An Orthogonality Property of the Legendre Polynomials

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Abstract

We give a remarkable additional othogonality property of the classical Legendre polynomials on the real interval $[-1,1]$: polynomials up to degree n from this family are mutually orthogonal under the arcsine measure weighted by the degree-n normalized Christoffel function.

Keywords: Legendre polynomials, Christoffel function, equilbrium measure

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Let $P_n(x)$ denote the classical Legendre polynomial of degree n and

$$P_n^*(x) := \frac{\sqrt{2n+1}}{\sqrt{2}} P_n(x)$$

its orthonormalized version. Thus, with $\delta_{i,j}$ the Kronecker delta, the family P_n^* satisfies

$$\int_{-1}^{1} P_i^*(x) P_j^*(x)\,\mathrm{d}x = \delta_{i,j}, \qquad\qquad i,j \geq 0$$

We consider the normalized (reciprocal of) the associated Christoffel function

$$K_n(x) := \frac{1}{n+1}\sum_{k=0}^{n}(P_k^*(x))^2. \tag{1}$$

As is well known, $K_n(x)dx$ tends weak-$*$ to the equilibrium measure of complex potential theory for the interval $[-1,1]$, and more precisely,

$$\lim_{n\to\infty} K_n(x) = \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}, \quad x \in (-1,1)$$

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locally uniformly. In other words,

$$\lim_{n\to\infty} \frac{1}{K_n(x)} \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}} = 1, \quad x \in (-1,1)$$

locally uniformly, and it would not be unexpected that, at least asymptotically,

$$\int_{-1}^{1} P_i^*(x)P_j^*(x) \frac{1}{K_n(x)} \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}} dx \approx \delta_{ij}, \quad 0 \le i,j \le n.$$

The purpose of this note is to show that the above is actually an identity.

Theorem 1 *With the above notation*

$$\int_{-1}^{1} P_i^*(x)P_j^*(x) \frac{1}{K_n(x)} \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}} dx = \delta_{ij}, \quad 0 \le i,j \le n. \tag{2}$$

We expect this result to have use in applied approximation problems. For example, one application lies in polynomial approximation of functions from point-evaluations. Our result indicates that the functions $\{Q_j(x)\}_{j=0}^n \triangleq \left\{ \frac{1}{\sqrt{K_n(x)}} P_j^*(x) \right\}_{j=0}^n$ are an orthonormal set on $(-1,1)$ under the Lebesgue density $\frac{1}{\pi\sqrt{1-x^2}}$. If we generate Monte Carlo samples from this density, evaluate an unknown function at these samples, and perform least-squares regression using Q_j as a basis, then a stability factor for this problem is given by $\max_{x\in[-1,1]} \sum_{j=0}^n Q_j^2(x) = n+1$ [1]. In fact, this is the smallest attainable stability factor, and therefore this approximation strategy has optimal stability.

The remainder of this document is devoted to the proof of (2).

Proof of Theorem 1. We change variables letting $x = \cos(\theta)$ to arrive at the equivalent expression

$$\frac{1}{\pi} \int_0^{\pi} \frac{P_i^*(\cos(\theta))P_j^*(\cos(\theta))}{K_n(\cos(\theta))} d\theta = \delta_{ij}, \quad 0 \le i,j \le n$$

which by symmetry is equivalent to

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{P_i^*(\cos(\theta))P_j^*(\cos(\theta))}{K_n(\cos(\theta))} d\theta = \delta_{ij}, \quad 0 \le i,j \le n. \tag{3}$$

Now, for $z \in \mathbb{C}$, let $J(z) := (z + 1/z)/2$. Then for $z = e^{i\theta}$ in the integral (3) we obtain, $d\theta = -iz^{-1}dz$, $\cos(\theta) = J(z)$, and the equation becomes

$$\frac{1}{2\pi i} \int_C \frac{z^{-1}P_i^*(J(z))P_j^*(J(z))}{K_n(J(z))} dz = \delta_{ij}, \quad 0 \le i,j \le n \tag{4}$$

where C is the unit circle, oriented in the counter-clockwise direction.

The proof is a direct calculation of (4) based on the following lemmas.

First note that $K_n(\cos(\theta))$ is a *positive* trigonometric polynomial (of degree $2n$). By the Féjer-Riesz Factorization Theorem there exists a trigonometric polynomial, $T_n(\theta)$ say, such that

$$K_n(\cos(\theta)) = |T_n(\theta)|^2.$$

In general the coefficients of the factor polynomial, $T_n(\theta)$ in this case, are algebraic functions of the coefficients of the original polynomial. However in our case we have the explict (essentially) rational factorization.

Proposition 1 *(Féjer-Riesz Factorization of $K_n(J(z))$) Let*

$$F_n(z) := \frac{d}{dz}\left(z^{n+1}P_n(J(z))\right) = (n+1)z^n P_n(J(z)) + \frac{z^{n-1}(z^2-1)}{2}P_n'(J(z)). \tag{5}$$

Then

$$K_n(J(z)) = \frac{1}{2(n+1)}F_n(z)F_n(1/z). \tag{6}$$

Proof. To begin, one may easily verify that

$$F_n(1/z) = z^{-2n}\{(n+1)z^n P_n(J(z)) - \frac{z^{n-1}(z^2-1)}{2}P_n'(J(z))\}. \tag{7}$$

Hence

$$\begin{aligned} F_n(z)F_n(1/z) &= z^{-2n}\{(n+1)^2 z^{2n}(P_n(J(z)))^2 - z^{2(n-1)}\left(\frac{z^2-1}{2}\right)^2 (P_n'(J(z))^2\} \\ &= (n+1)^2(P_n(J(z)))^2 - z^{-2}\left(\frac{z^2-1}{2}\right)^2 (P_n'(J(z))^2. \end{aligned}$$

Now notice that

$$\begin{aligned} z^{-2}\left(\frac{z^2-1}{2}\right)^2 &= \frac{1}{4}\left(z-\frac{1}{z}\right)^2 \\ &= \frac{1}{4}\left(z^2+2+\frac{1}{z^2}-4\right) \\ &= J^2(z)-1 \end{aligned}$$

so that

$$F_n(z)F_n(1/z) = (n+1)^2(P_n(J(z)))^2 - (J(z)^2-1)(P_n'(J(z)))^2.$$

The result follows then from Lemma 1, below. □

Lemma 1 *For all $x \in \mathbb{C}$, we have*

$$K_n(x) = \frac{1}{2(n+1)}\left((n+1)^2(P_n(x))^2 - (x^2-1)(P_n'(x))^2\right).$$

Proof. First, we collect the following known identities concerning Legendre polynomials [2]:

(Christoffel-Darboux formula)

$$\sum_{k=0}^{n} P_k^2(x) = \frac{n+1}{2}\left[P_{n+1}'(x)P_n(x) - P_{n+1}(x)P_n'(x)\right] \tag{8a}$$

(Three-term recurrence)

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \tag{8b}$$

(Differentiated three-term recurrence)

$$(n+1)P_{n+1}'(x) = (2n+1)\left(P_n(x)+xP_n'(x)\right) - nP_{n-1}'(x) \tag{8c}$$

$$(x^2-1)P_n'(x) = n\left(xP_n(x) - P_{n-1}(x)\right) \tag{8d}$$

$$(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x) \tag{8e}$$

We easily see from the Christoffel-Darboux formula that

$$K_n(x) = \frac{1}{2}\left(P'_{n+1}(x)P_n(x) - P_{n+1}(x)P'_n(x)\right).$$

Hence the result holds iff

$$\begin{aligned}(n+1)P'_{n+1}(x)P_n(x) - (n+1)P_{n+1}(x)P'_n(x)& \\ &= (n+1)^2(P_n(x))^2 - (x^2-1)(P'_n(x))^2\end{aligned}$$

$$\Updownarrow \text{(8c),(8b),(8d)}$$

$$\begin{aligned}\left\{(2n+1)\left(P_n(x) + xP'_n(x)\right) - nP'_{n-1}(x)\right\}P_n(x)& \\ - \left\{(2n+1)xP_n(x) - nP_{n-1}(x)\right\}P'_n(x)& \\ = (n+1)^2(P_n(x))^2 - n\left(xP_n(x) - P_{n-1}(x)\right)P'_n(x)&\end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned}(2n+1)\left(P_n(x)\right)^2 - nP'_{n-1}(x)P_n(x) + nP_{n-1}(x)P'_n(x)& \\ = (n+1)^2(P_n(x))^2 - nxP_n(x)P'_n(x) + nP_{n-1}(x)P'_n(x)&\end{aligned}$$

$$\Updownarrow$$

$$-n^2\left(P_n(x)\right)^2 - nP'_{n-1}(x)P_n(x) = -nxP_n(x)P'_n(x)$$

$$\Updownarrow$$

$$xP'_n(x) = nP_n(x) + P'_{n-1}(x)$$

$$\Updownarrow \text{(8c)}$$

$$\frac{1}{2n+1}\left((n+1)P'_{n+1}(x) + nP'_{n-1}(x)\right) - P_n(x) = nP_n(x) + P'_{n-1}(x)$$

$$\Updownarrow$$

$$(n+1)P'_{n+1}(x) = (n+1)P'_{n-1}(x) + (2n+1)(n+1)P_n(x),$$

and this last relation is the same as the relation (8e). □

There is somewhat more that can be said about $F_n(z)$.

Lemma 2 *Let $F_n(z)$ be the polynomial of degree $2n$ defined in (5). Then all of its zeros are simple and lie in the* **interior** *of the unit disk.*

Proof. The polynomial

$$Q_n(z) := z^{n+1}P_n(J(z)) = z\left\{z^nP_n(J(z))\right\}$$

has a zero at $z = 0$ and its other zeros are those of $P_n(J(z))$ which are those $z \in \mathbb{C}$

for which $J(z) = r \in (-1, 1)$, a zero of $P_n(x)$. But

$$\begin{aligned} & J(z) = r \in (-1,1) \\ \iff & (z + 1/z)/2 = r \\ \iff & z^2 - 2rz + 1 = 0 \\ \iff & z = r \pm i\sqrt{1-r^2}. \end{aligned}$$

In particular $|z| = 1$ for the zeros of $z^n P_n(J(z))$. It follows then from the Gauss-Lucas Theorem that the zeros of $F_n(z)$ are in the convex hull of $z = 0$ and certain points on the unit circle, i.e., are all in the *closed* unit disk.

Suppose a zero of $F_n(z)$ satisfies $|z| = 1$. By Proposition 1,

$$K_n(J(z)) = \frac{1}{2(n+1)} F_n(z) F_n(1/z),$$

so that $K_n(J(z))$ also vanishes. But $|z| = 1$ implies that $J(z) \in [-1, 1]$, and $K_n(J(z))$ thus cannot vanish. Therefore, no zeros of F_n lie on the unit circle.

To see that the zeros are all simple, an elemenary calculation and the ODE for Legendre polynomials gives us

$$F_n'(z) = 2n(n+1)z^{n-1}P_n(J(z)) + \{nz^n - (n+1)z^{n-2}\}P_n'(J(z)).$$

Hence $F_n(z) = F_n'(z) = 0$ if and only if

$$\begin{pmatrix} n+1 & \frac{z^2-1}{2} \\ \frac{2n(n+1)}{z} & nz - \frac{n+1}{z} \end{pmatrix} \begin{pmatrix} z^n P_n(J(z)) \\ z^{n-1}P_n'(J(z)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

But the determinant of this matrix is

$$(n+1)(nz - (n+1)/z) - n(n+1)(z^2-1)/z = -(n+1)/z \neq 0.$$

Hence $F_n(z) = F_n'(z) = 0$ if and only if $z^n P_n(J(z)) = z^{n-1}P_n'(J(z)) = 0$ if and only if $P_n(J(z)) = P_n'(J(z)) = 0$, which is not possible as $P_n(x)$ has only simple zeros. □

The integral (4) can therefore be expressed as

$$\begin{aligned} \frac{1}{2\pi i}\int_C \frac{z^{-1}P_i^*(J(z))P_j^*(J(z))}{K_n(J(z))}dz &= \frac{1}{2\pi i}\int_C \frac{2(n+1)z^{-1}P_i^*(J(z))P_j^*(J(z))}{F_n(z)F_n(1/z)}dz \\ &= \frac{1}{2\pi i}\int_C \frac{2(n+1)z^{2n-1}P_i^*(J(z))P_j^*(J(z))}{F_n(z)z^{2n}F_n(1/z)}dz \\ &= \frac{1}{2\pi i}\int_C \frac{2(n+1)z^{2n-1}P_i^*(J(z))P_j^*(J(z))}{F_n(z)G_n(z)}dz \end{aligned}$$

where we define the *polynomial* of degree $2n$,

$$G_n(z) := z^{2n}F_n(1/z). \tag{9}$$

As all the zeros of $F_n(z)$ are in the interior of the unit disk, the zeros of $G_n(z)$ are all exterior to the (closed) unit disk.

The following formulas for $F_n(z)$ and $G_n(z)$ will be useful.

Lemma 3 *We have*

$$F_n(z) = \frac{z^n}{z^2-1}\{((2n+1)z^2 - 1)P_n(J(z)) - 2nzP_{n-1}(J(z))\}$$

and

$$G_n(z) = \frac{z^n}{z^2-1}\{(z^2-(2n+1))P_n(J(z)) + 2nzP_{n-1}(J(z))\}.$$

Proof. From the formula (5) we have

$$F_n(z) = (n+1)z^n P_n(J(z)) + z^{n-1}\frac{z^2-1}{2}P_n'(J(z))$$

and from (7),

$$G_n(z) = (n+1)z^n P_n(J(z)) - z^{n-1}\frac{z^2-1}{2}P_n'(J(z)).$$

From the Legendre Polynomial identity (8d) with $x = J(z)$, we obtain

$$z^{n-1}\frac{z^2-1}{2}P_n'(J(z)) = 2n\frac{z^{n+1}}{z^2-1}J(z)P_n(J(z)) - 2n\frac{z^{n+1}}{z^2-1}P_{n-1}(J(z)).$$

Combining these gives the result. $\square$

It is also interesting to note that $F_n(z)$ is a certain Hypergeometric function.

Lemma 4 *We have*

1. *The polynomial $y = F_n(z)$ is a solution of the ODE*

$$(1-z^2)y'' + 2\frac{(n-2)z^2-n}{z}y' + 6ny = 0.$$

2. *If $F_n(z) =: f_n(z^2)$ then $y = f_n(z)$ is a solution of the Hypergeometric ODE*

$$z(1-z)y'' + (c-(a+b+1)z)y' - aby = 0$$

with $a = -n$, $b = 3/2$ and $c = 1/2 - n$.

3. $f_n(z) = 2^{-2n}\binom{2n}{n}\,{}_2F_1(a,b;c;z).$

4. $F_n(z) = 2^{-2n}\binom{2n}{n}\,{}_2F_1(a,b;c;z^2).$

5.

$$\begin{aligned} F_n(z) &= 2^{-2n}\binom{2n}{n}\sum_{k=0}^{n}\frac{(2k+1)\binom{n}{k}^2}{\binom{2n}{2k}}z^{2k} \\ &= 2^{-2n}\sum_{k=0}^{n}(2k+1)\binom{2k}{k}\binom{2n-2k}{n-k}z^{2k}. \end{aligned}$$

6. *If we write $F_n(z) = \sum_{k=0}^{n} c_k z^{2k}$, then*

$$\begin{aligned} G_n(z) &= \sum_{k=0}^{n} c_{n-k}z^{2k} \\ &= \sum_{k=0}^{n}\frac{2(n-k)+1}{2k+1}c_k z^{2k}. \end{aligned}$$

Proof. Equation (1) is easily verified using the ODE for $P_n(x)$ and the definition of $F_n(z)$, (5).

(2) follows by changing variables $z' = z^2$.

(3) follows as ${}_2F_1(a,b;c;z)$ is the only polynomial solution of the Hypergeometric ODE. The constant of proportionality is calculated by noting that the leading coefficient of ${}_2F_1(a,b;c;z)$ is $2n+1$ whereas that of $f_n(z)$ is $(2n+1)/2^n$ times the leading coefficient of $P_n(x)$, i.e., $(2n+1)/2^n \times \binom{2n}{n}/2^n$.

(4) is trivial from the defintion $f_n(z^2) := F_n(z)$.

(5) follows from the fact that

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{n} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, $$

with $(a)_k$ the rising factorial, and calculating

$$ (a)_k = (-1)^k \frac{n!}{(n-k)!}, \quad (b)_k = 2^{-2k} \frac{(2k+1)!}{k!}, \quad (c)_k = (-1)^k 2^{-2k} \frac{(2n)!(n-k)!}{(2n-2k)!n!} $$

so that

$$ \frac{1}{k!} \frac{(a)_k (b)_k}{(c)_k} = \frac{(2k+1)\binom{n}{k}^2}{\binom{2n}{2k}}. $$

(6) follows easy from the fact that $G_n(z) := z^n F_n(1/z) = \sum_{k=0}^{n} c_{n-k} z^k$ and that c_{n-k} is easily computed from the formula for c_k given in (5). $\square$

Returning the the proof of the Theorem, we will actually show that

(a) $$ \frac{1}{2\pi i} \int_C \frac{2(n+1) z^{2n-1} P_k(J(z))}{F_n(z) G_n(z)} dz = 0, \quad 1 \le k \le 2n, $$

(b) $$ \frac{1}{2\pi i} \int_C \frac{2(n+1) z^{2n-1} P_0(J(z))}{F_n(z) G_n(z)} dz = 2. $$

The Theorem follows directly as, for $i, j \le n$ we may expand

$$ P_i^*(x) P_j^*(x) = \sum_{k=0}^{2n} a_k P_k(x) $$

for certain coefficients a_k. From (a) and (b) we then conclude that

$$ \frac{1}{2\pi i} \int_C \frac{2(n+1) z^{2n-1} P_i^*(J(z)) P_j^*(J(z))}{F_n(z) G_n(z)} dz = 2a_0. $$

But for $i \ne j$,

$$ a_0 = \frac{1}{2} \int_{-1}^{1} P_i^*(x) P_j^*(x) P_0(x) dx = 0 $$

as $P_0(x) = 1$ and $P_i^*(x)$ and $P_j^*(x)$ are orthogonal. While for $i = j$, we have

$$ a_0 = \frac{1}{2} \int_{-1}^{1} (P_i^*(x))^2 P_0(x) dx = \frac{1}{2}. $$

We will now compute the partial fraction decomposition of the integrands in (a) and (b),

$$ \frac{2(n+1) z^{2n-1} P_k(J(z))}{F_n(z) G_n(z)}, $$

which will involve the following two pairs of families of functions.

Definition 1 *We define*

$$ A_0^{(n)}(z) := z^{n-1}, \quad A_1^{(n)}(z) := z^{n-2}, $$

$$B_0^{(n)}(z) := z^{n-1}, \quad B_1^{(n)}(z) := z^n,$$

with

$$(n+k+1)A_{k+1}^{(n)} := (2(n+k)+1)J(z)A_k^{(n)}(z) - (n+k)A_{k-1}^{(n)}(z), \quad k = 1, 2, \cdots$$

and

$$(n+k+1)B_{k+1}^{(n)} := (2(n+k)+1)J(z)B_k^{(n)}(z) - (n+k)B_{k-1}^{(n)}(z), \quad k = 1, 2, \cdots .$$

Further, we let

$$C_0^{(n)}(z) := z^{n-1}, \quad C_1^{(n)}(z) := z^{n-2}\frac{(2n+1)z^2-1}{2n},$$

$$D_0^{(n)}(z) := z^{n-1}, \quad D_1^{(n)}(z) := z^{n-2}\frac{(2n+1)-z^2}{2n},$$

with

$$(n-k)C_{k+1}^{(n)}(z) := (2(n-k)+1)J(z)C_k^{(n)}(z) - (n-k+1)C_{k-1}^{(n)}(z), \quad 1 \le k \le n-1$$

and

$$(n-k)D_{k+1}^{(n)}(z) := (2(n-k)+1)J(z)D_k^{(n)}(z) - (n-k+1)D_{k-1}^{(n)}(z), \quad 1 \le k \le n-1.$$

Proposition 2 *We have*

$$2(n+1)z^{2n-1}P_{n+k}(J(z)) = A_k^{(n)}(z)G_n(z) + B_k^{(n)}(z)F_n(z), \quad k = 0, 1, 2, \cdots \tag{10}$$

so that

$$\frac{2(n+1)z^{2n-1}P_{n+k}(J(z))}{F_n(z)G_n(z)} = \frac{A_k^{(n)}(z)}{F_n(z)} + \frac{B_k^{(n)}(z)}{G_n(z)}, \quad k = 0, 1, 2, \cdots$$

and

$$2(n+1)z^{2n-1}P_{n-k}(J(z)) = C_k^{(n)}(z)G_n(z) + D_k^{(n)}(z)F_n(z), \quad 0 \le k \le n \tag{11}$$

so that

$$\frac{2(n+1)z^{2n-1}P_{n-k}(J(z))}{F_n(z)G_n(z)} = \frac{C_k^{(n)}(z)}{F_n(z)} + \frac{D_k^{(n)}(z)}{G_n(z)}, \quad 0 \le k \le n.$$

Proof (by induction). As $A_0^{(n)} = C_0^{(n)} = z^{n-1}$ and $B_0^{(n)} = D_0^{(n)} = z^{n-1}$, the $k = 0$ case is in common and we calculate

$$\begin{aligned}
&= z^{n-1}(F_n(z) + G_n(z)) \\
&\overset{\text{(Lemma 3)}}{=} z^{n-1}\frac{z^n}{z^2-1}\{((2n+1)z^2-1) + (z^2-(2n+1))\}P_n(J(z)) \\
&= z^{2n-1}\frac{1}{z^2-1}\{2(n+1)z^2 - 2(n+1)\}P_n(J(z)) \\
&= 2(n+1)z^{2n-1}P_n(J(z)),
\end{aligned}$$

as desired.

We now prove (10) for the case $k = 1$. We calculate, using Lemma 3,

$$
\begin{aligned}
& A_1^{(n)}(z)G_n(z) + B_1^{(n)}(z)F_n(z) \\
& = z^{n-2}G_n(z) + z^n F_n(z) \\
& = z^{n-2}(G_n(z) + z^2 F_n(z)) \\
& = z^{n-2}\frac{z^n}{z^2-1}\{[(z^2-(2n+1))P_n(J(z)) + 2nzP_{n-1}(J(z))] \\
& \qquad + z^2[((2n+1)z^2-1)P_n(J(z)) - 2nzP_{n-1}(J(z))]\} \\
& = \frac{z^{2n-2}}{z^2-1}\{(2n+1)(z^4-1)P_n(J(z)) - 2nz(z^2-1)P_{n-1}(J(z))\} \\
& = z^{2n-2}\{(2n+1)(z^2+1)P_n(J(z)) - 2nzP_{n-1}(J(z))\} \\
& = z^{2n-1}\{(2n+1)(z+1/z)P_n(J(z)) - 2nP_{n-1}(J(z))\} \\
& = z^{2n-1}\{(2n+1)2J(z)P_n(J(z)) - 2nP_{n-1}(J(z))\} \\
& \overset{(8b)}{=} 2(n+1)z^{2n-1}P_{n+1}(J(z)).
\end{aligned}
$$

Now for (11) for $k = 1$. We calculate, again using Lemma 3,

$$
\begin{aligned}
& C_1^{(n)}(z)G_n(z) + D_1^{(n)}(z)F_n(z) \\
& = z^{n-2}\left\{\frac{(2n+1)z^2-1}{2n}G_n(z) + \frac{(2n+1)-z^2}{2n}F_n(z)\right\} \\
& = z^{n-2}\frac{z^n}{z^2-1}\left\{\frac{(2n+1)z^2-1}{2n}[(z^2-(2n+1))P_n(J(z)) + 2nzP_{n-1}(J(z))]\right. \\
& \qquad \left. + \frac{(2n+1)-z^2}{2n}[((2n+1)z^2-1)P_n(J(z)) - 2nzP_{n-1}(J(z))]\right\} \\
& = \frac{z^{2n-2}}{z^2-1}2nz\left\{\frac{(2n+1)z^2-1}{2n} - \frac{(2n+1)-z^2}{2n}\right\}P_{n-1}(J(z)) \\
& = \frac{z^{2n-1}}{z^2-1}((2(n+1)z^2 - 2(n+1))P_{n-1}(J(z)) \\
& = 2(n+1)z^{2n-1}P_{n-1}(J(z)).
\end{aligned}
$$

The rest of the proof proceeds by induction. Assuming that (10) and (11) hold from 0 up to a certain k, we prove that they also hold for $k+1$. To this end we calculate

$$
\begin{aligned}
& A_{k+1}^{(n)}(z)G_n(z) + B_{k+1}^{(n)}(z)F_n(z) \\
& = \frac{(2(n+k)+1)J(z)A_k^{(n)}(z) - (n+k)A_{k-1}^{(n)}(z)}{n+k+1}G_n(z) \\
& \quad + \frac{(2(n+k)+1)J(z)B_k^{(n)}(z) - (n+k)B_{k-1}^{(n)}(z)}{n+k+1}F_n(z) \\
& = \frac{1}{n+k+1}\{(2(n+k)+1)J(z)[A_k^{(n)}(z)G_n(z) + B_k^{(n)}(z)F_n(J(z))] \\
& \quad - (n+k)[A_{k-1}^{(n)}(z)G_n(z) + B_{k-1}^{(n)}F_n(J(z))] \\
& = 2(n+1)z^{2n-1}\frac{1}{n+k+1}\{(2(n+k)+1)J(z)P_{n+k}(J(z)) \\
& \quad - (n+k)P_{n+k-1}(J(z))\} \quad \text{(by the induction hypothesis)} \\
& = 2(n+1)z^{2n-1}P_{n+k+1}(J(z)),
\end{aligned}
$$

by the three-term recursion formula for Legendre polynomials (8b) with degree $m =$

$n+k$.

Similarly,

$$\begin{aligned}
&C_{k+1}^{(n)}(z)G_n(z)+D_{k+1}^{(n)}(z)F_n(z)\\
&=\frac{(2(n-k)+1)J(z)C_k^{(n)}(z)-(n-k+1)C_{k-1}^{(n)}(z)}{n-k}G_n(z)\\
&\quad+\frac{(2(n-k)+1)J(z)D_k^{(n)}(z)-(n-k+1)D_{k-1}^{(n)}(z)}{n-k}F_n(z)\\
&=\frac{1}{n-k}\{(2(n-k)+1)J(z)[C_k^{(n)}(z)G_n(z)+D_k^{(n)}(z)F_n(J(z))]\\
&\quad-(n-k+1)[C_{k-1}^{(n)}(z)G_n(z)+D_{k-1}^{(n)}F_n(J(z))]\\
&=2(n+1)z^{2n-1}\frac{1}{n-k}\{(2(n-k)+1)J(z)P_{n-k}(J(z))\\
&\quad-(n-k+1)P_{n-(k-1)}(J(z))\}\quad\text{(by the induction hypothesis)}\\
&=2(n+1)z^{2n-1}P_{n-(k+1)}(J(z)),
\end{aligned}$$

using the reverse three-term recursion,

$$mP_{m-1}(x)=(2m+1)xP_m(x)-(m+1)P_{m+1}(x)$$

with $m=n-k$. □

Due to the $J(z)$ factor in the recursive definitions of Definition 1, the functions $A_k^{(n)}(z)$, $B_k^{(n)}(z)$, $C_k^{(n)}(z)$ and $D_k^{(n)}(z)$ are all Laurent polynomials. It is easy to verify that they have the forms:

- $A_k^{(n)}(z)=\sum_{j=n-(k+1)}^{n+(k-3)}a_kz^k$, $k\geq 1$
- $B_k^{(n)}(z)=\sum_{j=n-(k-1)}^{n+(k-1)}b_kz^k$, $k\geq 1$
- $C_k^{(n)}(z)=\sum_{j=n-(k+1)}^{n+(k-1)}c_kz^k$, $1\leq k\leq n$
- $D_k^{(n)}(z)=\sum_{j=n-(k+1)}^{n+(k-1)}d_kz^k$, $1\leq k\leq n$.

In particular, for $0\leq k\leq n-1$ the functions $A_k^{(n)}(z)$, $B_k^{(n)}(z)$, $C_k^{(n)}(z)$ are all *polynomials* of degree at most $2n-2$.

Hence, for $1\leq k\leq 2n-1$,

$$\frac{1}{2\pi i}\int_C\frac{2(n+1)z^{2n-1}P_k(J(z))}{F_n(z)G_n(z)}dz=\frac{1}{2\pi i}\left\{\int_C\frac{p(z)}{F_n(z)}dz+\int_C\frac{q(z)}{G_n(z)}dz\right\}$$

for certain polynomials $p(z)$ and $q(z)$ of degree at most $2n-2$.

Now, $\int_C\frac{q(z)}{G_n(z)}dz=0$ as all the zeros of $G_n(z)$ lie outside the (closed) unit disk. Further, if we let z_j $1\leq j\leq 2n$, be the (simple) zeros of $F_n(z)$, we may write

$$\frac{p(z)}{F_n(z)}=\frac{p(z)/c_n}{F_n(z)/c_n}=\sum_{j=1}^{2n}\frac{R_j}{z-z_j}$$

where c_n is the leading coefficient of $F_n(z)$. Hence

$$\frac{1}{2\pi i}\int C\frac{A_n(z)}{F_n(z)}dz = \frac{1}{2\pi i}2\pi i\sum_{j=1}^{2n} R_j.$$

But $\sum_{j=1}^{2n} R_j$ is the leading coefficient (of z^{2n-1}) of $p(z)/c_n$, i.e., 0, as $p(z)$ is of degree at most $2n-2$. It follows that

$$\frac{1}{2\pi i}\int_C \frac{2(n+1)z^{2n-1}P_k(J(z))}{F_n(z)G_n(z)}dz = 0, \quad 1 \leq k \leq 2n-1.$$

The cases $P_0(J(z))$ and $P_{2n}(J(z))$ are special.

First consider the case $P_{2n}(J(z))$. We have from Proposition 2, with $k = n$,

$$2(n+1)z^{2n-1}P_{2n}(J(z)) = A_n^{(n)}(z)G_n(z) + B_n^{(n)}(z)F_n(z).$$

However, $A_n^{(n)}(z) = \sum_{j=-1}^{2n-3} a_j z^j$ has a z^{-1} while $B_n^{(n)}(z) = \sum_{j=+1}^{2n-1} b_j z^j$ is still a polynomial, of degree at most $2n-1$. Therefore it is still the case that $\int_C \frac{B_n^{(n)}(z)}{G_n(z)}dz = 0$ (the zeros of $G_n(z)$ being all oustide the unit disk). We need to show that $\int_C \frac{A_n^{(n)}(z)}{F_n(z)}dz = 0$. Write $A_n^{(n)}(z) = q(z) + c/z$ where $q(z)$ is a polynomial of degree $2n-3$. Then

$$\int_C \frac{A_n^{(n)}(z)}{F_n(z)}dz = \int_C \frac{q(z)}{F_n(z)}dz + c\int_C \frac{1}{zF_n(z)}dz.$$

The first integral on the right is zero as the coefficient in $q(z)$ of z^{2n-1} is 0. For the second integral, decompose

$$\begin{aligned} c\int_C \frac{1}{zF_n(z)}dz &= c\int_C \frac{1}{F_n(0)}\left\{\frac{1}{z} - \frac{(F_n(z)-F_n(0))/z}{F_n(z)}\right\}dz \\ &= \frac{1}{F_n(0)}\left\{\int_C \frac{1}{z}dz - \int_C \frac{(F_n(z)-F_n(0))/z}{F_n(z)}\right\}dz. \end{aligned}$$

The first integral is trivially $2\pi i$, while the second is $2\pi i$ times the coefficient of z^{2n-1} in $(F_n(z)-F_n(0))/z$ divided by the leading coefficient (of z^{2n}) in $F_n(z)$, i.e., $2\pi i \times 1$. Hence, indeed

$$\int_C \frac{A_n^{(n)}(z)}{F_n(z)}dz = 0.$$

Lastly we calculate

$$\frac{1}{2\pi i}\int_C \frac{2(n+1)z^{2n-1}P_0(J(z))}{F_n(z)G_n(z)}dz = \frac{1}{2\pi i}\int_C \frac{2(n+1)z^{2n-1}}{F_n(z)G_n(z)}dz.$$

We still have (11) from Proposition 2,

$$2(n+1)z^{2n-1}P_{n-n}(z) = C_n^{(n)}(z)G_n(z) + D_n^{(n)}(z)F_n(z)$$

However, $C_n^{(n)}(z)$ and $D_n^{(n)}(z)$ have the form

$$C_n^{(n)}(z) = \sum_{j=-1}^{2n-1} c_k z^k, \quad D_n^{(n)}(z) = \sum_{j=-1}^{2n-1} d_k z^k,$$

i.e., are both of the form $p(z) + c/z$ where $p(z)$ is a polynomial of degree $2n - 1$. Specifically, write $C_n^{(n)}(z) = p(z) + c/z$ and $D_n^{(n)}(z) = q(z) + d/z$. We thus have the following expression:

$$\begin{aligned}\frac{1}{2\pi i}\int_C \frac{2(n+1)z^{2n-1}P_0(z)}{F_n(z)G_n(z)}\mathrm{d}z &= \frac{1}{2\pi i}\int_C \frac{C_n^{(n)}(z)}{F_n(z)}\mathrm{d}z + \frac{1}{2\pi i}\int_C \frac{D_n^{(n)}(z)}{G_n(z)}\mathrm{d}z \\ &= \frac{1}{2\pi i}\left[\int_C \frac{p(z)}{F_n(z)}\mathrm{d}z + \int_C \frac{c}{zF_n(z)}\mathrm{d}z \right. \\ &\quad \left. + \int_C \frac{q(z)}{G_n(z)}\mathrm{d}z + \int_C \frac{d}{zG_n(z)}\mathrm{d}z\right].\end{aligned}$$

We need to show that this expression has value 2. Now we have already shown that $\int_C \frac{1}{zF_n(z)}dz = 0$ and also remarked that $\int_C \frac{q(z)}{G_n(z)}dz = 0$. Hence we need to calculate $\frac{1}{2\pi i}\int_C \frac{p(z)}{F_n(z)}dz$ and $\frac{1}{2\pi i}\int_C \frac{d}{zG_n(z)}dz$. But the first of these is just the (leading) coefficient of z^{2n-1} in $p(z)$, i.e., in $C_n^{(n)}(z)$ divided by the leading coefficient of $F_n(z)$. From Lemma 4 we have that

$$F_n(z) = 2^{-2n}(2n+1)\binom{2n}{n}z^{2n} + \cdots$$

and it is easy to verify by induction that also $C_n^{(n)}(z) = 2^{-2n}(2n+1)\binom{2n}{n}z^{2n} + \cdots$. Hence

$$\frac{1}{2\pi i}\int_C \frac{p(z)}{F_n(z)}dz = 1.$$

For the second integral, decompose as before

$$\frac{1}{2\pi i}\int_C \frac{d}{zG_n(z)}dz = \frac{d}{2\pi i}\frac{1}{G_n(0)}\int_C \left\{\frac{1}{z} - \frac{(G_n(z) - G_n(0))/z}{G_n(z)}\right\}dz.$$

The second integral above is zero as all the zeros of $G_n(z)$ are outside the closed unit disk. The first integral is trivially $d/G_n(0)$. From Lemma 4 parts 4. and 6. we have that

$$G_n(0) = (2n+1)F_n(0) = 2^{-2n}(2n+1)\binom{2n}{n}$$

whereas the coefficient of z^{-1} in $B_n^{(n)}(z)$ is easily verified by induction to have the same value. Hence

$$\frac{1}{2\pi i}\int_C \frac{d}{zG_n(z)}dz = 1$$

and we have shown that

$$\frac{1}{2\pi i}\int_C \frac{2(n+1)z^{2n-1}P_0(J(z))}{F_n(z)G_n(z)}dz = 1 + 1 = 2,$$

as claimed. □

References

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