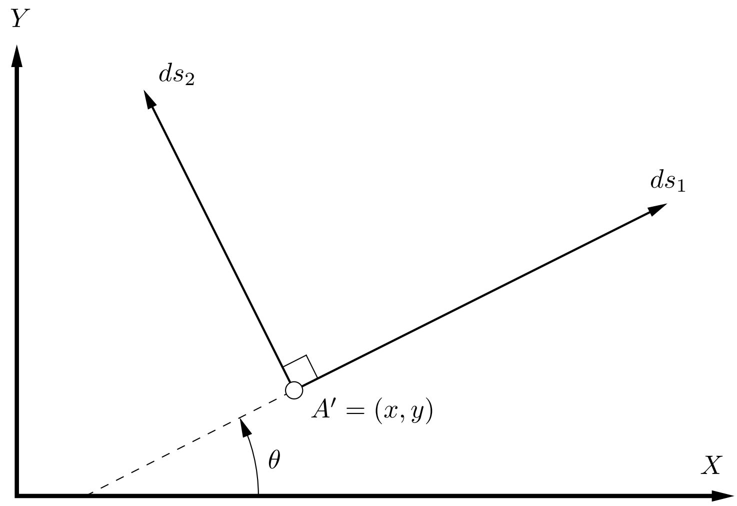

Figure 3.3: Infinitesimal displacement of the rear contact point.

In fact, an increment of the angle χ by the amount $d\chi$, considering θ and α constant, corresponds to a displacement of the point A' through an interval $ds_1 = (R + r\cos\alpha)d\chi$, whereas an increment of the roll angle α by the amount $d\alpha$, while θ and χ are constant, corresponds to a displacement of the point A' through the interval $ds_2 = rd\alpha$. Obviously, if the yaw angle θ varies, with α and χ constant, the point A does not move. Hence, we have

$$\begin{cases} dx = ds_1\cos\theta - ds_2\sin\theta, \\ dy = ds_1\sin\theta + ds_2\cos\theta, \end{cases}$$

and differentiating with respect to time, we obtain the kinematic constraints

$$\begin{cases} \dot{x} = -r\dot{\alpha}\sin\theta + (R + r\cos\alpha)\dot{\chi}\cos\theta, \\ \dot{y} = r\dot{\alpha}\cos\theta + (R + r\cos\alpha)\dot{\chi}\sin\theta. \end{cases} \tag{3.9}$$

Remark 3.4. Because of the sign conventions for the roll angle, the infinitesimal displacement are positive as depicted in Figure 3.3.

We note that the constraint equations (3.9) can also be written in a different way, that is,

$$\begin{cases} \dot{x}\cos\theta + \dot{y}\sin\theta = (R + r\cos\alpha)\dot{\chi}, \\ \dot{x}\sin\theta - \dot{y}\cos\theta = -r\dot{\alpha}. \end{cases} \tag{3.10}$$

Moreover, the front contact point constraints shall be expressed by analogy with those on the point A'. Therefore, using the auxiliary angles, the condition for rolling without slipping is simply given by

$$\begin{cases} \dot{x}_f\cos\widetilde{\theta} + \dot{y}_f\sin\widetilde{\theta} = r_f\cos\widetilde{\alpha}\dot{\chi}_f, \\ \dot{x}_f\sin\widetilde{\theta} - \dot{y}_f\cos\widetilde{\theta} = -r_f\dot{\widetilde{\alpha}}. \end{cases} \tag{3.11}$$

Remark 3.5. As mentioned before, the nonholonomic constraints introduced above reduce the free velocities of the systems. Indeed, our bicycle model has just three degrees of freedom. This is a particular feature of nonholonomic systems, which have less degrees of freedom than the number of generalized coordinates. Furthermore, the choice of the degrees of freedom is arbitrary and depends on the particular situation studied.

Because equations (3.11) are written by means of the auxiliary angles, we need to express them with respect to the generalized coordinates. First of all, comparing matrix $\mathcal{R}_D$ with $\widetilde{\mathcal{R}}_D$ and being $\mu \equiv \varphi = \lambda - \varepsilon$, we note that

$$\begin{aligned} \sin\widetilde{\theta}\cos\widetilde{\alpha} = {}& (\cos\alpha\cos\psi - \sin\alpha\sin\lambda\sin\psi)\sin\theta + \\ & + \cos\lambda\sin\psi\cos\theta, \end{aligned}$$

and

$$\begin{aligned} \cos\widetilde{\theta}\cos\widetilde{\alpha} = {}& (\cos\alpha\cos\psi - \sin\alpha\sin\lambda\sin\psi)\cos\theta + \\ & - \cos\lambda\sin\psi\sin\theta, \end{aligned}$$

whereas the front contact point velocity can be obtained by taking the derivative respect to time of (3.3) and (3.4). Moreover, deriving with respect to time equation (3.1), we have

$$\begin{aligned} \dot{\widetilde{\alpha}}\cos\widetilde{\alpha} = {}& -\dot{\alpha}\sin\alpha\sin\lambda\sin\psi + \dot{\psi}\cos\alpha\sin\lambda\cos\psi + \\ & + \dot{\alpha}\cos\alpha\cos\psi - \dot{\psi}\sin\alpha\sin\psi. \end{aligned} \tag{3.12}$$

We now have all the relations needed to express the nonholonomic constraints of the front contact point with respect to generalized velocities, eliminating the auxiliary angles. Therefore, starting form the second of (3.11), we write it as

$$\begin{aligned} & (\dot{x}\sin\theta - \dot{y}\cos\theta)(\cos\alpha\cos\psi - \sin\alpha\sin\lambda\sin\psi) + \\ & + (\dot{x}\cos\theta + \dot{y}\sin\theta)\cos\lambda\sin\psi + w\dot{\theta}(\sin\alpha\sin\lambda\sin\psi - \cos\alpha\cos\psi) = \\ & = -r\dot{\widetilde{\alpha}}\cos\widetilde{\alpha}, \end{aligned}$$

and, using equation (3.12) and constraints (3.10), we obtain

$$\begin{aligned} & (R + r\cos\alpha)\dot{\chi}\cos\lambda\sin\psi + r\dot{\psi}(\cos\alpha\sin\lambda\cos\psi - \sin\alpha\sin\psi) + \\ & + w\dot{\theta}(\sin\alpha\sin\lambda\sin\psi - \cos\alpha\cos\psi) = 0. \end{aligned} \tag{3.13}$$

Likewise, the first constraint of (3.11) becomes

$$\begin{aligned} & (\dot{x}\cos\theta + \dot{y}\sin\theta)(\cos\alpha\cos\psi - \sin\alpha\sin\lambda\sin\psi) + \\ & - (\dot{x}\sin\theta - \dot{y}\cos\theta)\cos\lambda\sin\psi + w\dot{\theta}\cos\lambda\sin\psi = r\cos^2\widetilde{\alpha}\dot{\chi}_f \end{aligned}$$

and, using constraints (3.10), we obtain

$$\begin{aligned}&(R + r\cos\alpha)\dot{\chi}(\cos\alpha\cos\psi - \sin\alpha\sin\lambda\sin\psi) + r\dot{\alpha}\cos\lambda\sin\psi + \\ &+ w\dot{\theta}\cos\lambda\sin\psi = r\cos^2\widehat{\alpha}\dot{\chi}_f.\end{aligned} \tag{3.14}$$

As we have stated before, choosing three degrees of freedom we can constrain four generalized velocities. In particular, riding a bicycle, one controls the roll angle, the steering angle and the forward velocity, therefore we opt for these as free coordinates. Consequently, relations (3.14) and (3.13) shall be expressed with respect to $\dot{\chi}_f$ and $\dot{\theta}$, respectively. For simplifying the computation, we introduce the nonlinear functions

$$\widehat{a}(\alpha, \psi) = w(\cos\alpha\cos\psi - \sin\lambda\sin\alpha\sin\psi)$$

and

$$\widehat{d}(\alpha, \psi) = (R + r\cos\alpha)(\cos\alpha\cos\psi - \sin\alpha\sin\lambda\sin\psi),$$

therefore, noting that $\widehat{a}(\alpha, \psi) > 0$ for a bicycle usual geometric parameter values, we can write (3.14) and (3.13) as

$$\dot{\chi}_f = \dot{\chi}\widehat{g}(\alpha, \psi) + \dot{\alpha}\widehat{h}(\alpha, \psi) + \dot{\psi}\widehat{l}(\alpha, \psi) \tag{3.15}$$

and

$$\dot{\theta} = \dot{\chi}\widehat{m}(\alpha, \psi) + \dot{\psi}\widehat{p}(\alpha, \psi), \tag{3.16}$$

respectively, where

$$\begin{cases} \widehat{g}(\alpha, \psi) = \dfrac{1}{r\cos^2\widetilde{\alpha}}\left[\widehat{d}(\alpha, \psi) + w\widehat{m}(\alpha, \psi)\cos\lambda\sin\psi\right], \\[2ex] \widehat{h}(\alpha, \psi) = \dfrac{\cos\lambda\sin\psi}{\cos^2\widetilde{\alpha}}, \\[2ex] \widehat{l}(\alpha, \psi) = \dfrac{w}{\cos^2\widetilde{\alpha}}\left[\dfrac{\cos\alpha\sin\lambda\cos\psi - \sin\alpha\sin\psi}{\widehat{a}(\alpha, \psi)}\cos\lambda\sin\psi\right], \end{cases}$$

and

$$\begin{cases} \widehat{m}(\alpha, \psi) = \dfrac{1}{\widehat{a}(\alpha, \psi)}(R + r\cos\alpha)\cos\lambda\sin\psi, \\[2ex] \widehat{p}(\alpha, \psi) = \dfrac{r}{\widehat{a}(\alpha, \psi)}\left[\cos\alpha\sin\lambda\cos\psi - \sin\alpha\sin\psi\right]. \end{cases}$$

Remark 3.6. It is clear that the explicit expression of constrain (3.15) is obtained substituting $\dot{\theta}$ with the relation (3.16), as it is not a free generalised velocity.

Remark 3.7. The nonlinear functions introduced in expressions (3.15) and (3.16) are the same used in [RF12], considering the assumptions $r = r_f$, $d = 0$, $R_f = 0$, and relation (3.6).

Chapter 4

Dynamics of the CPA bicycle

We now turn our attention to the dynamics of the CPA bicycle introduced in the previous chapter. The equations of motion will be derived by means of the Euler-Lagrange equations for nonholonomic systems. Because we need to write the Lagrangian for our system, we start by writing the kinetic energy and the potential associated with the CPA bicycle.

4.1 Kinetic energy and potential

The kinetic energy of the the CPA bicycle is clearly equal to the sum of the kinetic energy of each of the rigid bodies which compose the system. Each kinetic energy is computed by König's theorem, which states that the kinetic energy of each body is the sum of the kinetic energy associated to the movement of the centre of mass and the kinetic energy associated to the movement of the particles relative to the centre of mass, that is,

$$K_i = \frac{1}{2} m_1 v^2(P_i) + \frac{1}{2}\langle \vec{\omega}_i, \sigma_i(P_i)\vec{\omega}_i \rangle, \qquad i = 1, 2, 3, 4,$$

where $\vec{\omega}_i$ are the angular velocities introduced before, m_i is the mass of the i-th body, P_i its centre of mass and $\sigma_i(P_i)$ its inertia tensor in the local reference frame of the body.

For example, let us write the kinetic energy for the rear wheel. The velocity of the centre of mass of the rear wheel is given by equation (3.7), hence

$$\begin{aligned} v^2(B) = {} & \dot{x}^2 + \dot{y}^2 + R^2\dot{\alpha}^2 + R^2\dot{\theta}^2 \sin^2\alpha + \\ & + 2R\dot{\alpha}\cos\alpha(-\dot{x}\sin\theta + \dot{y}\cos\theta) - 2R\dot{\theta}\sin\alpha(\dot{x}\cos\theta + \dot{y}\sin\theta), \end{aligned}$$

while the inertia tensor of a torus is

$$\sigma_1(B) = \begin{pmatrix} I_{1xx} & 0 & 0 \\ 0 & I_{1yy} & 0 \\ 0 & 0 & I_{1xx} \end{pmatrix},$$

where

$$I_{1xx} = \left(\frac{5}{8}r^2 + \frac{1}{2}R^2\right) m_1 \qquad\qquad I_{1yy} = \left(\frac{3}{4}r^2 + R^2\right) m_1,$$

thus we have

$$\langle \vec{\omega}_{S_{Br}}, \sigma_1(B)\vec{\omega}_{S_{Br}} \rangle = I_{1xx}(\dot{\alpha}^2 + \dot{\theta}^2\cos^2\alpha) + I_{1yy}(\dot{\chi} + \dot{\theta}\sin\alpha)^2.$$

In conclusion, the kinetic energy of the rear wheel is

$$\begin{aligned}K_1 = &\frac{1}{2}m_1\left[\dot{x}^2+\dot{y}^2+R^2\dot{\alpha}^2+R^2\dot{\theta}^2\sin^2\alpha\right]+\\&+m_1R\left[\dot{\alpha}\cos\alpha(-\dot{x}\sin\theta+\dot{y}\cos\theta)-\dot{\theta}\sin\alpha(\dot{x}\cos\theta+\dot{y}\sin\theta)\right]+\\&+\frac{1}{2}(\dot{\alpha}^2+\dot{\theta}^2\cos^2\alpha)I_{1xx}+\frac{1}{2}(\dot{\chi}-\dot{\theta}\sin\alpha)^2I_{1yy}.\end{aligned}$$

Likewise, it would be possible to write down the expression for the kinetic energy of the other rigid bodies. However their expressions are very complicated. Therefore, we need to write the total kinetic energy in a more manageable form, which can be readily used to compute the equations of motion. In particular, we write the kinetic energy in a general form by means of nonlinear functions which are defined as the Hessian of the kinetic energy itself, that is, by computing the second partial derivative of the kinetic energy with respect to the velocities which multiply each of these functions. Furthermore, we write these functions such that they depend only on the roll and the steering angles, as this choice will simplify future computations. Thus, it can be easily prove that the kinetic energy has the following general form:

$$\begin{aligned}K(\theta,\alpha,\psi,\dot{q}) = &\frac{1}{2}M(\dot{x}^2+\dot{y}^2)+\frac{1}{2}\dot{\alpha}^2a(\alpha,\psi)+\frac{1}{2}\dot{\theta}^2b(\alpha,\psi)+\frac{1}{2}\dot{\psi}^2d(\alpha,\psi)+\\&+\frac{1}{2}\dot{\chi}^2E+\frac{1}{2}\dot{\chi}_fF+\dot{\alpha}\dot{\theta}g(\alpha,\psi)+\dot{\alpha}\dot{\psi}h(\alpha,\psi)+\dot{\alpha}\dot{\chi}_fk(\alpha,\psi)\\&+\dot{\theta}\dot{\psi}l(\alpha,\psi)+\dot{\psi}\dot{\chi}_fm(\alpha,\psi)+\dot{\theta}\dot{\chi}n(\alpha)+\dot{\theta}\dot{\chi}_fp(\alpha,\psi)+\\&+\dot{\theta}(-\dot{x}\sin\theta+\dot{y}\cos\theta)q(\psi)+\dot{\alpha}(-\dot{x}\sin\theta+\dot{y}\cos\theta)r(\alpha,\psi)+\\&+\dot{\psi}(-\dot{x}\sin\theta+\dot{y}\cos\theta)s(\alpha,\psi)+\dot{\theta}(\dot{x}\cos\theta+\dot{y}\sin\theta)u(\alpha,\psi)+\\&+\dot{\psi}(\dot{x}\cos\theta+\dot{y}\sin\theta)z(\psi).\end{aligned}$$

The nonlinear functions are evaluated by means of the Wolfram Mathematica code, reported in Appendix A. For instance, we have

$$M=m_1+m_2+m_3+m_4.$$

In the same way, the potential can be written as a general function of the roll and steering angles, that is, $U=U(\alpha,\psi)$. Assuming that the system is subjected to only the gravity force acting on the centres of mass, we have

$$\begin{aligned}U(\alpha,\psi) = &-g\,[Mr+(m_1+m_2+m_3)R\cos\alpha+\\&+m_2(l_2\cos\alpha\sin\varphi-d_2\sin\alpha+h_2\cos\alpha\cos\varphi)+\\&+m_3(l\cos\alpha\sin\varphi+l_3(\cos\alpha\sin\lambda\cos\psi-\sin\alpha\sin\psi)+\\&\;+d_3(-\cos\alpha\sin\lambda\sin\psi-\sin\alpha\cos\psi)+h_3\cos\alpha\cos\lambda)]\,.\end{aligned}$$

Moreover, we can easily consider the dissipation as a function of the generalised velocities. Indeed, it is possible to derive purely velocity dependent dissipative forces from a dissipation function, which we assume to be quadratic, symmetric and positive definite with respect to the generalised velocities themselves. For example, we can consider the dissipation due to the steering axis rotation by means of a Rayleigh dissipation such as

$$\mathcal{F}=\frac{1}{2}\nu\dot{\psi}^2, \tag{4.1}$$

where ν is the coefficient of viscous friction. In the same way, we could take into account the presence of suspensions in the CPA bicycle model by considering both the potentials due to the springs and the dissipative functions due to the dampers. However, for the purpose of this thesis, we examine only the case of the steering axis dissipation.

4.2 Equations of motion

According to the theory of nonholonomic systems exposed in Chapter2, we develop the equations of motion using the Ehresmann connection given by the constraints, and deriving the constrained Lagrangian. The equations are then written explicitly in terms of the constrained Lagrangian and the curvature of the connection.

We start writing the *classical Lagrangian*. This is taken to be of the form kinetic energy plus potential, that is,

$$L(\theta, \alpha, \psi, \dot{x}, \dot{y}, \dot{\theta}, \dot{\chi}_f, \dot{\alpha}, \dot{\psi}, \dot{\chi}) = K(\theta, \alpha, \psi, \dot{x}, \dot{y}, \dot{\theta}, \dot{\chi}_f, \dot{\alpha}, \dot{\psi}, \dot{\chi}) + U(\alpha, \psi). \tag{4.2}$$

Due to the symmetry properties of the wheels, we note that the Lagrangian depends neither on the position (x, y) of the rear contact point nor on the angles χ and χ_f.

Then we turn our attention to the nonholonomic constraints to write the constrained Lagrangian. Without considering the symmetry of the problem, we think of these constraints as the horizontal space of an Ehresmann connection. In particular, we have to choose a bundle $Q \to R$. As we have already remarked in the previous chapter, possible controls would be added to either the roll angle α, the steering angle ψ or the rear wheel rotation angle χ; therefore, we are motivated to choose the base R to be $S^1 \times S^1 \times S^1$ parametrised by (α, ψ, χ), where the projection to R is simply

$$q = (x, y, \theta, \chi_f, \alpha, \psi, \chi)^T \mapsto (\alpha, \psi, \chi)^T.$$

Then, identifying the *base* and the *fibre* velocities as $\dot{r} = (\dot{\alpha}, \dot{\psi}, \dot{\chi})$ and $\dot{s} = (\dot{x}, \dot{y}, \dot{\theta}, \dot{\chi}_f)$, respectively, such that $q = (s, r)$, the constraints derived in Section 3.3 can be written as

$$\dot{s}^a = -A^a_\alpha \dot{r}^\alpha,$$

where A^a_α are the components of the Ehresmann connection. In particular, these are

$$\begin{aligned}
&A^1_1 = r\sin\theta, && A^1_2 = 0, && A^1_3 = -(R + r\cos\alpha)\cos\theta,\\
&A^2_1 = -r\cos\theta, && A^2_2 = 0, && A^2_3 = -(R + r\cos\alpha)\sin\theta,\\
&A^3_1 = 0, && A^3_2 = -\widehat{p}(\alpha, \psi), && A^3_3 = -\widehat{m}(\alpha, \psi),\\
&A^4_1 = -\widehat{h}(\alpha, \psi), && A^4_2 = -\widehat{l}(\alpha, \psi), && A^4_3 = -\widehat{g}(\alpha, \psi).
\end{aligned}$$

We now define the *constrained Lagrangian* by substituting the nonholonomic constraints into the classical Lagrangian (4.2), hence

$$\begin{aligned}
L_c(r^\alpha, s^a, \dot{r}^\alpha) &= L_c(r^\alpha, s^a, \dot{r}^\alpha, -A^a_\alpha(r, s)\dot{r}^\alpha) =\\
&= L_c(\alpha, \psi, \dot{\alpha}, \dot{\psi}, \dot{\chi}).
\end{aligned}$$

We observe that the constrained Lagrangian does not depend on the fibre coordinates, that is, it is cyclic in the variable s. Furthermore, we observe that the substitution of the nonholonomic constraints into the Lagrangian influences only the kinetic energy expression, whilst the potential remains the same. For writing the equations of motion in a concise form, it is useful to write the constrained kinetic energy by means of a new set of nonlinear functions, that is,

$$\begin{aligned}
K_c(\alpha, \psi, \dot{\alpha}, \dot{\psi}, \dot{\chi}) = &\frac{1}{2}\dot{\alpha}\mathcal{A}(\alpha, \psi) + \frac{1}{2}\dot{\chi}\mathcal{C}(\alpha, \psi) + \frac{1}{2}\dot{\psi}\mathcal{E}(\alpha, \psi) +\\
&+ \dot{\alpha}\dot{\psi}\mathcal{G}(\alpha, \psi) + \dot{\alpha}\dot{\chi}\mathcal{M}(\alpha, \psi) + \dot{\psi}\dot{\chi}\mathcal{P}(\alpha, \psi),
\end{aligned}$$

where one can easily check that

$$
\begin{aligned}
\mathcal{A}(\alpha,\psi) &= Mr^2 + a(\alpha,\psi) + F\widehat{h}^2(\alpha,\psi) + 2\widehat{h}(\alpha,\psi)k(\alpha,\psi) + 2rr(\alpha,\psi),\\
\mathcal{C}(\alpha,\psi) &= M(R + r\,\mathrm{c}\,\alpha)^2 + \widehat{m}^2(\alpha,\psi)b(\alpha,\psi) + E + F\widehat{g}^2(\alpha,\psi) +\\
&\quad + 2\widehat{m}(\alpha,\psi)\left[n(\alpha) + \widehat{g}(\alpha,\psi)p(\alpha,\psi) + (R + r\cos\alpha)u(\alpha,\psi)\right],\\
\mathcal{E}(\alpha,\psi) &= d(\alpha,\psi) + \widehat{p}^2(\alpha,\psi)b(\alpha,\psi) + F\widehat{l}^2(\alpha,\psi) + 2\widehat{p}(\alpha,\psi)l(\alpha,\psi) +\\
&\quad + 2\widehat{l}(\alpha,\psi)\left[m(\alpha,\psi) + \widehat{p}(\alpha,\psi)p(\alpha,\psi)\right],\\
\mathcal{G}(\alpha,\psi) &= F\widehat{h}(\alpha,\psi)\widehat{l}(\alpha,\psi) + \widehat{p}(\alpha,\psi)g(\alpha,\psi) + h(\alpha,\psi) + \widehat{l}(\alpha,\psi)k(\alpha,\psi) +\\
&\quad + \widehat{h}(\alpha,\psi)m(\alpha,\psi) + \widehat{p}(\alpha,\psi)\widehat{h}(\alpha,\psi)p(\alpha,\psi) +\\
&\quad + r\widehat{p}(\alpha,\psi)q(\psi) + rs(\alpha,\psi),\\
\mathcal{M}(\alpha,\psi) &= F\widehat{g}(\alpha,\psi)\widehat{h}(\alpha,\psi) + \widehat{m}(\alpha,\psi)g(\alpha,\psi) + \widehat{g}(\alpha,\psi)k(\alpha,\psi) +\\
&\quad + \widehat{m}(\alpha,\psi)\widehat{h}(\alpha,\psi)p(\alpha,\psi) + r\widehat{m}(\alpha,\psi)q(\psi),\\
\mathcal{P}(\alpha,\psi) &= \widehat{m}(\alpha,\psi)\widehat{p}(\alpha,\psi)b(\alpha,\psi) + F\widehat{g}(\alpha,\psi)\widehat{l}(\alpha,\psi) + \widehat{m}(\alpha,\psi)l(\alpha,\psi) +\\
&\quad + \widehat{g}(\alpha,\psi)m(\alpha,\psi) + \widehat{p}(\alpha,\psi)n(\alpha) + \widehat{m}(\alpha,\psi)\widehat{l}(\alpha,\psi)p(\alpha,\psi) +\\
&\quad + \widehat{p}(\alpha,\psi)\widehat{g}(\alpha,\psi)p(\alpha,\psi) + (R + r\cos\alpha)\widehat{p}(\alpha,\psi)u(\alpha,\psi) +\\
&\quad + (R + r\cos\alpha)z(\psi).
\end{aligned}
$$

From the theory, we know that the equations of motion in terms of the constrained Lagrangian are given by expression (2.17), thus the equations of motion for the CPA bicycle are given by

$$\frac{d}{dt}\frac{\partial L_c}{\partial \dot{r}^\alpha} - \frac{\partial L_c}{\partial r^\alpha} = -\frac{\partial L}{\partial \dot{s}^b} B^b_{\alpha\beta}\dot{r}^\beta, \tag{4.3}$$

where

$$B^b_{\alpha\beta} = \frac{\partial A^b_\alpha}{\partial r^\beta} - \frac{\partial A^b_\beta}{\partial r^\alpha} + A^a_\alpha \frac{\partial A^b_\beta}{\partial s^a} - A^a_\beta \frac{\partial A^b_\alpha}{\partial s^a}, \tag{4.4}$$

are the coefficients of the curvature of the connection $A(r,s)$. In particular, after a straightforward computation, we have

$$
\begin{aligned}
B^1_{12} &= -B^1_{21} = \widehat{p}(\alpha,\psi)r\cos\theta,\\
B^1_{13} &= -B^1_{31} = -r\sin\alpha\cos\theta + \widehat{m}(\alpha,\psi)r\cos\theta,\\
B^1_{23} &= -B^1_{32} = -\widehat{p}(\alpha,\psi)(R + r\cos\alpha)\sin\theta,
\end{aligned}
$$

$$
\begin{aligned}
B^2_{12} &= -B^2_{21} = \widehat{p}(\alpha,\psi)r\sin\theta,\\
B^2_{13} &= -B^2_{31} = -r\sin\alpha\sin\theta + \widehat{m}(\alpha,\psi)r\sin\theta,\\
B^2_{23} &= -B^2_{32} = \widehat{p}(\alpha,\psi)(R + r\cos\alpha)\cos\theta,
\end{aligned}
$$

$$
\begin{aligned}
B^3_{12} &= -B^3_{21} = \frac{\partial \widehat{p}(\alpha,\psi)}{\partial \alpha},\\
B^3_{13} &= -B^3_{31} = \frac{\partial \widehat{m}(\alpha,\psi)}{\partial \alpha},\\
B^3_{23} &= -B^3_{32} = \frac{\partial \widehat{m}(\alpha,\psi)}{\partial \psi},
\end{aligned}
$$

$$
\begin{aligned}
B^4_{12} = -B^4_{21} &= -\frac{\partial \widehat{h}(\alpha,\psi)}{\partial \psi} + \frac{\partial \widehat{l}(\alpha,\psi)}{\partial \alpha},\\
B^4_{13} = -B^4_{31} &= \frac{\partial \widehat{g}(\alpha,\psi)}{\partial \alpha},\\
B^4_{23} = -B^4_{32} &= \frac{\partial \widehat{g}(\alpha,\psi)}{\partial \psi},
\end{aligned}
$$

with the remaining $B^b_{\alpha\beta}$ zero.

Therefore, the equation for the roll angle is

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L_c}{\partial \dot\alpha} - \frac{\partial L_c}{\partial \alpha} = &- \left.\frac{\partial L}{\partial \dot x}\right|_c (B^1_{12}\dot\psi + B^1_{13}\dot\chi) - \left.\frac{\partial L}{\partial \dot y}\right|_c (B^2_{12}\dot\psi + B^2_{13}\dot\chi) +\\
&- \left.\frac{\partial L}{\partial \dot\theta}\right|_c (B^3_{12}\dot\psi + B^3_{13}\dot\chi) - \left.\frac{\partial L}{\partial \dot\chi_f}\right|_c (B^4_{12}\dot\psi + B^4_{13}\dot\chi),
\end{aligned}
$$

which we can write explicitly in the form

$$
\begin{aligned}
&\ddot\alpha\mathcal{A}(\alpha,\psi) + \ddot\psi\mathcal{G}(\alpha,\psi) + \ddot\chi\mathcal{M}(\alpha,\psi) + \frac{1}{2}\dot\alpha^2\frac{\partial\mathcal{A}(\alpha,\psi)}{\partial\alpha} +\\
&\quad + \dot\alpha\dot\chi\widetilde{a}(\alpha,\psi) + \dot\alpha\dot\psi\widetilde{b}(\alpha,\psi) + \dot\psi^2\widetilde{d}(\alpha,\psi) + \dot\chi\dot\psi\widetilde{e}(\alpha,\psi) + \dot\chi^2\widetilde{f}(\alpha,\psi) =\\
&\quad = \frac{\partial U(\alpha,\psi)}{\partial\alpha},
\end{aligned}
\tag{4.5}
$$

where

$$
\begin{aligned}
\widetilde{a}(\alpha,\psi) = &\,(k(\alpha,\psi) + F\widehat{h}(\alpha,\psi))\frac{\partial\widehat{g}(\alpha,\psi)}{\partial\alpha} +\\
&+ (rq(\psi) + g(\alpha,\psi) + \widehat{h}(\alpha,\psi)p(\alpha,\psi))\frac{\partial\widehat{m}(\alpha,\psi)}{\partial\alpha},\\
\widetilde{b}(\alpha,\psi) = &\,(rq(\psi) + g(\alpha,\psi) + \widehat{h}(\alpha,\psi)p(\alpha,\psi))\frac{\partial\widehat{p}(\alpha,\psi)}{\partial\alpha} + \frac{\partial a(\alpha,\psi)}{\partial\psi} +\\
&+ (k(\alpha,\psi) + F\widehat{h}(\alpha,\psi))\frac{\partial\widehat{l}(\alpha,\psi)}{\partial\alpha} + F\widehat{h}(\alpha,\psi)\frac{\partial\widehat{h}(\alpha,\psi)}{\partial\psi} +\\
&+ \frac{\partial\widehat{h}(\alpha,\psi)}{\partial\psi}k(\alpha,\psi) + 2\widehat{h}(\alpha,\psi)\frac{\partial k(\alpha,\psi)}{\partial\psi} + 2r\frac{\partial r(\alpha,\psi)}{\partial\psi},\\
\widetilde{d}(\alpha,\psi) = &\,(z(\psi) + u(\alpha,\psi)\widehat{p}(\alpha,\psi))\widehat{p}(\alpha,\psi)r + F\widehat{h}(\alpha,\psi)\frac{\partial\widehat{l}(\alpha,\psi)}{\partial\psi} + \frac{\partial h(\alpha,\psi)}{\partial\psi} +\\
&+ \frac{\partial\widehat{p}(\alpha,\psi)}{\partial\psi}g(\alpha,\psi) + \widehat{p}(\alpha,\psi)\frac{\partial g(\alpha,\psi)}{\partial\psi} + \frac{\partial\widehat{l}(\alpha,\psi)}{\partial\psi}k(\alpha,\psi) +\\
&+ \widehat{l}(\alpha,\psi)\frac{\partial k(\alpha,\psi)}{\partial\psi} + \widehat{h}(\alpha,\psi)\frac{\partial m(\alpha,\psi)}{\partial\psi} + \frac{\partial\widehat{p}(\alpha,\psi)}{\partial\psi}\widehat{h}(\alpha,\psi)p(\alpha,\psi) +\\
&+ \widehat{p}(\alpha,\psi)\widehat{h}(\alpha,\psi)\frac{\partial p(\alpha,\psi)}{\partial\psi} + r\frac{\partial\widehat{p}(\alpha,\psi)}{\partial\psi}q(\psi) + r\widehat{p}(\alpha,\psi)\frac{\partial q(\psi)}{\partial\psi} +\\
&+ r\frac{\partial s(\alpha,\psi)}{\partial\psi} - \frac{1}{2}\frac{\partial d(\alpha,\psi)}{\partial\alpha} - \frac{1}{2}\widehat{p}^2(\alpha,\psi)\frac{\partial b(\alpha,\psi)}{\partial\alpha} - \widehat{p}(\alpha,\psi)\frac{\partial l(\alpha,\psi)}{\partial\alpha} +\\
&- \widehat{l}(\alpha,\psi)\frac{\partial m(\alpha,\psi)}{\partial\alpha} - \widehat{p}(\alpha,\psi)\widehat{l}(\alpha,\psi)\frac{\partial p(\alpha,\psi)}{\partial\alpha},\\
\widetilde{e}(\alpha,\psi) = &\,(u(\alpha,\psi)\widehat{m}(\alpha,\psi) + M(R + r\cos\alpha))\widehat{p}(\alpha,\psi)r +
\end{aligned}
$$

$$
\begin{aligned}
&+ (z(\psi) + u(\alpha,\psi)\widehat{p}(\alpha,\psi))\widehat{m}(\alpha,\psi)r + F\frac{\partial \widehat{g}(\alpha,\psi)}{\partial\psi}\widehat{h}(\alpha,\psi)+\\
&+ \frac{\partial \widehat{m}(\alpha,\psi)}{\partial\psi} g(\alpha,\psi) + \widehat{m}(\alpha,\psi)\frac{\partial g(\alpha,\psi)}{\partial\psi} + \frac{\partial \widehat{g}(\alpha,\psi)}{\partial\psi}k(\alpha,\psi)+\\
&+ \widehat{g}(\alpha,\psi)\frac{\partial k(\alpha,\psi)}{\partial\psi} + \frac{\partial \widehat{m}(\alpha,\psi)}{\partial\psi}\widehat{h}(\alpha,\psi)p(\alpha,\psi) + \widehat{m}(\alpha,\psi)\widehat{h}(\alpha,\psi)\frac{\partial p(\alpha,\psi)}{\partial\psi}+\\
&+ r\frac{\partial \widehat{m}(\alpha,\psi)}{\partial\psi}q(\psi) + r\widehat{m}(\alpha,\psi)\frac{\partial q(\psi)}{\partial\psi} - \widehat{m}(\alpha,\psi)\widehat{p}(\alpha,\psi)\frac{\partial b(\alpha,\psi)}{\partial\alpha}+\\
&- \widehat{m}(\alpha,\psi)\frac{\partial l(\alpha,\psi)}{\partial\alpha} - \widehat{g}(\alpha,\psi)\frac{\partial m(\alpha,\psi)}{\partial\alpha} - \widehat{p}(\alpha,\psi)\frac{n(\alpha)}{\partial\alpha}+\\
&- \widehat{m}(\alpha,\psi)\widehat{l}(\alpha,\psi)\frac{\partial p(\alpha,\psi)}{\partial\alpha} - \widehat{p}(\alpha,\psi)\widehat{g}(\alpha,\psi)\frac{\partial p(\alpha,\psi)}{\partial\alpha}+\\
&- (R + r\cos\alpha)\widehat{p}(\alpha,\psi)\frac{\partial u(\alpha,\psi)}{\partial\alpha},
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{f}(\alpha,\psi) = {}& (u(\alpha,\psi)\widehat{m}(\alpha,\psi) + M(R + r\cos\alpha))\widehat{m}(\alpha,\psi)r - \frac{1}{2}\widehat{m}^2(\alpha,\psi)\frac{\partial b(\alpha,\psi)}{\partial\alpha}+\\
&- \widehat{m}(\alpha,\psi)\frac{\partial n(\alpha)}{\partial\alpha} - \widehat{m}(\alpha,\psi)\widehat{g}(\alpha,\psi)\frac{\partial p(\alpha,\psi)}{\partial\alpha}+\\
&- \widehat{m}(\alpha,\psi)(R + r\cos\alpha)\frac{\partial u(\alpha,\psi)}{\partial\alpha}.
\end{aligned}
$$

Then, taking into account the dissipative function (4.1) for the steering axis rotation, the equation for the steering angle becomes

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L_c}{\partial\dot\psi} - \frac{\partial L_c}{\partial\psi} + \frac{\partial\mathcal{F}}{\partial\dot\psi} = {}& - \left.\frac{\partial L}{\partial\dot x}\right|_c (B^1_{21}\dot\alpha + B^1_{23}\dot\chi) - \left.\frac{\partial L}{\partial\dot y}\right|_c (B^2_{21}\dot\alpha + B^2_{23}\dot\chi)+\\
&- \left.\frac{\partial L}{\partial\dot\theta}\right|_c (B^3_{21}\dot\alpha + B^3_{23}\dot\chi) - \left.\frac{\partial L}{\partial\dot\chi_f}\right|_c (B^4_{21}\dot\alpha + B^4_{23}\dot\chi),
\end{aligned}
$$

and writing it explicitly, we have

$$
\begin{aligned}
&\ddot\psi\mathcal{E}(\alpha,\psi) + \ddot\alpha\mathcal{G}(\alpha,\psi) + \ddot\chi\mathcal{P}(\alpha,\psi) + \frac{1}{2}\dot\psi^2\frac{\partial\mathcal{E}(\alpha,\psi)}{\partial\psi}+\\
&\quad + \dot\psi\dot\chi\widetilde{g}(\alpha,\psi) + \dot\alpha\dot\psi\widetilde{h}(\alpha,\psi) + \dot\alpha^2\widetilde{l}(\alpha,\psi) + \dot\alpha\dot\chi\widetilde{m}(\alpha,\psi) + \dot\chi^2\widetilde{n}(\alpha,\psi) = \\
&\quad = \frac{\partial U(\alpha,\psi)}{\partial\psi} - \nu\dot\psi,
\end{aligned}
\tag{4.6}
$$

where

$$
\begin{aligned}
\widetilde{g}(\alpha,\psi) = {}& (m(\alpha,\psi) + F\widehat{l}(\alpha,\psi) + \widehat{p}(\alpha,\psi)p(\alpha,\psi))\frac{\partial\widehat{g}(\alpha,\psi)}{\partial\psi}+\\
&+ (s(\alpha,\psi) + q(\psi)\widehat{p}(\alpha,\psi))\widehat{p}(\alpha,\psi)(R + r\cos\alpha)+\\
&+ (l(\alpha,\psi) + \widehat{p}(\alpha,\psi)b(\alpha,\psi) + \widehat{l}(\alpha,\psi)p(\alpha,\psi))\frac{\partial\widehat{m}(\alpha,\psi)}{\partial\psi},\\
\widetilde{h}(\alpha,\psi) = {}& \frac{\partial d(\alpha,\psi)}{\partial\alpha} + \widehat{p}(\alpha,\psi)b(\alpha,\psi)\frac{\partial\widehat{p}(\alpha,\psi)}{\partial\alpha} + \widehat{p}^2(\alpha,\psi)\frac{\partial b(\alpha,\psi)}{\partial\alpha}+\\
&+ F\widehat{l}(\alpha,\psi)\frac{\partial\widehat{l}(\alpha,\psi)}{\partial\alpha} + \frac{\partial\widehat{p}(\alpha,\psi)}{\partial\alpha}l(\alpha,\psi) + 2\widehat{p}(\alpha,\psi)\frac{\partial l(\alpha,\psi)}{\partial\alpha}+\\
&+ \frac{\partial\widehat{l}(\alpha,\psi)}{\partial\alpha}m(\alpha,\psi) + 2\widehat{l}(\alpha,\psi)\frac{\partial m(\alpha,\psi)}{\partial\alpha} + \frac{\partial\widehat{p}(\alpha,\psi)}{\partial\alpha}\widehat{l}(\alpha,\psi)p(\alpha,\psi)+
\end{aligned}
$$

$$
\begin{aligned}
&+\widehat{p}(\alpha,\psi)\frac{\partial\widehat{l}(\alpha,\psi)}{\partial\alpha}p(\alpha,\psi)+2\widehat{p}(\alpha,\psi)\widehat{l}(\alpha,\psi)\frac{\partial p(\alpha,\psi)}{\partial\alpha}+\\
&+(m(\alpha,\psi)+F\widehat{l}(\alpha,\psi)+\widehat{p}(\alpha,\psi)p(\alpha,\psi))\frac{\partial\widehat{h}(\alpha,\psi)}{\partial\psi}+\\
&-(z(\psi)+u(\alpha,\psi)\widehat{p}(\alpha,\psi))\widehat{p}(\alpha,\psi)r,\\
\widetilde{l}(\alpha,\psi)=\ &F\frac{\partial\widehat{h}(\alpha,\psi)}{\partial\alpha}\widehat{l}(\alpha,\psi)+\widehat{p}(\alpha,\psi)\frac{\partial g(\alpha,\psi)}{\partial\alpha}+\frac{\partial h(\alpha,\psi)}{\partial\alpha}+\widehat{l}(\alpha,\psi)\frac{\partial k(\alpha,\psi)}{\partial\alpha}+\\
&+r\frac{\partial s(\alpha,\psi)}{\partial\alpha}+\frac{\partial\widehat{h}(\alpha,\psi)}{\partial\alpha}m(\alpha,\psi)+\widehat{h}(\alpha,\psi)\frac{\partial m(\alpha,\psi)}{\partial\alpha}+\\
&+\widehat{p}(\alpha,\psi)\frac{\partial\widehat{h}(\alpha,\psi)}{\partial\alpha}p(\alpha,\psi)+\widehat{p}(\alpha,\psi)\widehat{h}(\alpha,\psi)\frac{\partial p(\alpha,\psi)}{\partial\alpha}+\\
&-\frac{1}{2}\frac{\partial a(\alpha,\psi)}{\partial\psi}-\widehat{h}(\alpha,\psi)\frac{\partial k(\alpha,\psi)}{\partial\psi}-r\frac{\partial r(\alpha,\psi)}{\partial\psi},\\
\widetilde{m}(\alpha,\psi)=\ &\frac{\partial\widehat{m}(\alpha,\psi)}{\partial\alpha}\widehat{p}(\alpha,\psi)b(\alpha,\psi)+\widehat{m}(\alpha,\psi)\widehat{p}(\alpha,\psi)\frac{\partial b(\alpha,\psi)}{\partial\alpha}+F\frac{\widehat{g}(\alpha,\psi)}{\partial\alpha}\widehat{l}(\alpha,\psi)+\\
&+\frac{\partial\widehat{m}(\alpha,\psi)}{\partial\alpha}l(\alpha,\psi)+\widehat{m}(\alpha,\psi)\frac{\partial l(\alpha,\psi)}{\partial\alpha}+\\
&+\frac{\partial g(\alpha,\psi)}{\partial\alpha}m(\alpha,\psi)+\widehat{g}(\alpha,\psi)\frac{\partial m(\alpha,\psi)}{\partial\alpha}+\\
&+\widehat{p}(\alpha,\psi)\frac{n(\alpha)}{\partial\alpha}+\frac{\partial\widehat{m}(\alpha,\psi)}{\partial\alpha}\widehat{l}(\alpha,\psi)p(\alpha,\psi)+\widehat{m}(\alpha,\psi)\widehat{l}(\alpha,\psi)\frac{\partial p(\alpha,\psi)}{\partial\alpha}+\\
&+\widehat{p}(\alpha,\psi)\frac{\partial\widehat{g}(\alpha,\psi)}{\partial\alpha}p(\alpha,\psi)+\widehat{p}(\alpha,\psi)\widehat{g}(\alpha,\psi)\frac{\partial p(\alpha,\psi)}{\partial\alpha}+\\
&-r\sin\alpha\widehat{p}(\alpha,\psi)u(\alpha,\psi)+\\
&+(R+r\cos\alpha)\widehat{p}(\alpha,\psi)\frac{\partial u(\alpha,\psi)}{\partial\alpha}-r\sin\alpha z(\psi)-\widehat{m}(\alpha,\psi)\frac{\partial g(\alpha,\psi)}{\partial\psi}+\\
&-\widehat{g}(\alpha,\psi)\frac{\partial k(\alpha,\psi)}{\partial\psi}-\widehat{m}(\alpha,\psi)\widehat{h}(\alpha,\psi)\frac{\partial p(\alpha,\psi)}{\partial\psi}-r\widehat{m}(\alpha,\psi)\frac{\partial q(\psi)}{\partial\psi}+\\
&-(u(\alpha,\psi)\widehat{m}(\alpha,\psi)+M(R+r\cos\alpha))\widehat{p}(\alpha,\psi)r+\\
&+(r(\alpha,\psi)+Mr)\widehat{p}(\alpha,\psi)(R+r\cos\alpha),\\
\widetilde{n}(\alpha,\psi)=\ &q(\psi)\widehat{m}(\alpha,\psi)\widehat{p}(\alpha,\psi)(R+r\cos\alpha)-\frac{1}{2}\widehat{m}^2(\alpha,\psi)\frac{\partial b(\alpha,\psi)}{\partial\psi}+\\
&-\widehat{m}(\alpha,\psi)\widehat{g}(\alpha,\psi)\frac{\partial p(\alpha,\psi)}{\partial\psi}-\widehat{m}(\alpha,\psi)(R+r\cos\alpha)\frac{\partial u(\alpha,\psi)}{\partial\psi}.
\end{aligned}
$$

Finally, the equation for the rear wheel rotation angle is

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L_c}{\partial\dot{\chi}}=&-\left.\frac{\partial L}{\partial\dot{x}}\right|_c(B^1_{31}\dot{\alpha}+B^1_{32}\dot{\psi})-\left.\frac{\partial L}{\partial\dot{y}}\right|_c(B^2_{31}\dot{\alpha}+B^2_{32}\dot{\psi})+\\
&-\left.\frac{\partial L}{\partial\dot{\theta}}\right|_c(B^3_{31}\dot{\alpha}+B^3_{32}\dot{\psi})-\left.\frac{\partial L}{\partial\dot{\chi}_f}\right|_c(B^4_{31}\dot{\alpha}+B^4_{32}\dot{\psi})
\end{aligned}
$$

which becomes

$$
\begin{aligned}
\ddot{\chi}\mathcal{C}(\alpha,\psi)+\ddot{\alpha}\mathcal{M}(\alpha,\psi)+\ddot{\psi}\mathcal{P}(\alpha,\psi)+\dot{\alpha}\dot{\chi}\widetilde{p}(\alpha,\psi)+\dot{\psi}\dot{\chi}\widetilde{q}(\alpha,\psi)+\\
+\dot{\alpha}^2\widetilde{r}(\alpha,\psi)+\dot{\alpha}\dot{\psi}\widetilde{s}(\alpha,\psi)+\dot{\psi}^2\widetilde{u}(\alpha,\psi)=0,
\end{aligned} \tag{4.7}
$$

where

$$\begin{aligned}
\widetilde{p}(\alpha,\psi) &= -M(R+r\cos\alpha)r\sin\alpha + \widehat{m}(\alpha,\psi)b(\alpha,\psi)\frac{\partial\widehat{m}(\alpha,\psi)}{\partial\alpha}+\\
&+\widehat{m}^2(\alpha,\psi)\frac{\partial b(\alpha,\psi)}{\partial\alpha}+F\widehat{g}(\alpha,\psi)\frac{\partial\widehat{g}(\alpha,\psi)}{\partial\alpha}+\frac{\widehat{m}(\alpha,\psi)}{\partial\alpha}n(\alpha)+\\
&+2\widehat{m}(\alpha,\psi)\frac{\partial n(\alpha)}{\partial\alpha}+\frac{\partial\widehat{m}(\alpha,\psi)}{\partial\alpha}\widehat{g}(\alpha,\psi)p(\alpha,\psi)+\widehat{m}(\alpha,\psi)\frac{\partial\widehat{g}(\alpha,\psi)}{\partial\alpha}p(\alpha,\psi)+\\
&+2\widehat{m}(\alpha,\psi)\widehat{g}(\alpha,\psi)\frac{\partial p(\alpha,\psi)}{\partial\alpha}+\frac{\partial\widehat{m}(\alpha,\psi)}{\partial\alpha}(R+r\cos\alpha)u(\alpha,\psi)+\\
&-\widehat{m}(\alpha,\psi)r\sin\alpha u(\alpha,\psi)+2\widehat{m}(\alpha,\psi)(R+r\cos\alpha)\frac{\partial u(\alpha,\psi)}{\partial\alpha}+\\
&-(u(\alpha,\psi)\widehat{m}(\alpha,\psi)+M(R+r\cos\alpha)\widehat{m}(\alpha,\psi)r,\\
\widetilde{q}(\alpha,\psi) &= \widehat{m}(\alpha,\psi)b(\alpha,\psi)\frac{\partial\widehat{m}(\alpha,\psi)}{\partial\psi}+\widehat{m}^2(\alpha,\psi)\frac{\partial b(\alpha,\psi)}{\partial\psi}+F\widehat{g}(\alpha,\psi)\frac{\partial\widehat{g}(\alpha,\psi)}{\partial\psi}+\\
&+\frac{\partial\widehat{m}(\alpha,\psi)}{\partial\psi}n(\alpha)+\frac{\partial\widehat{m}(\alpha,\psi)}{\partial\psi}\widehat{g}(\alpha,\psi)p(\alpha,\psi)+\widehat{m}(\alpha,\psi)\frac{\partial\widehat{g}(\alpha,\psi)}{\partial\psi}p(\alpha,\psi)+\\
&+2\widehat{m}(\alpha,\psi)\widehat{g}(\alpha,\psi)\frac{\partial p(\alpha,\psi)}{\partial\psi}+\frac{\partial\widehat{m}(\alpha,\psi)}{\partial\psi}(R+r\cos\alpha)u(\alpha,\psi)+\\
&+2\widehat{m}(\alpha,\psi)(R+r\cos\alpha)\frac{\partial u(\alpha,\psi)}{\partial\psi}-q(\psi)\widehat{m}(\alpha,\psi)\widehat{p}(\alpha,\psi)(R+r\cos\alpha),\\
\widetilde{r}(\alpha,\psi) &= F\widehat{g}(\alpha,\psi)\frac{\partial\widehat{h}(\alpha,\psi)}{\partial\alpha}+\widehat{m}(\alpha,\psi)\frac{\partial g(\alpha,\psi)}{\partial\alpha}+\widehat{g}(\alpha,\psi)\frac{\partial k(\alpha,\psi)}{\partial\alpha}+\\
&+\widehat{m}(\alpha,\psi)\frac{\partial\widehat{h}(\alpha,\psi)}{\partial\alpha}p(\alpha,\psi)+\widehat{m}(\alpha,\psi)\widehat{h}(\alpha,\psi)\frac{\partial p(\alpha,\psi)}{\partial\alpha},\\
\widetilde{s}(\alpha,\psi) &= F\widehat{g}(\alpha,\psi)\frac{\partial\widehat{h}(\alpha,\psi)}{\partial\psi}+\widehat{m}(\alpha,\psi)\frac{\partial g(\alpha,\psi)}{\partial\psi}+\widehat{g}(\alpha,\psi)\frac{\partial k(\alpha,\psi)}{\partial\psi}+\\
&+\widehat{m}(\alpha,\psi)\frac{\partial\widehat{h}(\alpha,\psi)}{\partial\psi}p(\alpha,\psi)+\widehat{m}(\alpha,\psi)\widehat{h}(\alpha,\psi)\frac{\partial p(\alpha,\psi)}{\partial\psi}+r\widehat{m}(\alpha,\psi)\frac{\partial q(\psi)}{\partial\psi}+\\
&+\widehat{m}(\alpha,\psi)\frac{\partial\widehat{p}(\alpha,\psi)}{\partial\alpha}b(\alpha,\psi)+\widehat{m}(\alpha,\psi)\widehat{p}(\alpha,\psi)\frac{\partial b(\alpha,\psi)}{\partial\alpha}+F\widehat{g}(\alpha,\psi)\frac{\partial\widehat{l}(\alpha,\psi)}{\partial\alpha}+\\
&+\widehat{m}(\alpha,\psi)\frac{\partial l(\alpha,\psi)}{\partial\alpha}+\widehat{g}(\alpha,\psi)\frac{\partial m(\alpha,\psi)}{\partial\alpha}+\frac{\partial\widehat{p}(\alpha,\psi)}{\partial\alpha}n(\alpha)+\widehat{p}(\alpha,\psi)\frac{\partial n(\alpha)}{\partial\alpha}+\\
&+\widehat{m}(\alpha,\psi)\frac{\partial\widehat{l}(\alpha,\psi)}{\partial\alpha}p(\alpha,\psi)+\widehat{m}(\alpha,\psi)\widehat{l}(\alpha,\psi)\frac{\partial p(\alpha,\psi)}{\partial\alpha}+\\
&+\frac{\partial\widehat{p}(\alpha,\psi)}{\partial\alpha}\widehat{g}(\alpha,\psi)p(\alpha,\psi)+\widehat{p}(\alpha,\psi)\widehat{g}(\alpha,\psi)\frac{\partial p(\alpha,\psi)}{\partial\alpha}+\\
&+(R+r\cos\alpha)\left(\frac{\partial\widehat{p}(\alpha,\psi)}{\partial\alpha}u(\alpha,\psi)+\widehat{p}(\alpha,\psi)\frac{\partial u(\alpha,\psi)}{\partial\alpha}\right)+\\
&-(z(\psi)+u(\alpha,\psi)\widehat{p}(\alpha,\psi))\widehat{m}(\alpha,\psi)r+\\
&-(r(\alpha,\psi)+Mr)\widehat{p}(\alpha,\psi)(R+r\cos\alpha),\\
\widetilde{u}(\alpha,\psi) &= \widehat{m}(\alpha,\psi)\frac{\partial\widehat{p}(\alpha,\psi)}{\partial\psi}b(\alpha,\psi)+\widehat{m}(\alpha,\psi)\widehat{p}(\alpha,\psi)\frac{\partial b(\alpha,\psi)}{\partial\psi}+F\widehat{g}(\alpha,\psi)\frac{\partial\widehat{l}(\alpha,\psi)}{\partial\psi}+\\
&+\widehat{m}(\alpha,\psi)\frac{\partial l(\alpha,\psi)}{\partial\psi}+\widehat{g}(\alpha,\psi)\frac{\partial m(\alpha,\psi)}{\partial\psi}+\frac{\partial\widehat{p}(\alpha,\psi)}{\partial\psi}n(\alpha)+\\
&+\widehat{m}(\alpha,\psi)\frac{\partial\widehat{l}(\alpha,\psi)}{\partial\psi}p(\alpha,\psi)+\widehat{m}(\alpha,\psi)\widehat{l}(\alpha,\psi)\frac{\partial p(\alpha,\psi)}{\partial\psi}+
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\partial \widehat{p}(\alpha,\psi)}{\partial \psi}\widehat{g}(\alpha,\psi)p(\alpha,\psi)+\widehat{p}(\alpha,\psi)\widehat{g}(\alpha,\psi)\frac{\partial p(\alpha,\psi)}{\partial \psi}+\\
&+(R+r\cos\alpha)\left(\frac{\partial \widehat{p}(\alpha,\psi)}{\partial \psi}u(\alpha,\psi)+\widehat{p}(\alpha,\psi)\frac{\partial u(\alpha,\psi)}{\partial \psi}\right)+\\
&+(R+r\cos\alpha)\frac{\partial z(\psi)}{\partial \psi}-(s(\alpha,\psi)+\\
&+q(\psi)\widehat{p}(\alpha,\psi))\widehat{p}(\alpha,\psi)(R+r\cos\alpha).
\end{aligned}
$$

4.3 Particular solutions

After having written explicitly the equations of motion, we consider two classes of particular solutions which can be written in closed form. Furthermore, special solutions are the starting point for studying the stability of a system as well as they are a good source of computational example.

First of all, we consider the trivial *rectilinear motion* of the system with constant velocity, which is obtained by choosing

$$
\begin{aligned}
\alpha(t) &= 0,\\
\psi(t) &= 0,\\
\chi(t) &= \chi_0 t.
\end{aligned}
$$

Consequently, equation (4.7) is clearly satisfied, whist equations (4.5) and (4.6) become

$$\chi_0^2\widetilde{f}(0,0)=\left.\frac{\partial U(\alpha,\psi)}{\partial \alpha}\right|_{(0,0)}$$

and

$$\chi_0^2\widetilde{n}(0,0)=\left.\frac{\partial U(\alpha,\psi)}{\partial \psi}\right|_{(0,0)},$$

respectively. Then, it is easy to check that $\widetilde{f}(0,0)=\widetilde{n}(0,0)=0$, as well as

$$\left.\frac{\partial U(\alpha,\psi)}{\partial \alpha}\right|_{(0,0)}=\left.\frac{\partial U(\alpha,\psi)}{\partial \psi}\right|_{(0,0)}=0,$$

hence all the equations of motion are satisfied and we have a solution for the system. Now, using the nonholonomic constraints, we can also determine how the other generalised coordinates evolve in time. First of all, we note that the only term different from zero in relation (3.15) is

$$\widehat{g}(0,0)=\frac{R+r}{r},$$

therefore

$$\dot{\chi}_f=\frac{R+r}{r}\chi_0.$$

Furthermore, being α and ψ equal to zero, equation (3.16) becomes

$$\dot{\theta}=0 \implies \theta(t)=\theta_0.$$

Finally, constraints (3.9) for the rear contact point are

$$
\begin{cases}
\dot{x}=(R+r)\chi_0\cos\theta_0,\\
\dot{y}=(R+r)\chi_0\sin\theta_0,
\end{cases}
$$

that is, the trajectory is linear in time and the direction of the motion depends on the initial value θ_0.

The second class of solutions is given by *circular motions*, which are a generalisation of the rectilinear motion presented above. In particular, we now choose a solution to be

$$\alpha(t) = \alpha_0,$$
$$\psi(t) = \psi_0,$$
$$\chi(t) = \chi_0 t,$$

where α_0, ψ_0 and χ_0 are constants. Although equation (4.7) is still satisfied, equations (4.5) and (4.6) become

$$\chi_0^2 \widetilde{f}(\alpha_0, \psi_0) = \left.\frac{\partial U(\alpha, \psi)}{\partial \alpha}\right|_{(\alpha_0, \psi_0)} \tag{4.8}$$

and

$$\chi_0^2 \widetilde{n}(\alpha_0, \psi_0) = \left.\frac{\partial U(\alpha, \psi)}{\partial \psi}\right|_{(\alpha_0, \psi_0)}, \tag{4.9}$$

where the equality does not hold in general. Therefore, fixed either α_0, ψ_0 or χ_0, we need to solve these two nonlinear equations together to determine the remaining constants. Then, from the nonholonomic constraints, we get

$$\dot{\chi}_f = \chi_0 \widehat{g}_0$$

and

$$\dot{\theta} = \chi_0 \widehat{m}_0 \implies \theta(t) = (\chi_0 \widehat{m}_0)t = \theta_0 t,$$

where

$$\widehat{g}_0 = \widehat{g}(\alpha_0, \psi_0)$$

and

$$\widehat{m}_0 = \widehat{m}(\alpha_0, \psi_0).$$

Furthermore, constraints (3.9) become

$$\begin{cases} \dot{x} = (R + r\cos\alpha_0)\chi_0 \cos\theta_0 t, \\ \dot{y} = (R + r\cos\alpha_0)\chi_0 \sin\theta_0 t, \end{cases}$$

and integrating with respect to time, we have

$$\begin{cases} x(t) = \dfrac{1}{\theta_0}(R + r\cos\alpha_0)\chi_0 \sin\theta_0 t, \\ y(t) = -\dfrac{1}{\theta_0}(R + r\cos\alpha_0)\chi_0 \cos\theta_0 t. \end{cases}$$

For instance, let us consider a CPA bicycle defined by the geometric parameters in Table 4.1, where the parameters not listed are equal to zero. Then, fixed $\chi_0 = 65\,\mathrm{rad\,s^{-1}}$ and solving equations (4.8) and (4.9), we find that the system describes a circular motion if

$$\alpha_0 \simeq 0.380\,725\,\mathrm{rad}$$

and

$$\psi_0 \simeq 0.199\,096\,\mathrm{rad}.$$

In Figure 4.1 it is shown the trajectory described by the rear contact point on the ground plane. The initial position of the contact point coincides with the axis origin, and the initial yaw angle $\theta(0)$ is set to zero.

Remark 4.1. As stated before, we note that the rectilinear motion is a particular case of the circular one whenever $\alpha_0 = 0$ and $\psi_0 = 0$.

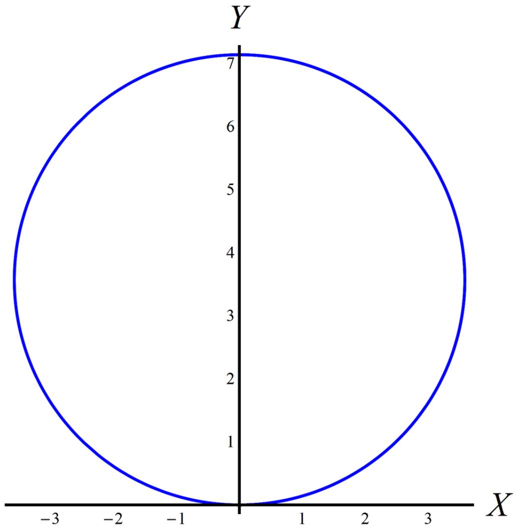

Figure 4.1: Circular trajectory of the CPA bicycle on the ground plane.

Table 4.1: Geometric parameter values for the example CPA bicycle.

Symbol	**Meaning**	**Value**
w	wheelbase	$0.750\,\mathrm{m}$
λ	caster angle	$20°$
r	rear wheel crown radius	$0.01\,\mathrm{m}$
R	rear wheel major radius	$0.05\,\mathrm{m}$
m_1	rear wheel mass	$0.35\,\mathrm{kg}$
(x_2, z_2)	position of rear frame centre of mass	$(0.5044\,\mathrm{m},\ 0.4279\,\mathrm{m})$
m_2	rear frame mass	$6.425\,\mathrm{kg}$
I_{2xx} I_{2yy} I_{2zz} I_{2xz}	rear frame moments of inertia	$0.064\,60\,\mathrm{kgm}^2$ $2.592\,62\,\mathrm{kgm}^2$ $2.546\,42\,\mathrm{kgm}^2$ $0.231\,02\,\mathrm{kgm}^2$
m_3	front frame mass	$2.412\,\mathrm{kg}$
(x_3, z_3)	position of rear frame centre of mass	$(0.7338\,\mathrm{m},\ 0.3022\,\mathrm{m})$
I_{3xx} I_{3yy} I_{3zz} I_{3xz}	front frame moments of inertia	$0.037\,97\,\mathrm{kgm}^2$ $0.038\,07\,\mathrm{kgm}^2$ $0.001\,85\,\mathrm{kgm}^2$ $-0.003\,93\,\mathrm{kgm}^2$
m_4	front wheel mass	$0.3\,\mathrm{kg}$

Chapter 5

Numerical solutions and stability analysis

In the last section of the previous chapter, we determined two classes of particular solutions for the CPA dynamics which can be expressed in closed form. However, due to their complexity, in general we need to numerically integrate the equations of motion. All the numerical solution presented in this chapter will be evaluated by considering the CPA bicycle characterised by the parameters in Table 4.1. The coefficient of viscous friction μ is assumed equal to zero.

5.1 Numerical integration of equations of motion

In order to integrate the equations of motion derived before, we consider a particular solution, that is, either rectilinear or circular motion, and perturb the initial conditions.

For example, we can consider a circular motion for $\chi_0 = 70\,\mathrm{rad\,s^{-1}}$, and after having solved the two algebraic equations (4.8) and (4.9), we approximate the other two angle as $\alpha_0 = 0.2928\,\mathrm{rad}$ and $\psi_0 = 0.1299\,\mathrm{rad}$. As shown in Figure 5.1, we note that the roll and the steering angles are initially constant, that is, the system describes a circular path; then, due to the perturbation, the solutions oscillate around the rectilinear stable motion. In Figure 5.2, the path of the rear contact point on the ground clarifies this situation.

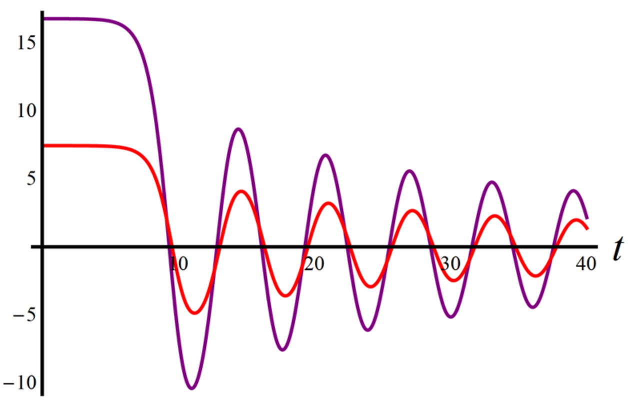

Figure 5.1: Evolution of roll angle (purple) and steering angle (red).

Let us now consider the rectilinear motion of the system with perturbed initial condition.

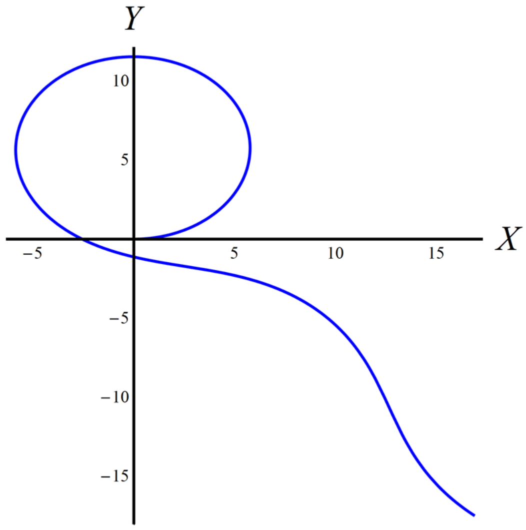

Figure 5.2: Position of the rear contact point.

First of all, we consider the initial conditions

$$\alpha(0) = 10^\circ,$$
$$\psi(0) = -5^\circ,$$
$$\dot{\chi}(0) = 70\,\mathrm{rad\,s^{-1}}.$$

In this case, the system is asymptotically stable, as shown in Figure 5.3, and the trajectory described by the rear contact point tends to be rectilinear after a certain time, as illustrated in Figure 5.4.

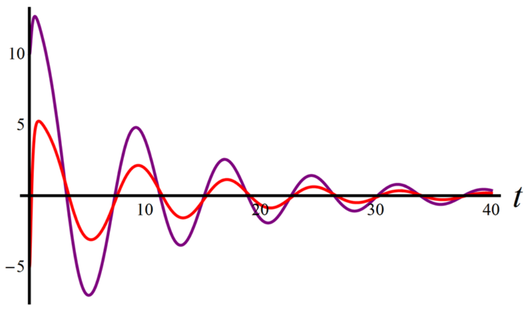

Figure 5.3: Evolution of roll angle (purple) and steering angle (red).

However, this asymptotically stable behaviour of the solution depends on the initial condition. For instance, if we reduce the initial angular velocity of the rear wheel, that is, $\dot{\chi}(0) = 65\,\mathrm{rad\,s^{-1}}$, the CPA bicycle has a limit cycle, as shown in Figure 5.5, where is represented the phase space of the roll angle.

If the initial angular velocity $\dot{\chi}(0)$ is further reduced, the system becomes unstable and the bicycle hits the ground in finite time. We note that the case of the limit cycle is possible only if the system has no friction, that is, $\mu = 0$.

5.2 Considerations about the stability and conclusions

As we have seen above, the bicycle can be either asymptotically stable, simple stable or unstable. These stability conditions depends on the geometric parameters of the system as

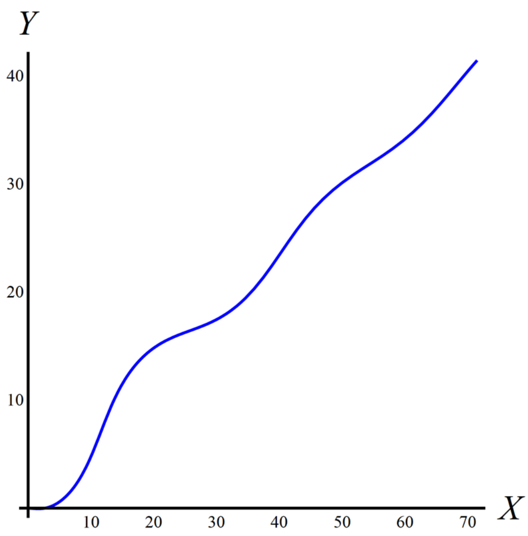

Figure 5.4: Position of the rear contact point.

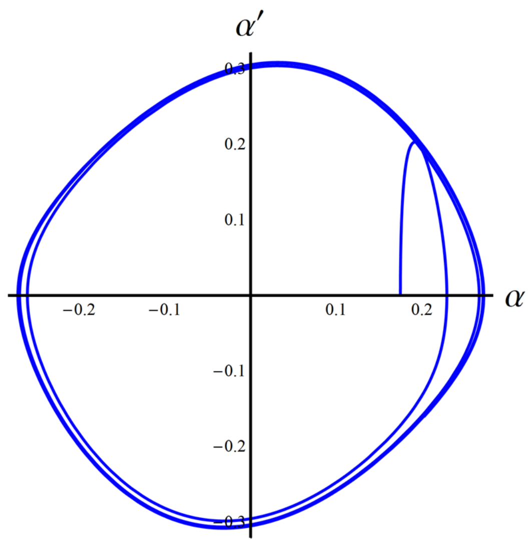

Figure 5.5: Roll angle phase space.

well as on the initial conditions. For instance, the example proposed before shows that, fixed the initial values of roll and steering angles, the system is initially unstable, then becomes simple stable increasing the initial angular velocity, and finally asymptotically stable. However, if this velocity is further increased, the system becomes unstable again.

In the future, we want to study the stability of the CPA bicycle by considering the dependence on the the geometric parameters. In this case, it is necessary a parametric study of the equations of motion.

Appendix A

Code

Listing A.1: Example code

```
Az := {{Cos[\[Theta][t]], Sin[\[Theta][t]], 0}, {-Sin[\[Theta][t]], Cos[\[Theta][t]], 0}, {0, 0, 1}}
Ax1 := {{1, 0, 0}, {0, Cos[\[Alpha][t]], -Sin[\[Alpha][t]]}, {0, Sin[\[Alpha][t]], Cos[\[Alpha][t]]}}
Ay1 := {{Cos[\[Chi][t]], 0, -Sin[\[Chi][t]]}, {0, 1, 0}, {Sin[\[Chi][t]], 0, Cos[\[Chi][t]]}}
Aym := {{Cos[\[Phi]], 0, Sin[\[Phi]]}, {0, 1, 0}, {-Sin[\[Phi]], 0, Cos[\[Phi]]}}
Aye := {{Cos[\[Lambda]], 0, Sin[\[Lambda]]}, {0, 1, 0}, {-Sin[\[Lambda]], 0, Cos[\[Lambda]]}}
Az2 := {{Cos[\[Psi][t]], Sin[\[Psi][t]], 0}, {-Sin[\[Psi][t]], Cos[\[Psi][t]], 0}, {0, 0, 1}}
Ay2 := {{Cos[\[Chi]a[t]], 0, -Sin[\[Chi]a[t]]}, {0, 1, 0}, {Sin[\[Chi]a[t]], 0, Cos[\[Chi]a[t]]}}

R1 := Ax1.Az
R1r := Ay1.Ax1.Az
R2 := Aym.Ax1.Az
R3 := Aye.Ax1.Az
R4 := Az2.Aye.Ax1.Az
R4r := Ay2.Az2.Aye.Ax1.Az

i1[t_] := Inverse[R1].{1, 0, 0}
j1[t_] := Inverse[R1].{0, 1, 0}
k1[t_] := Inverse[R1].{0, 0, 1}
i1r[t_] := Inverse[R1r].{1, 0, 0}
j1r[t_] := Inverse[R1r].{0, 1, 0}
k1r[t_] := Inverse[R1r].{0, 0, 1}
i2[t_] := Inverse[R2].{1, 0, 0}
j2[t_] := Inverse[R2].{0, 1, 0}
k2[t_] := Inverse[R2].{0, 0, 1}
i3[t_] := Inverse[R3].{1, 0, 0}
j3[t_] := Inverse[R3].{0, 1, 0}
k3[t_] := Inverse[R3].{0, 0, 1}
i4[t_] := Inverse[R4].{1, 0, 0}
j4[t_] := Inverse[R4].{0, 1, 0}
k4[t_] := Inverse[R4].{0, 0, 1}
i4r[t_] := Inverse[R4r].{1, 0, 0}
j4r[t_] := Inverse[R4r].{0, 1, 0}
k4r[t_] := Inverse[R4r].{0, 0, 1}


Angular velocities definition

\[Omega]1[t] = Simplify[{D[j1[t], t].k1[t], D[k1[t], t].i1[t], D[i1[t], t].j1[t]}];
\[Omega]1r[t] = Simplify[{D[j1r[t], t].k1r[t], D[k1r[t], t].i1r[t], D[i1r[t], t].j1r[t]}];
\[Omega]2[t] = Simplify[{D[j2[t], t].k2[t], D[k2[t], t].i2[t], D[i2[t], t].j2[t]}];
\[Omega]4[t] = Simplify[{D[j4[t], t].k4[t], D[k4[t], t].i4[t], D[i4[t], t].j4[t]}];
\[Omega]4r[t] = {D[j4r[t], t].k4r[t], D[k4r[t], t].i4r[t], D[i4r[t], t].j4r[t]};
```

```
Auxiliary angles definition

senat[t_] := Cos[\[Alpha][t]] Sin[\[Lambda]] Sin[\[Psi][t]] + Sin[\[Alpha][t]] Cos[\[Psi][t]]
cosat[t_] := Sqrt[1 - senat[t]^2]
dota[t_] := D[senat[t], t]/Sqrt[1 - senat[t]^2]
sentt[t_] := (Cos[\[Theta][t]] Cos[\[Lambda]] Sin[\[Psi][t]] + Sin[\[Theta][t]] (Cos[\[Alpha][t]] Cos
    [\[Psi][t]] - Sin[\[Alpha][t]] Sin[\[Lambda]] Sin[\[Psi][t]]) )/Sqrt[1 - senat[t]^2]
costt[t_] := (-Cos[\[Lambda]] Sin[\[Theta][t]] Sin[\[Psi][t]] + Cos[\[Theta][t]] (Cos[\[Alpha][t]]
    Cos[\[Psi][t]] - Sin[\[Alpha][t]] Sin[\[Lambda]] Sin[\[Psi][t]]) )/Sqrt[1 - senat[t]^2]
dottt[t_] := D[sentt[t], t]/costt[t]


Front wheel angular velocities defined by means of auxiliary angles

omegat[t_] := {-dota[t], -dottt[t] senat[t], dottt[t] cosat[t]}
omegatr[t_] := {-dota[t] Cos[\[Chi]a[t]] + dottt[t] cosat[t] Sin[\[Chi]a[t]], \[Chi]a'[t] - dottt[t]
    senat[t], dottt[t] cosat[t] Cos[\[Chi]a[t]] - dota[t] Sin[\[Chi]a[t]]}


Front contact point coordinates

zeta[t_] := x[t] + w Cos[\[Theta][t]]
doppiav[t_] := y[t] + w Sin[\[Theta][t]]


Rear wheel kinetic energy

g1[t_] = Simplify[Inverse[R1]].{0, 0, Rp} + {0, 0, rp};
v1[t_] := {x'[t], y'[t], 0} + D[g1[t], t]
sigma1 := {{Ix1, 0, 0}, {0, Iy1, 0}, {0, 0, Ix1}}
K1[t_] = Simplify[
   Expand[1/2 m1 v1[t].v1[t] + 1/2 \[Omega]1r[t].(sigma1.\[Omega]1r[t]) ]];


Rear frame kinetic energy

g2[t_] := Simplify[Inverse[R2]].{ l2, d2, 0}
v2[t_] := D[g2[t], t] + v1[t]
sigma2 := {{Ixx2, Ixy2, Ixz2}, {Ixy2, Iyy2, Izy2}, {Ixz2, Izy2, Izz2}}
K2[t_] = 1/2 m2 v2[t].v2[t] + 1/2 \[Omega]2[t].(sigma2.\[Omega]2[t]);


Front frame kinetic energy

g3[t_] := Simplify[Inverse[R4]].{ l3, d3, h3}
v3[t_] := v1[t] + D[Simplify[Inverse[R3]].{w Cos[\[Lambda]], 0, 0}, t] +
  D[g3[t], t]
sigma3 := {{Ix3, Ixy3, Ixz3}, {Ixy3, Iy3, Izy3}, {Ixz3, Izy3, Iz3}}
K3[t_]  = 1/2 m3 v3[t].v3[t] + 1/2 \[Omega]4[t].(sigma3.\[Omega]4[t]);


Front wheel kinetic energy

K4[t_] = 1/2 m4 (xf'[t]^2 + yf'[t]^2 ) +
   1/2 I4 (dota[t]^2 + dottt[t]^2 + \[Chi]a'[t]^2 -
      2 \[Chi]a'[t] dottt[t] senat[t]);


Potentials
```

```
U1[t_] := − m1 grav (rp + Rp Cos[\[Alpha][t]] )
U2[t_] := −
   m2 grav (rp + Rp Cos[\[Alpha][t]] + l2 Cos[\[Alpha][t]] Sin[\[Phi]] −
    d2 Sin[\[Alpha][t]])
U3[t_] := −
   m3 grav (rp + Rp Cos[\[Alpha][t]] +
    w Cos[\[Lambda]] Cos[\[Alpha][t]] Sin[\[Lambda]] +
    h3 Cos[\[Lambda]] Cos[\[Alpha][t]] +
    l3 (Cos[\[Alpha][t]] Sin[\[Lambda]] Cos[\[Psi][t]] −
       Sin[\[Alpha][t]] Sin[\[Psi][t]]) −
    d3 (Cos[\[Alpha][t]] Sin[\[Lambda]] Sin[\[Psi][t]] +
       Sin[\[Alpha][t]] Cos[\[Psi][t]]) )
U4[t_] := − m4 grav (rp)

Utot[t_] = U1[t] + U2[t] + U3[t] + U4[t];


Ktot[t_] = (K1[t] + K2[t] + K3[t] + K4[t]);


Definition of nonlinear FUNCTIONS

M = D[D[Ktot[t], x'[t], x'[t]]] /. {\[Theta][t] −> 0, \[Chi][t] −> 0};
a[\[Alpha][t], \[Psi][t]] =
  D[D[Ktot[t], \[Alpha]'[t], \[Alpha]'[t]]] /. {\[Theta][t] −> 0, \[Chi][t] −>
      0};
bb[\[Alpha][t], \[Psi][t]] =
  D[D[Ktot[t], \[Theta]'[t], \[Theta]'[t]]] /. {\[Theta][t] −> 0, \[Chi][t] −>
      0};
dd[\[Alpha][t], \[Psi][t]] =
  D[D[Ktot[t], \[Psi]'[t], \[Psi]'[t]]] /. {\[Theta][t] −> 0, \[Chi][t] −> 0};
EE = D[D[Ktot[t], \[Chi]'[t], \[Chi]'[t]]] /. {\[Theta][t] −> 0, \[Chi][t] −>
     0};
F = D[D[Ktot[t], \[Chi]a'[t], \[Chi]a'[t]]] /. {\[Theta][t] −> 0, \[Chi][t] −>
      0};
g[\[Alpha][t], \[Psi][t]] =
  D[D[Ktot[t], \[Theta]'[t], \[Alpha]'[t]]] /. {\[Theta][t] −> 0, \[Chi][t] −>
      0};
h[\[Alpha][t], \[Psi][t]] =
  D[D[Ktot[t], \[Psi]'[t], \[Alpha]'[t]]] /. {\[Theta][t] −> 0, \[Chi][t] −>
     0};
k[\[Alpha][t], \[Psi][t]] =
  D[D[Ktot[t], \[Chi]a'[t], \[Alpha]'[t]]] /. {\[Theta][t] −> 0, \[Chi][t] −>
     0};
ll[\[Alpha][t], \[Psi][t]] =
  D[D[Ktot[t], \[Psi]'[t], \[Theta]'[t]]] /. {\[Theta][t] −> 0, \[Chi][t] −>
     0};
m[\[Alpha][t], \[Psi][t]] =
  D[D[Ktot[t], \[Psi]'[t], \[Chi]a'[t]]] /. {\[Theta][t] −> 0, \[Chi][t] −> 0};
n[\[Alpha][t]] =
  D[D[Ktot[t], \[Theta]'[t], \[Chi]'[t]]] /. {\[Theta][t] −> 0, \[Chi][t] −>
     0};
p[\[Alpha][t], \[Psi][t]] =
  D[D[Ktot[t], \[Theta]'[t], \[Chi]a'[t]]] /. {\[Theta][t] −> 0, \[Chi][t] −>
     0};
q[\[Psi][t]] =
  D[D[Ktot[t], y'[t], \[Theta]'[t]]] /. {\[Theta][t] −> 0, \[Chi][t] −> 0};
r[\[Alpha][t], \[Psi][t]] =
  D[D[Ktot[t], y'[t], \[Alpha]'[t]]] /. {\[Theta][t] −> 0, \[Chi][t] −> 0};
s[\[Alpha][t], \[Psi][t]] =
  D[D[Ktot[t], y'[t], \[Psi]'[t]]] /. {\[Theta][t] −> 0, \[Chi][t] −> 0};
```

```
u[\[Alpha][t], \[Psi][t]] =
  D[D[Ktot[t], x'[t], \[Theta]'[t]]] /. {\[Theta][t] -> 0, \[Chi][t] -> 0};
z[\[Psi][t]] =
  D[D[Ktot[t], x'[t], \[Psi]'[t]]] /. {\[Theta][t] -> 0, \[Chi][t] -> 0};


ac[\[Alpha][t], \[Psi][
     t]] = -w Sin[\[Lambda]] Sin[\[Alpha][t]] Sin[\[Psi][t]] +
    w Cos[\[Alpha][t]] Cos[\[Psi][t]];
cc[\[Alpha][t], \[Psi][t]] =
  rp (Cos[\[Alpha][t]] Sin[\[Lambda]] Cos[\[Psi][t]] -
     Sin[\[Alpha][t]] Sin[\[Psi][t]]);
mc[\[Alpha][t], \[Psi][t]] =
  1/ac[\[Alpha][t], \[Psi][t]] (Rp +
     rp Cos[\[Alpha][t]]) Cos[\[Lambda]] Sin[\[Psi][t]];
pc[\[Alpha][t], \[Psi][t]] = cc[\[Alpha][t], \[Psi][t]]/
  ac[\[Alpha][t], \[Psi][t]];

dc[\[Alpha][t], \[Psi][
     t]] = (Rp + rp Cos[\[Alpha][t]]) (Cos[\[Alpha][t]] Cos[\[Psi][t]] -
     Sin[\[Alpha][t]] Sin[\[Lambda]] Sin[\[Psi][t]]);
ec[\[Alpha][t], \[Psi][t]] = rp Cos[\[Lambda]] Sin[\[Psi][t]] cosat[t]^2;
gc[\[Alpha][t], \[Psi][t]] =
  1/(rp cosat[t]^2) (dc[\[Alpha][t], \[Psi][t]] +
     w mc[\[Alpha][t], \[Psi][t]] Cos[\[Lambda]] Sin[ \[Psi][t]]);
hc[\[Alpha][t], \[Psi][t]] = 1/ cosat[t]^2 Cos[\[Lambda]] Sin[\[Psi][t]];
lc[\[Alpha][t], \[Psi][t]] = (w Cos[\[Lambda]] Sin[ \[Psi][t]])/(
    ac[\[Alpha][t], \[Psi][t]] cosat[
      t]^2) (Cos[\[Alpha][t]] Sin[\[Lambda]] Cos[\[Psi][t]] -
     Sin[\[Alpha][t]] Sin[\[Psi][t]]);


Acors[\[Alpha][t], \[Psi][t]] =
  M rp^2 + a[\[Alpha][t], \[Psi][t]] + F hc[\[Alpha][t], \[Psi][t]]^2 +
   2 hc[\[Alpha][t], \[Psi][t]] k[\[Alpha][t], \[Psi][t]] +
   2 rp r[\[Alpha][t], \[Psi][t]];
Ccors[\[Alpha][t], \[Psi][t]] =
  M (Rp + rp Cos[\[Alpha][t]])^2 +
   mc[\[Alpha][t], \[Psi][t]]^2 bb[\[Alpha][t], \[Psi][t]] + EE +
   F gc[\[Alpha][t], \[Psi][t]]^2 +
   2 mc[\[Alpha][t], \[Psi][t]] (n[\[Alpha][t]] +
      gc[\[Alpha][t], \[Psi][t]] p[\[Alpha][t], \[Psi][t]] + (Rp +
         rp Cos[\[Alpha][t]]) u[\[Alpha][t], \[Psi][t]]);
Ecors[\[Alpha][t], \[Psi][t]] =
  dd[\[Alpha][t], \[Psi][t]] +
   pc[\[Alpha][t], \[Psi][t]]^2 bb[\[Alpha][t], \[Psi][t]] +
   F lc[\[Alpha][t], \[Psi][t]]^2 +
   2 pc[\[Alpha][t], \[Psi][t]] ll[\[Alpha][t], \[Psi][t]] +
   2 lc[\[Alpha][t], \[Psi][t]] (m[\[Alpha][t], \[Psi][t]] +
      pc[\[Alpha][t], \[Psi][t]] p[\[Alpha][t], \[Psi][t]]);
Gcors[\[Alpha][t], \[Psi][t]] =
  F hc[\[Alpha][t], \[Psi][t]] lc[\[Alpha][t], \[Psi][t]] +
   pc[\[Alpha][t], \[Psi][t]] g[\[Alpha][t], \[Psi][t]] +
   h[\[Alpha][t], \[Psi][t]] +
   lc[\[Alpha][t], \[Psi][t]] k[\[Alpha][t], \[Psi][t]] +
   hc[\[Alpha][t], \[Psi][t]] m[\[Alpha][t], \[Psi][t]] +
   pc[\[Alpha][t], \[Psi][t]] hc[\[Alpha][t], \[Psi][t]] p[\[Alpha][
      t], \[Psi][t]] + rp pc[\[Alpha][t], \[Psi][t]] q[\[Psi][t]] +
   rp s[\[Alpha][t], \[Psi][t]];
Mcors[\[Alpha][t], \[Psi][t]] =
  F gc[\[Alpha][t], \[Psi][t]] hc[\[Alpha][t], \[Psi][t]] +
```

```
   mc[\[Alpha][t], \[Psi][t]] g[\[Alpha][t], \[Psi][t]] +
   gc[\[Alpha][t], \[Psi][t]] k[\[Alpha][t], \[Psi][t]] +
   mc[\[Alpha][t], \[Psi][t]] hc[\[Alpha][t], \[Psi][t]] p[\[Alpha][
      t], \[Psi][t]] + rp mc[\[Alpha][t], \[Psi][t]] q[\[Psi][t]];
Pcors[\[Alpha][t], \[Psi][t]] =
  mc[\[Alpha][t], \[Psi][t]] pc[\[Alpha][t], \[Psi][t]] bb[\[Alpha][
      t], \[Psi][t]] +
   F gc[\[Alpha][t], \[Psi][t]] lc[\[Alpha][t], \[Psi][t]] +
   mc[\[Alpha][t], \[Psi][t]] ll[\[Alpha][t], \[Psi][t]] +
   gc[\[Alpha][t], \[Psi][t]] m[\[Alpha][t], \[Psi][t]] +
   pc[\[Alpha][t], \[Psi][t]] n[\[Alpha][t]] +
   mc[\[Alpha][t], \[Psi][t]] lc[\[Alpha][t], \[Psi][t]] p[\[Alpha][
      t], \[Psi][t]] +
   pc[\[Alpha][t], \[Psi][t]] gc[\[Alpha][t], \[Psi][t]] p[\[Alpha][
      t], \[Psi][t]] + (Rp +
      rp Cos[\[Alpha][t]]) (pc[\[Alpha][t], \[Psi][t]] u[\[Alpha][t], \[Psi][
         t]] + z[\[Psi][t]]);


DaDa[\[Alpha][t], \[Psi][t]] = D[a[\[Alpha][t], \[Psi][t]], \[Alpha][t]];
DaDp[\[Alpha][t], \[Psi][t]] = D[a[\[Alpha][t], \[Psi][t]], \[Psi][t]];
DbDa[\[Alpha][t], \[Psi][t]] = D[bb[\[Alpha][t], \[Psi][t]], \[Alpha][t]];
DbDp[\[Alpha][t], \[Psi][t]] = D[bb[\[Alpha][t], \[Psi][t]], \[Psi][t]];
DdDa[\[Alpha][t], \[Psi][t]] = D[dd[\[Alpha][t], \[Psi][t]], \[Alpha][t]];
DdDp[\[Alpha][t], \[Psi][t]] = D[dd[\[Alpha][t], \[Psi][t]], \[Psi][t]];
DgDa[\[Alpha][t], \[Psi][t]] = D[g[\[Alpha][t], \[Psi][t]], \[Alpha][t]];
DgDp[\[Alpha][t], \[Psi][t]] = D[g[\[Alpha][t], \[Psi][t]], \[Psi][t]];
DhDa[\[Alpha][t], \[Psi][t]] = D[h[\[Alpha][t], \[Psi][t]], \[Alpha][t]];
DhDp[\[Alpha][t], \[Psi][t]] = D[h[\[Alpha][t], \[Psi][t]], \[Psi][t]];
DkDa[\[Alpha][t], \[Psi][t]] = D[k[\[Alpha][t], \[Psi][t]], \[Alpha][t]];
DkDp[\[Alpha][t], \[Psi][t]] = D[k[\[Alpha][t], \[Psi][t]], \[Psi][t]];
DlDa[\[Alpha][t], \[Psi][t]] = D[ll[\[Alpha][t], \[Psi][t]], \[Alpha][t]];
DlDp[\[Alpha][t], \[Psi][t]] = D[ll[\[Alpha][t], \[Psi][t]], \[Psi][t]];
DmDa[\[Alpha][t], \[Psi][t]] = D[m[\[Alpha][t], \[Psi][t]], \[Alpha][t]];
DmDp[\[Alpha][t], \[Psi][t]] = D[m[\[Alpha][t], \[Psi][t]], \[Psi][t]];
DnDa[\[Alpha][t]] = D[n[\[Alpha][t]], \[Alpha][t]];
DpDa[\[Alpha][t], \[Psi][t]] = D[p[\[Alpha][t], \[Psi][t]], \[Alpha][t]];
DpDp[\[Alpha][t], \[Psi][t]] = D[p[\[Alpha][t], \[Psi][t]], \[Psi][t]];
DqDp[\[Psi][t]] = D[q[\[Psi][t]], \[Psi][t]];
DrDa[\[Alpha][t], \[Psi][t]] = D[r[\[Alpha][t], \[Psi][t]], \[Alpha][t]];
DrDp[\[Alpha][t], \[Psi][t]] = D[r[\[Alpha][t], \[Psi][t]], \[Psi][t]];
DsDa[\[Alpha][t], \[Psi][t]] = D[s[\[Alpha][t], \[Psi][t]], \[Alpha][t]];
DsDp[\[Alpha][t], \[Psi][t]] = D[s[\[Alpha][t], \[Psi][t]], \[Psi][t]];
DuDa[\[Alpha][t], \[Psi][t]] = D[u[\[Alpha][t], \[Psi][t]], \[Alpha][t]];
DuDp[\[Alpha][t], \[Psi][t]] = D[u[\[Alpha][t], \[Psi][t]], \[Psi][t]];
DzDp[\[Psi][t]] = D[z[\[Psi][t]], \[Psi][t]];
DgcDa[\[Alpha][t], \[Psi][t]] = D[gc[\[Alpha][t], \[Psi][t]], \[Alpha][t]];
DgcDp[\[Alpha][t], \[Psi][t]] = D[gc[\[Alpha][t], \[Psi][t]], \[Psi][t]];
DhcDa[\[Alpha][t], \[Psi][t]] = D[hc[\[Alpha][t], \[Psi][t]], \[Alpha][t]];
DhcDp[\[Alpha][t], \[Psi][t]] = D[hc[\[Alpha][t], \[Psi][t]], \[Psi][t]];
DlcDa[\[Alpha][t], \[Psi][t]] = D[lc[\[Alpha][t], \[Psi][t]], \[Alpha][t]];
DlcDp[\[Alpha][t], \[Psi][t]] = D[lc[\[Alpha][t], \[Psi][t]], \[Psi][t]];
DmcDa[\[Alpha][t], \[Psi][t]] = D[mc[\[Alpha][t], \[Psi][t]], \[Alpha][t]];
DmcDp[\[Alpha][t], \[Psi][t]] = D[mc[\[Alpha][t], \[Psi][t]], \[Psi][t]];
DpcDa[\[Alpha][t], \[Psi][t]] = D[pc[\[Alpha][t], \[Psi][t]], \[Alpha][t]];
DpcDp[\[Alpha][t], \[Psi][t]] = D[pc[\[Alpha][t], \[Psi][t]], \[Psi][t]];
DAcorsDa[\[Alpha][t], \[Psi][t]] =
  D[Acors[\[Alpha][t], \[Psi][t]], \[Alpha][t]];
DEcorsDp[\[Alpha][t], \[Psi][t]] =
  D[Ecors[\[Alpha][t], \[Psi][t]], \[Psi][t]];
```

```
atilde [\[Alpha][t],  \[Psi][
   t]]  := (k[\[Alpha][t],  \[Psi][t]]  +
      F hc[\[Alpha][t],  \[Psi][t]])  DgcDa[\[Alpha][t],  \[Psi][
      t]]  + (rp q[\[Psi][t]]  + g[\[Alpha][t],  \[Psi][t]]  +
      hc[\[Alpha][t],  \[Psi][t]]  p[\[Alpha][t],  \[Psi][t]])  DmcDa[\[Alpha][
      t],  \[Psi][t]]
btilde [\[Alpha][t],  \[Psi][
   t]]  := (rp  q[\[Psi][t]]  + g[\[Alpha][t],  \[Psi][t]]  +
      hc[\[Alpha][t],  \[Psi][t]]  p[\[Alpha][t],  \[Psi][t]])  DpcDa[\[Alpha][
      t],  \[Psi][t]]  + (k[\[Alpha][t],  \[Psi][t]]  +
      F hc[\[Alpha][t],  \[Psi][t]])  DlcDa[\[Alpha][t],  \[Psi][t]]  +
  DaDp[\[Alpha][t],  \[Psi][t]]  +
  F hc[\[Alpha][t],  \[Psi][t]]  DhcDp[\[Alpha][t],  \[Psi][t]]  +
  DhcDp[\[Alpha][t],  \[Psi][t]]  k[\[Alpha][t],  \[Psi][t]]  +
  2 hc[\[Alpha][t],  \[Psi][t]]  DkDp[\[Alpha][t],  \[Psi][t]]  +
  2 rp  DrDp[\[Alpha][t],  \[Psi][t]]
dtilde [\[Alpha][t],  \[Psi][
   t]]  := (z[\[Psi][t]]  +
      u[\[Alpha][t],  \[Psi][t]]  pc[\[Alpha][t],  \[Psi][t]])  pc[\[Alpha][
      t],  \[Psi][t]]  rp  +
  F hc[\[Alpha][t],  \[Psi][t]]  DlcDp[\[Alpha][t],  \[Psi][t]]  +
  DpcDp[\[Alpha][t],  \[Psi][t]]  g[\[Alpha][t],  \[Psi][t]]  +
  pc[\[Alpha][t],  \[Psi][t]]  DgDp[\[Alpha][t],  \[Psi][t]]  +
  DhDp[\[Alpha][t],  \[Psi][t]]  +
  DlcDp[\[Alpha][t],  \[Psi][t]]  k[\[Alpha][t],  \[Psi][t]]  +
  lc[\[Alpha][t],  \[Psi][t]]  DkDp[\[Alpha][t],  \[Psi][t]]  +
  hc[\[Alpha][t],  \[Psi][t]]  DmDp[\[Alpha][t], \[Psi][t]]  +
  DpcDp[\[Alpha][t],  \[Psi][t]]  hc[\[Alpha][t],  \[Psi][t]]  p[\[Alpha][
      t],  \[Psi][t]]  +
  pc[\[Alpha][t],  \[Psi][t]]  hc[\[Alpha][t],  \[Psi][t]]  DpDp[\[Alpha][
      t],  \[Psi][t]]  + rp DpcDp[\[Alpha][t],  \[Psi][t]]  q[\[Psi][t]]  +
  rp  pc[\[Alpha][t],  \[Psi][t]]  DqDp[\[Psi][t]]  +
  rp  DsDp[\[Alpha][t],  \[Psi][t]]  - 1/2 DdDa[\[Alpha][t], \[Psi][t]]  -
  1/2 pc[\[Alpha][t],  \[Psi][t]]^2  DbDa[\[Alpha][t],  \[Psi][t]]  -
  pc[\[Alpha][t],  \[Psi][t]]  DlDa[\[Alpha][t],  \[Psi][t]]  -
  lc[\[Alpha][t],  \[Psi][t]]  DmDa[\[Alpha][t], \[Psi][t]]  -
  pc[\[Alpha][t],  \[Psi][t]]  lc[\[Alpha][t],  \[Psi][t]]  DpDa[\[Alpha][
      t],  \[Psi][t]]
etilde [\[Alpha][t],  \[Psi][
   t]]  := (u[\[Alpha][t],  \[Psi][t]]  mc[\[Alpha][t],  \[Psi][t]]  +
      M (Rp + rp Cos[\[Alpha][t]]))  pc[\[Alpha][t],  \[Psi][
      t]]   rp + (z[\[Psi][t]]   +
      u[\[Alpha][t],  \[Psi][t]]  pc[\[Alpha][t],  \[Psi][t]]  ) mc[\[Alpha][
      t],  \[Psi][t]]   rp +
  F DgcDp[\[Alpha][t],  \[Psi][t]]  hc[\[Alpha][t],  \[Psi][t]]  +
  DmcDp[\[Alpha][t], \[Psi][t]]  g[\[Alpha][t],  \[Psi][t]]  +
  mc[\[Alpha][t],  \[Psi][t]]  DgDp[\[Alpha][t],  \[Psi][t]]  +
  DgcDp[\[Alpha][t],  \[Psi][t]]  k[\[Alpha][t],  \[Psi][t]]  +
  gc[\[Alpha][t],  \[Psi][t]]  DkDp[\[Alpha][t],  \[Psi][t]]  +
  DmcDp[\[Alpha][t], \[Psi][t]]  hc[\[Alpha][t],  \[Psi][t]]  p[\[Alpha][
      t],  \[Psi][t]]  +
  mc[\[Alpha][t],  \[Psi][t]]  hc[\[Alpha][t],  \[Psi][t]]  DpDp[\[Alpha][
      t],  \[Psi][t]]  + rp DmcDp[\[Alpha][t], \[Psi][t]]  q[\[Psi][t]]  +
  rp  mc[\[Alpha][t],  \[Psi][t]]  DqDp[\[Psi][t]]  -
  mc[\[Alpha][t],  \[Psi][t]]  pc[\[Alpha][t],  \[Psi][t]]  DbDa[\[Alpha][
      t],  \[Psi][t]]  -
  mc[\[Alpha][t],  \[Psi][t]]  DlDa[\[Alpha][t],  \[Psi][t]]  -
  gc[\[Alpha][t],  \[Psi][t]]  DmDa[\[Alpha][t], \[Psi][t]]  -
  pc[\[Alpha][t],  \[Psi][t]]  DnDa[\[Alpha][t]]  -
  mc[\[Alpha][t],  \[Psi][t]]  lc[\[Alpha][t],  \[Psi][t]]  DpDa[\[Alpha][
```

```
      t], \[Psi][t]] -
   pc[\[Alpha][t], \[Psi][t]] gc[\[Alpha][t], \[Psi][t]] DpDa[\[Alpha][
      t], \[Psi][t]] - (Rp + rp Cos[\[Alpha][t]]) pc[\[Alpha][t], \[Psi][
      t]] DuDa[\[Alpha][t], \[Psi][t]]
 ftilde [\[Alpha][t], \[Psi][
    t]] := (u[\[Alpha][t], \[Psi][t]] mc[\[Alpha][t], \[Psi][t]] +
      M (Rp + rp Cos[\[Alpha][t]])) mc[\[Alpha][t], \[Psi][t]] rp -
   1/2 mc[\[Alpha][t], \[Psi][t]]^2 DbDa[\[Alpha][t], \[Psi][t]] -
   mc[\[Alpha][t], \[Psi][t]] DnDa[\[Alpha][t]] -
   mc[\[Alpha][t], \[Psi][t]] gc[\[Alpha][t], \[Psi][t]] DpDa[\[Alpha][
      t], \[Psi][t]] -
   mc[\[Alpha][t], \[Psi][t]] (Rp + rp Cos[\[Alpha][t]]) DuDa[\[Alpha][
      t], \[Psi][t]]

gtilde [\[Alpha][t], \[Psi][
    t]] := (s[\[Alpha][t], \[Psi][t]] +
      q[\[Psi][t]] pc[\[Alpha][t], \[Psi][t]]) pc[\[Alpha][t], \[Psi][
      t]] (Rp + rp Cos[\[Alpha][t]]) + (m[\[Alpha][t], \[Psi][t]] +
      F lc[\[Alpha][t], \[Psi][t]] +
      pc[\[Alpha][t], \[Psi][t]] p[\[Alpha][t], \[Psi][t]]) DgcDp[\[Alpha][
      t], \[Psi][t]] + (ll[\[Alpha][t], \[Psi][t]] +
      pc[\[Alpha][t], \[Psi][t]] bb[\[Alpha][t], \[Psi][t]] +
       lc[\[Alpha][t], \[Psi][t]] p[\[Alpha][t], \[Psi][t]]) DmcDp[\[Alpha][
      t], \[Psi][t]]
htilde [\[Alpha][t], \[Psi][t]] :=
 DdDa[\[Alpha][t], \[Psi][t]] +
  pc[\[Alpha][t], \[Psi][t]] bb[\[Alpha][t], \[Psi][t]] DpcDa[\[Alpha][
      t], \[Psi][t]] +
  pc[\[Alpha][t], \[Psi][t]]^2 DbDa[\[Alpha][t], \[Psi][t]] +
  F lc[\[Alpha][t], \[Psi][t]] DlcDa[\[Alpha][t], \[Psi][t]] +
  DpcDa[\[Alpha][t], \[Psi][t]] ll[\[Alpha][t], \[Psi][t]] +
  2 pc[\[Alpha][t], \[Psi][t]] DlDa[\[Alpha][t], \[Psi][t]] +
  DlcDa[\[Alpha][t], \[Psi][t]] m[\[Alpha][t], \[Psi][t]] +
  2 lc[\[Alpha][t], \[Psi][t]] DmDa[\[Alpha][t], \[Psi][t]] +
  DpcDa[\[Alpha][t], \[Psi][t]] lc[\[Alpha][t], \[Psi][t]] p[\[Alpha][
      t], \[Psi][t]] +
  pc[\[Alpha][t], \[Psi][t]] DlcDa[\[Alpha][t], \[Psi][t]] p[\[Alpha][
      t], \[Psi][t]] +
  2 pc[\[Alpha][t], \[Psi][t]] lc[\[Alpha][t], \[Psi][t]] DpDa[\[Alpha][
      t], \[Psi][t]] - (z[\[Psi][t]] +
      u[\[Alpha][t], \[Psi][t]] pc[\[Alpha][t], \[Psi][t]]) pc[\[Alpha][
      t], \[Psi][t]] rp + (m[\[Alpha][t], \[Psi][t]] +
      F lc[\[Alpha][t], \[Psi][t]] +
      pc[\[Alpha][t], \[Psi][t]] p[\[Alpha][t], \[Psi][t]]) DhcDp[\[Alpha][
      t], \[Psi][t]]
 ltilde [\[Alpha][t], \[Psi][t]] :=
 F DhcDa[\[Alpha][t], \[Psi][t]] lc[\[Alpha][t], \[Psi][t]] +
  pc[\[Alpha][t], \[Psi][t]] DgDa[\[Alpha][t], \[Psi][t]] +
  DhDa[\[Alpha][t], \[Psi][t]] +
   lc[\[Alpha][t], \[Psi][t]] DkDa[\[Alpha][t], \[Psi][t]] +
  DhcDa[\[Alpha][t], \[Psi][t]] m[\[Alpha][t], \[Psi][t]] +
  hc[\[Alpha][t], \[Psi][t]] DmDa[\[Alpha][t], \[Psi][t]] +
  pc[\[Alpha][t], \[Psi][t]] DhcDa[\[Alpha][t], \[Psi][t]] p[\[Alpha][
      t], \[Psi][t]] +
  pc[\[Alpha][t], \[Psi][t]] hc[\[Alpha][t], \[Psi][t]] DpDa[\[Alpha][
      t], \[Psi][t]] + rp DsDa[\[Alpha][t], \[Psi][t]] -
  1/2 DaDp[\[Alpha][t], \[Psi][t]] -
  hc[\[Alpha][t], \[Psi][t]] DkDp[\[Alpha][t], \[Psi][t]] -
  rp DrDp[\[Alpha][t], \[Psi][t]]
mtilde[\[Alpha][t], \[Psi][t]] :=
 DmcDa[\[Alpha][t], \[Psi][t]] pc[\[Alpha][t], \[Psi][t]] bb[\[Alpha][
```

```
      t], \[Psi][t]] +
  mc[\[Alpha][t], \[Psi][t]] pc[\[Alpha][t], \[Psi][t]] DbDa[\[Alpha][
      t], \[Psi][t]] +
  F DgcDa[\[Alpha][t], \[Psi][t]] lc[\[Alpha][t], \[Psi][t]] +
  DmcDa[\[Alpha][t], \[Psi][t]] ll[\[Alpha][t], \[Psi][t]] +
  mc[\[Alpha][t], \[Psi][t]] DlDa[\[Alpha][t], \[Psi][t]] +
  DgcDa[\[Alpha][t], \[Psi][t]] m[\[Alpha][t], \[Psi][t]] +
  gc[\[Alpha][t], \[Psi][t]] DmDa[\[Alpha][t], \[Psi][t]] +
  pc[\[Alpha][t], \[Psi][t]] DnDa[\[Alpha][t]] +
  DmcDa[\[Alpha][t], \[Psi][t]] lc[\[Alpha][t], \[Psi][t]] p[\[Alpha][
      t], \[Psi][t]] +
  mc[\[Alpha][t], \[Psi][t]] lc[\[Alpha][t], \[Psi][t]] DpDa[\[Alpha][
      t], \[Psi][t]] +
  pc[\[Alpha][t], \[Psi][t]] DgcDa[\[Alpha][t], \[Psi][t]] p[\[Alpha][
      t], \[Psi][t]] +
  pc[\[Alpha][t], \[Psi][t]] gc[\[Alpha][t], \[Psi][t]] DpDa[\[Alpha][
      t], \[Psi][t]] -
  rp Sin[\[Alpha][t]] pc[\[Alpha][t], \[Psi][t]] u[\[Alpha][t], \[Psi][
      t]] + (Rp + rp Cos[\[Alpha][t]]) pc[\[Alpha][t], \[Psi][
      t]] DuDa[\[Alpha][t], \[Psi][t]] - rp Sin[\[Alpha][t]] z[\[Psi][t]] -
  mc[\[Alpha][t], \[Psi][t]] DgDp[\[Alpha][t], \[Psi][t]] -
  gc[\[Alpha][t], \[Psi][t]] DkDp[\[Alpha][t], \[Psi][t]] -
  mc[\[Alpha][t], \[Psi][t]] hc[\[Alpha][t], \[Psi][t]] DpDp[\[Alpha][
      t], \[Psi][t]] -
  rp mc[\[Alpha][t], \[Psi][t]] DqDp[\[Psi][
      t]] - (u[\[Alpha][t], \[Psi][t]] mc[\[Alpha][t], \[Psi][t]] +
      M (Rp + rp Cos[\[Alpha][t]])) pc[\[Alpha][t], \[Psi][
      t]] rp + (r[\[Alpha][t], \[Psi][t]] + M rp) pc[\[Alpha][t], \[Psi][
      t]] (Rp + rp Cos[\[Alpha][t]])
ntilde[\[Alpha][t], \[Psi][t]] :=
 q[\[Psi][t]] mc[\[Alpha][t], \[Psi][t]] pc[\[Alpha][t], \[Psi][t]] (Rp +
      rp Cos[\[Alpha][t]]) -
  1/2 mc[\[Alpha][t], \[Psi][t]]^2 DbDp[\[Alpha][t], \[Psi][t]] -
  mc[\[Alpha][t], \[Psi][t]] gc[\[Alpha][t], \[Psi][t]] DpDp[\[Alpha][
      t], \[Psi][t]] -
  mc[\[Alpha][t], \[Psi][t]] (Rp + rp Cos[\[Alpha][t]]) DuDp[\[Alpha][
      t], \[Psi][t]]

ptilde[\[Alpha][t], \[Psi][
    t]] := -M rp Sin[\[Alpha][t]] (Rp + rp Cos[\[Alpha][t]]) +
  mc[\[Alpha][t], \[Psi][t]] bb[\[Alpha][t], \[Psi][t]] DmcDa[\[Alpha][
      t], \[Psi][t]] +
  mc[\[Alpha][t], \[Psi][t]]^2 DbDa[\[Alpha][t], \[Psi][t]] +
  F gc[\[Alpha][t], \[Psi][t]] DgcDa[\[Alpha][t], \[Psi][t]] +
  DmcDa[\[Alpha][t], \[Psi][t]] n[\[Alpha][t]] +
  2 mc[\[Alpha][t], \[Psi][t]] DnDa[\[Alpha][t]] +
  DmcDa[\[Alpha][t], \[Psi][t]] gc[\[Alpha][t], \[Psi][t]] p[\[Alpha][
      t], \[Psi][t]] +
  mc[\[Alpha][t], \[Psi][t]] DgcDa[\[Alpha][t], \[Psi][t]] p[\[Alpha][
      t], \[Psi][t]] +
  2 mc[\[Alpha][t], \[Psi][t]] gc[\[Alpha][t], \[Psi][t]] DpDa[\[Alpha][
      t], \[Psi][t]] +
  DmcDa[\[Alpha][t], \[Psi][t]] (Rp + rp Cos[\[Alpha][t]]) u[\[Alpha][
      t], \[Psi][t]] -
  mc[\[Alpha][t], \[Psi][t]] rp Sin[\[Alpha][t]] u[\[Alpha][t], \[Psi][t]] +
  2 mc[\[Alpha][t], \[Psi][t]] (Rp + rp Cos[\[Alpha][t]]) DuDa[\[Alpha][
      t], \[Psi][t]] - (u[\[Alpha][t], \[Psi][t]] mc[\[Alpha][t], \[Psi][t]] +
      M (Rp + rp Cos[\[Alpha][t]])) mc[\[Alpha][t], \[Psi][t]] rp
qtilde[\[Alpha][t], \[Psi][t]] :=
 mc[\[Alpha][t], \[Psi][t]] bb[\[Alpha][t], \[Psi][t]] DmcDp[\[Alpha][
      t], \[Psi][t]] +
```

```
mc[\[Alpha][t], \[Psi][t]]^2 DbDp[\[Alpha][t], \[Psi][t]] +
F gc[\[Alpha][t], \[Psi][t]] DgcDp[\[Alpha][t], \[Psi][t]] +
DmcDp[\[Alpha][t], \[Psi][t]] n[\[Alpha][t]] +
DmcDp[\[Alpha][t], \[Psi][t]] gc[\[Alpha][t], \[Psi][t]] p[\[Alpha][
    t], \[Psi][t]] +
mc[\[Alpha][t], \[Psi][t]] DgcDp[\[Alpha][t], \[Psi][t]] p[\[Alpha][
    t], \[Psi][t]] +
2 mc[\[Alpha][t], \[Psi][t]] gc[\[Alpha][t], \[Psi][t]] DpDp[\[Alpha][
    t], \[Psi][t]] +
DmcDp[\[Alpha][t], \[Psi][t]] (Rp + rp Cos[\[Alpha][t]]) u[\[Alpha][
    t], \[Psi][t]] +
2 mc[\[Alpha][t], \[Psi][t]] (Rp + rp Cos[\[Alpha][t]]) DuDp[\[Alpha][
    t], \[Psi][t]] -
q[\[Psi][t]] mc[\[Alpha][t], \[Psi][t]] pc[\[Alpha][t], \[Psi][t]] (Rp +
    rp Cos[\[Alpha][t]])
rtilde[\[Alpha][t], \[Psi][t]] :=
F gc[\[Alpha][t], \[Psi][t]] DhcDa[\[Alpha][t], \[Psi][t]] +
mc[\[Alpha][t], \[Psi][t]] DgDa[\[Alpha][t], \[Psi][t]] +
gc[\[Alpha][t], \[Psi][t]] DkDa[\[Alpha][t], \[Psi][t]] +
mc[\[Alpha][t], \[Psi][t]] DhcDa[\[Alpha][t], \[Psi][t]] p[\[Alpha][
    t], \[Psi][t]] +
mc[\[Alpha][t], \[Psi][t]] hc[\[Alpha][t], \[Psi][t]] DpDa[\[Alpha][
    t], \[Psi][t]]
stilde[\[Alpha][t], \[Psi][t]] :=
F gc[\[Alpha][t], \[Psi][t]] DhcDp[\[Alpha][t], \[Psi][t]] +
mc[\[Alpha][t], \[Psi][t]] DgDp[\[Alpha][t], \[Psi][t]] +
gc[\[Alpha][t], \[Psi][t]] DkDp[\[Alpha][t], \[Psi][t]] +
mc[\[Alpha][t], \[Psi][t]] DhcDp[\[Alpha][t], \[Psi][t]] p[\[Alpha][
    t], \[Psi][t]] +
mc[\[Alpha][t], \[Psi][t]] hc[\[Alpha][t], \[Psi][t]] DpDp[\[Alpha][
    t], \[Psi][t]] + rp mc[\[Alpha][t], \[Psi][t]] DqDp[\[Psi][t]] +
mc[\[Alpha][t], \[Psi][t]] DpcDa[\[Alpha][t], \[Psi][t]] bb[\[Alpha][
    t], \[Psi][t]] +
mc[\[Alpha][t], \[Psi][t]] pc[\[Alpha][t], \[Psi][t]] DbDa[\[Alpha][
    t], \[Psi][t]] +
F gc[\[Alpha][t], \[Psi][t]] DlcDa[\[Alpha][t], \[Psi][t]] +
mc[\[Alpha][t], \[Psi][t]] DlDa[\[Alpha][t], \[Psi][t]] +
gc[\[Alpha][t], \[Psi][t]] DmDa[\[Alpha][t], \[Psi][t]] +
DpcDa[\[Alpha][t], \[Psi][t]] n[\[Alpha][t]] +
pc[\[Alpha][t], \[Psi][t]] DnDa[\[Alpha][t]] +
mc[\[Alpha][t], \[Psi][t]] DlcDa[\[Alpha][t], \[Psi][t]] p[\[Alpha][
    t], \[Psi][t]] +
mc[\[Alpha][t], \[Psi][t]] lc[\[Alpha][t], \[Psi][t]] DpDa[\[Alpha][
    t], \[Psi][t]] +
DpcDa[\[Alpha][t], \[Psi][t]] gc[\[Alpha][t], \[Psi][t]] p[\[Alpha][
    t], \[Psi][t]] +
pc[\[Alpha][t], \[Psi][t]] gc[\[Alpha][t], \[Psi][t]] DpDa[\[Alpha][
    t], \[Psi][t]] + (Rp + rp Cos[\[Alpha][t]]) DpcDa[\[Alpha][t], \[Psi][
    t]] u[\[Alpha][t], \[Psi][t]] + (Rp + rp Cos[\[Alpha][t]]) pc[\[Alpha][
    t], \[Psi][t]] DuDa[\[Alpha][t], \[Psi][t]] - (z[\[Psi][t]] +
    u[\[Alpha][t], \[Psi][t]] pc[\[Alpha][t], \[Psi][t]]) mc[\[Alpha][
    t], \[Psi][t]] rp - (r[\[Alpha][t], \[Psi][t]] + M rp) pc[\[Alpha][
    t], \[Psi][t]] (Rp + rp Cos[\[Alpha][t]])
utilde[\[Alpha][t], \[Psi][t]] :=
mc[\[Alpha][t], \[Psi][t]] DpcDp[\[Alpha][t], \[Psi][t]] bb[\[Alpha][
    t], \[Psi][t]] +
mc[\[Alpha][t], \[Psi][t]] pc[\[Alpha][t], \[Psi][t]] DbDp[\[Alpha][
    t], \[Psi][t]] +
F gc[\[Alpha][t], \[Psi][t]] DlcDp[\[Alpha][t], \[Psi][t]] +
mc[\[Alpha][t], \[Psi][t]] DlDp[\[Alpha][t], \[Psi][t]] +
gc[\[Alpha][t], \[Psi][t]] DmDp[\[Alpha][t], \[Psi][t]] +
```

```
  DpcDp[\[Alpha][t], \[Psi][t]] n[\[Alpha][t]] +
  mc[\[Alpha][t], \[Psi][t]] DlcDp[\[Alpha][t], \[Psi][t]] p[\[Alpha][
     t], \[Psi][t]] +
  mc[\[Alpha][t], \[Psi][t]] lc[\[Alpha][t], \[Psi][t]] DpDp[\[Alpha][
     t], \[Psi][t]] +
  DpcDp[\[Alpha][t], \[Psi][t]] gc[\[Alpha][t], \[Psi][t]] p[\[Alpha][
     t], \[Psi][t]] +
  pc[\[Alpha][t], \[Psi][t]] gc[\[Alpha][t], \[Psi][t]] DpDp[\[Alpha][
     t], \[Psi][t]] + (Rp + rp Cos[\[Alpha][t]]) DpcDp[\[Alpha][t], \[Psi][
     t]] u[\[Alpha][t], \[Psi][t]] + (Rp + rp Cos[\[Alpha][t]]) pc[\[Alpha][
     t], \[Psi][t]] DuDp[\[Alpha][t], \[Psi][t]] + (Rp +
     rp Cos[\[Alpha][t]]) DzDp[\[Psi][t]] - (s[\[Alpha][t], \[Psi][t]] +
     q[\[Psi][t]] pc[\[Alpha][t], \[Psi][t]]) pc[\[Alpha][t], \[Psi][
     t]] (Rp + rp Cos[\[Alpha][t]])


EQUATIONS OF MOTION

grav := 9.81
w := 0.750
\[Lambda] := 20 Pi /180
Rp := 0.05
rp := 0.01
m1 := 0.35
Ix1 := (5/8 rp^2 + 1/2 Rp^2) m1
Iy1 := (3/4 rp^2 + Rp^2) m1
xb := 0.5044
zb := 0.4279
\[Phi] := ArcTan[(zb - (Rp + rp))/xb]
\[Epsilon] := \[Lambda] - \[Phi]
l := -(Rp /Cos[\[Epsilon]]) Sin[\[Lambda]] + w Cos[\[Lambda]]/Cos[\[Epsilon]]
b := 1/Cos[\[Epsilon]] (Rp Cos[\[Phi]] + w Sin[\[Phi]])
m2 := 6.425
Ixx2 := 0.06460
Iyy2 := 2.59262
Izz2 := 2.54642
Ixz2 := 0.23102
Ixy2 := 0
Izy2 := 0
l2 := xb/Cos[\[Phi]]
d2 := 0
m3 := 2.412
xh := 0.7338
zh := 0.3022
ws := w Cos[\[Lambda]]
ks := Rp + rp + ws Sin[\[Lambda]]
bs := ws Cos[\[Lambda]]
l3 := (xh - bs) Cos[\[Lambda]] + (zh - ks) Sin[\[Lambda]]
d3 := 0
h3 := -(xh - bs) Sin[\[Lambda]] + (zh - ks) Cos[\[Lambda]]
Ix3 := 0.03797
Iy3 := 0.03807
Iz3 := 0.00185
Ixz3 := -0.00393
Ixy3 := 0
Izy3 := 0
m4 := 0.3
I4 := 2/5 rp^2 m4
 trail := rp Sin[\[Lambda]]/Cos[\[Lambda]]
```

```
Conditions for circular motion

\[Chi]0 := 70

FindRoot[{\[Chi]0^2 Evaluate[
      ftilde [\[Alpha][t], \[Psi][
        t]] /. {\[Alpha][t] -> \[Alpha]0, \[Psi][t] -> \[Psi]0}] ==
   Evaluate[(D[
       Utot[t], \[Alpha][t]]) /. {\[Alpha][t] -> \[Alpha]0, \[Psi][
        t] -> \[Psi]0}], \[Chi]0^2 Evaluate[
      ntilde [\[Alpha][t], \[Psi][
        t]] /. {\[Alpha][t] -> \[Alpha]0, \[Psi][t] -> \[Psi]0}] ==
   Evaluate[(D[
       Utot[t], \[Psi][t]]) /. {\[Alpha][t] -> \[Alpha]0, \[Psi][
        t] -> \[Psi]0}]}, {{\[Alpha]0, 0.5}, {\[Psi]0, 0.5}},
 MaxIterations -> 100]


ODE = { \[Alpha]''[t] Acors[\[Alpha][t], \[Psi][t]] + \[Psi]''[
       t] Gcors[\[Alpha][t], \[Psi][t]] + \[Chi]''[
       t] Mcors[\[Alpha][t], \[Psi][t]] +
     1/2 \[Alpha]'[t]^2 DAcorsDa[\[Alpha][t], \[Psi][t]] + \[Alpha]'[
       t] \[Chi]'[t] atilde [\[Alpha][t], \[Psi][t]] + \[Alpha]'[t] \[Psi]'[
       t] btilde [\[Alpha][t], \[Psi][t]] + \[Psi]'[t]^2 dtilde [\[Alpha][
        t], \[Psi][t]] + \[Chi]'[t] \[Psi]'[
       t] etilde [\[Alpha][t], \[Psi][t]] + \[Chi]'[t]^2 ftilde [\[Alpha][
        t], \[Psi][t]] == D[Utot[t], \[Alpha][t]],
   \[Alpha]''[t] Gcors[\[Alpha][t], \[Psi][t]] + \[Psi]''[
       t] Ecors[\[Alpha][t], \[Psi][t]] + \[Chi]''[
       t] Pcors[\[Alpha][t], \[Psi][t]] +
     1/2 \[Psi]'[t]^2 DEcorsDp[\[Alpha][t], \[Psi][t]] + \[Chi]'[t] \[Psi]'[
       t] gtilde [\[Alpha][t], \[Psi][t]] + \[Alpha]'[t] \[Psi]'[
       t] htilde [\[Alpha][t], \[Psi][t]] +
     \[Alpha]'[t]^2 ltilde [\[Alpha][t], \[Psi][t]] + \[Alpha]'[t] \[Chi]'[
       t] mtilde [\[Alpha][t], \[Psi][t]] + \[Chi]'[t]^2 ntilde [\[Alpha][
        t], \[Psi][t]] == D[Utot[t], \[Psi][t]],
   \[Alpha]''[t] Mcors[\[Alpha][t], \[Psi][t]] + \[Psi]''[
       t] Pcors[\[Alpha][t], \[Psi][t]] + \[Chi]''[
       t] Ccors[\[Alpha][t], \[Psi][t]] + \[Alpha]'[t] \[Chi]'[
       t] ptilde [\[Alpha][t], \[Psi][t]] + \[Psi]'[t] \[Chi]'[
       t] qtilde [\[Alpha][t], \[Psi][t]] + \[Alpha]'[t]^2 rtilde [\[Alpha][
        t], \[Psi][t]] + \[Alpha]'[t] \[Psi]'[
       t] stilde [\[Alpha][t], \[Psi][t]] + \[Psi]'[t]^2 utilde [\[Alpha][
        t], \[Psi][t]] == 0,
   x'[t] == - rp \[Alpha]'[
       t] Sin[\[Theta][t]] + (Rp + rp Cos[\[Alpha][t]]) \[Chi]'[
       t] Cos[\[Theta][t]],
   y'[t] ==
    rp \[Alpha]'[t] Cos[\[Theta][t]] + (Rp + rp Cos[\[Alpha][t]]) \[Chi]'[
       t] Sin[\[Theta][t]],
   \[Chi]a'[
     t] == \[Chi]'[t] gc[\[Alpha][t], \[Psi][t]] + \[Alpha]'[
       t] hc[\[Alpha][t], \[Psi][t]] + \[Psi]'[t] lc[\[Alpha][t], \[Psi][t]],
   \[Theta]'[
     t] == \[Chi]'[t] mc[\[Alpha][t], \[Psi][t]] + \[Psi]'[
       t] pc[\[Alpha][t], \[Psi][t]]};


Needs["DifferentialEquations`NDSolveProblems`"];
Needs["DifferentialEquations`NDSolveUtilities`"];
Needs["FunctionApproximations`"];
```

```
Needs["DifferentialEquations`InterpolatingFunctionAnatomy`"];

sol = NDSolve[{ODE, \[Alpha][0] == 10 Pi/180, \[Alpha]'[0] ==
    0 Pi/180, \[Psi][0] == -5 Pi/180, \[Psi]'[0] == 0, \[Chi]'[0] ==
    65, \[Chi][0] == 0, x[0] == 0,
   y[0] == 0, \[Theta][0] == 0 Pi/180, \[Chi]a[0] ==
    0}, {\[Alpha], \[Psi], \[Chi], \[Chi]a, \[Theta], x, y}, {t, 0, 40},
  Method -> "Automatic", SolveDelayed -> True]
```

Bibliography

[AKN02] V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt. *Mathematical Aspects of Classical and Celestial Mechanics.* Springer-Verlag, Berlin, 2006, ed. or. 2002.

[AMR88] R. Abraham, J. E. Marsden, and T. S. Ratiu. *Manifolds, Tensor Analysis, and Applications.* Springer-Verlag, 1988.

[BBCM03] A. M. Bloch, J. Baillieul, P. Crouch, and J. E. Marsden. *Nonholonomic Mechanics and Control.* Springer-Verlag, Berlin, 2003.

[BKMM96] A. M. Bloch, P. S. Krishnaprasad, J. E. Marsden, and R. M. Murray. Nonholonomic Mechanical Systems with Symmetry. *Arch. Rat. Mech. An.*, 136:21–99, 1996.

[BMCP07] P. Basu-Mandal, A. Chatterjee, and J. M. Papadopoulos. Hands-free Circular Motions of a Benchmark Bicycle. *Proc. R. Soc. A*, 463:1983–2003, 2007.

[Boo75] W. M. Boothby. *An Introduction to Differentiable Manifolds and Riemannian Geometry.* New York Academic Press, 1975.

[Bou99] C. Bourlet. Étude Théorique sur la Bicyclette. *Bulletin de la S. M. F.*, XXVII:47–67, 76–96, 1899.

[BS93] L. Bates and J. Sniatycki. Nonholonomic Reduction. *Rep. Math. Phys.*, 32(1):99–115, 1993.

[CLM11] V. Cossalter, R. Lot, and M. Massaro. An Advanced Multibody Code for Handling and Stability Analysis of Motorcycles. *Meccanica*, 46:943–958, 2011.

[Cos06] V. Cossalter. *Motorcycle Dynamics.* Lulu, 2006.

[GPV95] G. Gentili, F. Podestà, and E. Vesentini. *Lezioni di Geometria Differenziale.* Bollati-Boringhieri, 1995.

[Han88] R. S. Hand. Comparisons and Stability Analysis of Linearized Equations of Motion for a Basic Bicycle Model. Master's thesis, Cornell University, May 1988.

[Her94] H. R. Hertz. *Der Prinzipien der Mechanik in neuem Zusammenhange dargestellt.* Barth, Leipzig, 1894.

[Hol08a] D. D. Holm. *Geometric Mechanics Part 1: Dynamics and Symmetry.* Imperial College Press, 2008.

[Hol08b] D. D. Holm. *Geometric Mechanics Part 2: Rotating, Translating and Rolling.* Imperial College Press, 2008.

[HSCS09] D. D. Holm, T. Schmah, and C. C. Stoica. *Geometric Mechanics and Symmetry: From Finite to Infinite Dimensions*. Oxford University Press, 2009.

[KM97a] W. S. Koon and J. E. Marsden. Optimal Control for Holonomic and Nonholonomic Mechanical Systems with Symmetry. *SIAM J. Control and Optim.*, 35:901–929, apr 1997.

[KM97b] W. S. Koon and J. E. Marsden. The Hamiltonian and Lagrangian Approaches to the Dynamics of Nonholonomic Systems. *Reports on Math. Phys.*, 40:21–62, nov 1997.

[KM98] W. S. Koon and J. E. Marsden. The Poisson Reduction of Nonholonomic Mechanical Systems. *Reports on Math. Phys.*, 42:101–134, 1998.

[KMP+11] J. D. K. Kooijman, J. P. Meijaard, J. M. Papadopoulos, A. Ruina, and A. L. Schwab. A Bicycle can be Self-stable Without Gyroscopic or Caster Effects. *Science*, 332(6027):339–342, April 2011.

[Koo97] W. S. Koon. Reduction, Reconstruction and Optimal Control for Nonholonomic Mechanical System with Symmetry. Doctor in philosophy in mahematics, University of Califonia, Berkeley, 1997.

[LM82] J. Lowell and H. D. McKell. The Stability of Bicycle. *Am. J. Phys.*, 50(1106), 1982.

[Mar96] C. M. Marle. Kinematic and Geometric Constraints, Servomechanism and Control of Mechanical Systems. *Rend. Sem. Mat. Univ. Pol. Torino*, 54:353–364, 1996.

[Mar98] C. M. Marle. Various Approaches to Conservative and Nonconservative Nonholonomic Systems. *Reports on Mathematical Physics*, 42:211–229, 1998.

[Mel12] E. Meli. Analisi della cinematica e della dinamica della bicicletta e sviluppo di sistemi di controllo innovativi. Technical report, Università degli Studi di Firenze, July 2012.

[MG95] J. E. Marsden and N. H. Getz. Control for an Autonomous Bicycle. In *IEEE, International Conference on Robotics and Automation*, May 1995.

[Mon02] J. C. Monforte. *Geometric, Control and Numerical Aspects of Nonholonomic Systems*. Springer-Verlag, 2002.

[MPRS07] J. P. Meijaard, J. M. Papadopoulos, A. Ruina, and A. L. Schwab. Linearized dynamics equations for the balance and steer of a bicycle: a benchmark and review. *Proc. R. Soc. A*, 463:1955–1982, 2007.

[MPRS11] J. P. Meijaard, J. M. Papadopoulos, A. Ruina, and A. L. Schwab. Historical Review of Thoughts on Bicycle Self-Stability. Technical report, April 2011. http://ecommons.library.cornell.edu/bitstream/1813/22497/2/StableHistoryv32Arend.pdf.

[MS06] J. P. Meijaard and A. L. Schwab. Linearized Equations for an Extended Bicycle Model. In *Proceedings of III European Conference on Computational Mechanics Solids, Structures and Coupled Problems in Engineering*, page pp. 18, June 2006.

[NF67] J. I. Neimark and N. A. Fufaev. *Dynamics of Nonholonomic Systems*. America Mathematical Society, Providence, Rhode Island, 1972, ed. or. 1967.

[RF12] F. Ricci and G. Frosali. Kinematics of a bicycle with toroidal wheels. *Communications in Applied and Industrial Mathematics*, 3(1), 2012.

[Ric11] F. Ricci. Studio della Dinamica e Controllo di un Modello Non-Lineare per la Bicicletta. Master's thesis, Univertità degli Studi di Firenze, April 2011.

[RR00] P. J. Rabier and W. C. Rheinboldt. *Nonholonomic Motion of Rigid Mechanical Systems from a DAE Viewpoint*. SIAM, 2000.

[TMS] Two-Mass-Skate (TMS) Bicycle on TUDelft website. http://bicycle.tudelft.nl/stablebicycle/.

[War71] F. W. Warner. *Foundations of Differentiable manifolds and Lie Groups*. Springer-Verlag, 1971.

[Whi99] F. J. W. Whipple. The Stability of the Motion of a Bicycle. *The Quarterly Journal of Pure and Applied Mathematics*, XXX:312–384, 1899.